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Discrete Halanay-type inequalities and applications

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Abstract

In this paper we derive new discrete Halanay-type inequalities and show some applications in the investigation of the asymptotic behavior of nonlinear difference equations. In particular, difference equations involving the "maximum" functional are considered.

Keywords: Global asymptotic stability; Difference equations; Halanay inequality

1. Introduction

It is well known that in the theory of functional differential equations, it is useful to employ differential inequalities involving the max functional (see e.g. [3,4,8]) to investigate the asymptotic stability.

In [7], we showed that some discrete versions of these max inequalities can be applied to study the global asymptotic stability of generalized difference equations (see also [10]). In particular, we include here the two main results in [7] for convenience of the reader.

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Theorem 1 (Liz and Ferreiro [7, Theorem 1]). *Let $r > 0$ be a natural number, and let $\{x_n\}_{n \geqslant -r}$ be a sequence of real numbers satisfying the inequality*

$$\Delta x_n \leqslant -ax_n + b\max\{x_n, x_{n-1}, \ldots, x_{n-r}\}, \quad n \geqslant 0, \tag{1}$$

where $\Delta x_n = x_{n+1} - x_n$.

If $0 < b < a \leqslant 1$, then there exists a constant $\lambda_0 \in (0,1)$ such that

$$x_n \leqslant \max\{0, x_0, x_{-1}, \ldots, x_{-r}\}\lambda_0^n, \quad n \geqslant 0.$$

Moreover, λ_0 can be chosen as the root in the interval $(0,1)$ of the equation

$$\lambda^{r+1} + (a-1)\lambda^r - b = 0. \tag{2}$$

We shall refer to this result as the *discrete Halanay lemma*. By a simple use of Theorem 1, we also demonstrated the following statement:

Theorem 2 (Liz and Ferreiro [7, Theorem 2]). *Assume that $0 < a \leqslant 1$ and there exists a positive constant $b < a$ such that*

$$|f(n, x_n, \ldots, x_{n-r})| \leqslant b\|(x_n, \ldots, x_{n-r})\|_\infty, \quad \forall (x_n, \ldots, x_{n-r}) \in \mathbb{R}^{r+1}. \tag{3}$$

Then there exists $\lambda_0 \in (0,1)$ such that

$$|x_n| \leqslant \left(\max_{-r \leqslant i \leqslant 0} \{|x_i|\}\right)\lambda_0^n, \quad n \geqslant 0,$$

for every solution $\{x_n\}$ of

$$\Delta x_n = -ax_n + f(n, x_n, x_{n-1}, \ldots x_{n-r}), \quad a > 0, \tag{4}$$

where λ_0 can be calculated in the form established in Theorem 1.

In this paper we prove a generalization of Theorem 1, which allows us to obtain new conditions for the asymptotic stability of a family of difference equations. In particular, in Section 3 we introduce a discrete analog of the Yorke condition (see [5, Section 4.5]).

A class of difference equations closely related with the discrete-type Halanay inequalities are the difference equations with maxima (see [4] for a discussion in the continuous case). The asymptotic stability of these equations is addressed in Section 4.

Finally, in Section 5 we show the application of Theorem 2 to the Lozi map, which is a piecewise linear version of the Hénon map (see [6,9] and references therein).

2. Generalized discrete Halanay inequality

In this section, we give a generalization of the discrete Halanay lemma.

Let us consider the following inequalities:

$$\Delta u_n \leqslant -Au_n + B\widetilde{u_n} + Cv_n + D\widehat{v_n}, \quad n \geqslant 0, \tag{5}$$

$$u_n \leqslant (1-A)^n u_0 + \sum_{i=0}^{n-1} (1-A)^{n-i-1}[B\widetilde{u_i} + Cv_i + D\widehat{v_i}], \quad n \geqslant 0, \tag{6}$$

$$v_n \leqslant Eu_n + F\widetilde{u_n}, \quad n \geqslant 0, \tag{7}$$

where $\Delta u_n = u_{n+1} - u_n$, $\widetilde{u_n} = \max\{u_n, \dots, u_{n-r}\}$, $\widehat{v_n} = \max\{v_{n-1}, \dots, v_{n-r}\}$, $r \geqslant 1$, and A, B, C, D, E, F are real constants.

Denote $u = \{u_n\}_{n \geqslant -r}$, $v = \{v_n\}_{n \geqslant -r}$. First we observe that for $A \leqslant 1$ it is not difficult to prove by induction that if the pair (u, v) satisfies inequality (5), then it also satisfies (6).

Theorem 3. *Assume that (u, v) satisfies the system of inequalities (6)–(7). If $B, C, D, E, F \geqslant 0$, $FD + B > 0$, $E + F > 0$ and*

$$B + (E+F)(C+D) < A \leqslant 1, \tag{8}$$

then there exist constants $K_1 \geqslant 0$, $K_2 \geqslant 0$, and $\lambda_0 \in (0,1)$ such that

$$u_n \leqslant K_1 \lambda_0^n, \quad v_n \leqslant K_2 \lambda_0^n, \quad n \geqslant 0.$$

Moreover, λ_0 can be chosen as the smallest root in the interval $(0,1)$ of the equation $h(\lambda) = 0$, where

$$h(\lambda) = \lambda^{2r+1} - (1 - A + CE)\lambda^{2r} - (B + FC + ED)\lambda^r - FD. \tag{9}$$

Proof. Let (x, y) be a solution of the system

$$\left.\begin{aligned} &x_n = (1-A)^n x_0 + \sum_{i=0}^{n-1} (1-A)^{n-i-1}[B\widetilde{x_i} + Cy_i + D\widehat{y_i}], \quad n \geqslant 0, \\ &y_n = Ex_n + F\widetilde{x_n}, \quad n \geqslant 0, \end{aligned}\right\} \tag{10}$$

Since $A \leqslant 1$, it is easy to prove by induction that if $u_n \leqslant x_n$ and $v_n \leqslant y_n$ for $n = -r, \dots, 0$, then $u_n \leqslant x_n$ and $v_n \leqslant y_n$ for all $n \geqslant 0$.

On the other hand, System (10) is equivalent to

$$\left.\begin{aligned} &\Delta x_n = -Ax_n + B\widetilde{x_n} + Cy_n + D\widehat{y_n}, \quad n \geqslant 0, \\ &y_n = Ex_n + F\widetilde{x_n}, \quad n \geqslant 0. \end{aligned}\right\} \tag{11}$$

Next we prove that, under the assumptions of the theorem, there exists a solution (x, y) to system (11) in the form $x_n = \lambda_0^n$, $y_n = \alpha\lambda_0^n$, with $\alpha > 0$, $\lambda_0 \in (0,1)$. Indeed, such (x, y) is a solution of (11) if and only if

$$\left.\begin{aligned} &\lambda_0^{n+1} = (1-A)\lambda_0^n + B\lambda_0^{n-r} + C\alpha\lambda_0^n + D\alpha\lambda_0^{n-r}, \\ &\alpha\lambda_0^n = E\lambda_0^n + F\lambda_0^{n-r}. \end{aligned}\right\} \tag{12}$$

This is equivalent to the existence of a solution $\lambda_0 \in (0,1)$ of the equation $h(\lambda) = 0$, where h is the polynomial defined by (9).

Now, $h(0) = -FD < 0$ (if $FD = 0$, we consider $g(\lambda) = h(\lambda)\lambda^{-r}$ and thus $g(0) = -(B + FC + ED) < 0$). On the other hand, $h(1) = A - B - (E + F)(C + D) > 0$ in view of (8). As a consequence, there exists $\lambda_0 \in (0,1)$ such that $h(\lambda_0) = 0$. Hence (λ_0, α) is a solution of (12) with $\alpha = E + F\lambda_0^{-r} > 0$.

For this value of λ_0, the pair (Kx, Ky) is a solution of (11) for all $K \geqslant 0$. Thus, choosing $K = \max\{0, u_{-r}, \ldots, u_0, \alpha^{-1}v_{-r}, \ldots, \alpha^{-1}v_0\}$, we have that $u_n \leqslant K\lambda_0^n$ and $v_n \leqslant K\alpha\lambda_0^n$ for $n = -r, \ldots, 0$. Hence, using the first part of the proof, we conclude that $u_n \leqslant K\lambda_0^n$ and $v_n \leqslant K\alpha\lambda_0^n$ for all $n \geqslant 0$. $\square$

Remark. For $E = F = 0$ and $v_n \leqslant 0$, $n = -r, \ldots, 0$, we can obtain the same result with $\alpha = 0$, $K = \max\{0, u_{-r}, \ldots, u_0\}$. In particular, since (5) implies (6) for $A \leqslant 1$, setting $v \equiv 0$ we get the conclusion of Theorem 1.

3. Global asymptotic stability of difference equations

In order to show the applicability of the previous result, in this section we consider the following generalized difference equation

$$\Delta x_n = -ax_n - bf(n, x_n, x_{n-1}, \ldots, x_{n-r}), \qquad b > 0. \tag{13}$$

Given $r + 1$ points $\{x_{-r}, x_{-r+1}, \ldots, x_0\}$ there is a unique solution $\{x_n\}$ of Eq. (13), which can be explicitly calculated by iterations. Next, we study the asymptotic behavior of its solutions by using Theorem 3.

We will assume that f satisfies the following conditions:

(H1) $|f(n, x_n, \ldots, x_{n-r})| \leqslant \|(x_n, \ldots, x_{n-r})\|_\infty$, $\forall (x_n, \ldots, x_{n-r}) \in \mathbb{R}^{r+1}$.
(H2) $|f(n, x_n, \ldots, x_{n-r}) - x_n| \leqslant r\|(\Delta x_{n-1}, \ldots, \Delta x_{n-r})\|_\infty$, $\forall (x_n, \ldots, x_{n-r}) \in \mathbb{R}^{r+1}$.

Hypotheses (H1) and (H2) are satisfied for some important linear and nonlinear difference equations. Since

$$|\max\{x_n, \ldots, x_{n-r}\}| \leqslant \max\{|x_n|, \ldots, |x_{n-r}|\}$$

and

$$x_n - x_i = \sum_{j=i}^{n-1} \Delta x_j \leqslant r \max\{\Delta x_{n-r}, \ldots, \Delta x_{n-1}\}, \qquad \forall i = n - r, \ldots, n,$$

it is easy to check that (H1) and (H2) hold if the following condition is satisfied

$$\min\{x_n, \ldots, x_{n-r}\} \leqslant f(n, x_n, \ldots, x_{n-r}) \leqslant \max\{x_n, \ldots, x_{n-r}\}. \tag{14}$$

Condition (14) is an analog of the so-called Yorke condition for functional differential equations (see [4,5]).

Example. Let us consider the following difference equation of order 5.

$$x_{n+1} = -a(n)x_n - b(n)x_{n-4}. \tag{15}$$

Theorem 2 provides the global asymptotic stability of the trivial solution if

$$\sup_{n\in\mathbb{N}} \{|a(n)| + |b(n)|\} < 1. \tag{16}$$

Considering the case of constant coefficients $a(n) \equiv a$, $b(n) \equiv b > 0$, we distinguish two cases. If $a > 0$, condition (16) becomes $a + b < 1$, which is the exact condition for the asymptotic stability of zero obtained by applying the Schur–Cohn criterion (see [1, Theorem 4.12]). However, if $a < 0$, the exact condition for the asymptotic stability is $|a + b| < 1$, $b(b - a) < 1$, so in this case the condition $b - a < 1$ given by (16) is not sharp.

Next we obtain new conditions for the asymptotic stability of Eq. (13) using Theorem 3.

Theorem 4. *Assume that f satisfies hypotheses* (H1) *and* (H2). *If either*

(a) $0 \leqslant a \leqslant 1 - b$ *and* $0 < br < 1$, *or*
(b) $a < 0$ *and* $0 < br < (a + b)(-a + b)^{-1}$

holds, then there exist $K > 0$ and $\lambda_0 \in (0, 1)$ such that for every solution $\{x_n\}$ of (13), *we have*

$$|x_n| \leqslant K\lambda_0^n, \qquad n \geqslant 0,$$

where λ_0 can be calculated in the form established in Theorem 3.

As a consequence, the trivial solution of Eq. (13) *is globally asymptotically stable.*

Proof. Eq. (13) can be written in the form

$$\Delta x_n = -(a + b)x_n - b(f(n, x_n, x_{n-1}, \ldots, x_{n-r}) - x_n).$$

Hence,

$$x_n = (1 - (a + b))^n x_0 + \sum_{i=0}^{n-1} (1 - (a + b))^{n-1-i}(-b)(f(i, x_i, x_{i-1}, \ldots, x_{i-r}) - x_i).$$

Note that if either condition (a) or (b) is satisfied, then $1 - (a + b) \geqslant 0$. Now, using (H2), we have

$$\begin{aligned} |x_n| &\leqslant (1 - (a + b))^n |x_0| \\ &\quad + \sum_{i=0}^{n-1} (1 - (a + b))^{n-1-i} br \max\{|\Delta x_{i-1}|, \ldots, |\Delta x_{i-r}|\}. \end{aligned} \tag{17}$$

On the other hand, using hypothesis (H1) in Eq. (13),

$$|\Delta x_n| \leqslant |a||x_n| + b \max\{|x_n|, |x_{n-1}|, \ldots, |x_{n-r}|\}. \tag{18}$$

Denote $u_n = |x_n|$, $v_n = |\Delta x_n|$. We can apply Theorem 3 to the system of inequalities (17) and (18), with $A = a + b$, $B = 0$, $C = 0$, $D = br$, $E = |a|$, $F = b$. The proof is completed by noting that condition (8) is satisfied if

$$(|a| + b)br < a + b \leqslant 1. \qquad \square$$

Remark. While the discrete Halanay lemma only applies when $|b| < a$, Theorem 4 allows us to obtain the global asymptotic stability in (13) for some values of $b > a > 0$ and also for $a < 0$.

In particular, when we apply Theorem 4 to Eq. (15) with constant coefficients $a < 0$, $b > 0$, we obtain the asymptotic stability under any of the following conditions:

(i) $a \geqslant -1$, $a + b \leqslant 0$ and $4b < 1$,
(ii) $a < -1$ and $4b(b - a - 1) < (1 + a + b)$,

improving the region of the plane of parameters (a, b) given by (16). (Notice that in this case, Eq. (15) can be rewritten as $\Delta x_n = -(a + 1)x_n - bx_{n-4}$.)

We also emphasize that Theorem 2 provides delay-independent conditions for the global asymptotic stability, while the conditions of Theorem 4 are dependent of the delay r in Eq. (13).

4. Difference equations with maxima

In this section we study some asymptotic properties of the following class of difference equations:

$$\Delta x_n = -ax_n + b\max\{x_n, x_{n-1}, \ldots, x_{n-r}\}. \tag{19}$$

For recent papers on difference equations involving the *max* functional, see [2].

On the other hand, Eq. (19) can be regarded as a discrete analog of differential equations with maxima (see [4,11]).

Since function f defined by $f(x_1, \ldots, x_n) = \max\{x_1, \ldots, x_n\}$ obviously satisfies (14), we can apply Theorem 4 to obtain some relations between coefficients a and b that ensure the global asymptotic stability of the zero solution. Moreover, from Theorem 2 we know that Eq. (19) is globally exponentially stable if $|b| < a \leqslant 1$. However, it seems very difficult to find a criterion for the global asymptotic stability as it was established in [11] for the corresponding continuous time equation

$$x'(t) = -ax(t) + b\max_{t-r \leqslant s \leqslant t}\{x(s)\}. \tag{20}$$

For example, it can be seen that if $a \neq b$, Eq. (20) does not have periodic solutions different from zero [11]. In fact, all nonzero solutions are eventually strictly monotone. This is not the case for Eq. (19) as it is shown in the following example.

Example. For $a = 2$, $b = 1$, and $r = 1$ Eq. (19) becomes

$$x_{n+1} = -x_n + \max\{x_n, x_{n-1}\}.$$

In this case, we have the two-periodic solution $\{x_n\}$ defined by $x_n = 0$ if n is odd, and $x_n = 1$ if n is even.

Some properties can be easily proved for the general case.

Theorem 5. *The trivial solution of Eq.* (19) *is unstable if one of the following conditions hold*:

(i) $b > a$;
(ii) $a > 2$, $a > b > 0$.

Proof. (*i*) If $b > a$, for any $x_0 > 0$ the solution defined by $x_n = (1-a+b)^n x_0$ diverges to $+\infty$.

(*ii*) Assume first that $a > b+2$. Choose an initial string $x_{-r}, \dots, x_{-1}, x_0$ such that $x_0 > 0$ and $x_0 \geqslant \max\{x_{-r}, \dots, x_{-1}\}$. Then $x_1 = -\lambda x_0 < 0$ where $\lambda = a-b-1 > 1$. Therefore, $x_1 < x_0$ and $x_2 = (1-a)x_1 + bx_0 > (1-a)x_1 \geqslant -x_1 = \lambda x_0 > x_0$. By repeating this argument we derive a sequence $\{x_{2n}\}$ such that $x_{2n} > \lambda^n x_0$ for all $n \geqslant 1$. Since $\lambda > 1$, it diverges and the instability follows.

Assume next that $a = b+2$, and let the initial string $x_{-r}, \dots, x_{-1}, x_0$ be as above. Then $x_1 = -x_0 < 0$, and $x_2 = (2b+1)x_0 > x_0$. By induction one finds $x_{2n+1} = -x_{2n}$, $x_{2n+2} = (2b+1)^{n+1} x_0$, $n \geqslant 0$, and the instability follows.

Finally, let $2 < a < b+2$, $b < a$. Choose the initial string $x_{-r}, \dots, x_{-1}$, x_0 as above. Since $-1 < 1-a+b < 1$ we have $x_1 = (1-a+b)x_0 < x_0$. Next we find $x_2 = (1-a)x_1 + bx_0 = [(1-a)(1-a+b)+b]x_0$. Denote $\phi(a,b) = (1-a)(1-a+b)+b$. It is easy to see that $\phi(a,b) > 0$ if $0 < b < a$, and, in this case, $\phi(a,b) > 1$ if and only if $a > 2$. Therefore, in the case $a > 2$ we have $x_2 = \phi(a,b)x_0 > x_0$. By the induction argument, $x_{2n} = [\phi(a,b)]^n x_0$, and the instability follows. □

Remark. If $a = b$ then all constant sequences are solutions of Eq. (19). Moreover, if $a = b > 0$ then all solutions are constant for $n \geqslant 1$. As a consequence, the zero solution of Eq. (19) is stable but not asymptotically stable. Indeed, if $a = b > 0$ then $\Delta x_n = b(\max\{x_n, x_{n-1}, \dots, x_{n-r}\} - x_n) \geqslant 0$, for all $n \geqslant 0$. Thus x_n is nondecreasing and therefore $\max\{x_n, x_{n-1}, \dots, x_{n-r}\} = x_n$ for $n \geqslant 0$. Hence Eq. (19) takes the form $\Delta x_n = 0$, that is, $x_{n+1} = x_n$ for all $n \geqslant 0$.

If $a = b < 0$ then $\Delta x_n \leqslant 0$, and any solution is nonincreasing. Hence solutions $\{x_n\}$ of (19) solve the linear difference equation

$$x_{n+1} = (1-a)x_n + ax_{n-r}. \tag{21}$$

Thus, in this case the set of solutions of (19) consists of the nonincreasing solutions of (21), which are determined by the positive real roots of the characteristic polynomial $p(\lambda) = \lambda^{r+1} + (a-1)\lambda^r - a$. Since $p(0) = -a > 0$, $p(1) = 0$, $\lim_{\lambda \to +\infty} p(\lambda) = +\infty$, and the unique positive solution of $p'(\lambda) = 0$ is $c = (r - ra)/(r+1)$, we conclude that $p(\lambda)$ has only two positive roots $\lambda_1 = 1$ and λ_2. We distinguish three cases: if $|a| > 1/r$ then $\lambda_2 > c > 1$ and the zero solution of (19) is unstable. The same happens for $|a| = 1/r$, since in that case $\lambda_2 = c = 1$ is a root of multiplicity 2, and therefore $\{x_n\} = \{-kn\}$ is a solution of (19) for all $k > 0$. Finally, if $|a| < 1/r$ then $\lambda_2 < c < 1$ and the zero solution is stable.

Theorems 2, 4 and 5 provide some sufficient conditions for the stability or instability of Eq. (19). For the general $r \in \mathbb{N}$ we do not know the stability nature of Eq. (19) in the remaining case, $a > b$ and $1 < a \leqslant 2$, when both a and b are positive. However, this problem can be solved in the special case of $r = 1$, that is for the equation

$$\Delta x_n = -ax_n + b\max\{x_n, x_{n-1}\}, \quad a > 0, \; b > 0. \tag{22}$$

Theorem 6. *The trivial solution of Eq.* (22) *is globally asymptotically stable, if and only if* $b < a < 2$.

Proof. In view of Theorems 2 and 5 we only need to consider the case $1 < a \leqslant 2$.

Assume first that the initial string x_{-1}, x_0 satisfies $x_0 \geqslant x_{-1}$ and $x_0 > 0$. Exactly as in the last part of the proof of Theorem 5 we find that $x_1 = (1 - a + b)x_0 < x_0$ since $-1 < 1-a+b < 1$. Then $x_2 = (1-a)x_1 + bx_0 = [(1-a)(1-a+b)+b]x_0 = \phi(a,b)x_0$, where $\phi(a,b)$ was introduced in the proof of Theorem 5. On the other hand, $x_2 = \phi(a,b)x_0 = \phi(a,b)(1 - a + b)^{-1}x_1$ if $a - b \neq 1$. Recall that if $b < a < 2$ then $0 < \phi(a,b) < 1$. Thus, if $1 < a-b < 2$ then $x_2 > 0 > x_1$, and, if $0 < a-b < 1$ then $a > 1$ implies that $\phi(a,b) > 1-a+b$ and hence $0 < x_1 < x_2$. In any case, $x_3 = (1-a+b)x_2 = \phi(a,b)x_1$. By induction one sees that $x_n = \phi(a,b)x_{n-2}$, for all $n \geqslant 2$. In consequence, $\lim_{n\to+\infty} x_n = 0$ if $a < 2$, and $\{x_n\}$ oscillates between x_1 and $(b-1)x_1$ if $a = 2$.

If $a - b = 1$, it is easy to prove that $x_{2n-1} = 0$ and $x_{2n} = b^n x_0 = (a-1)^n x_0$, $n \geqslant 1$, and hence $\lim_{n\to+\infty} x_n = 0$ if and only if $a < 2$.

If $x_0 \geqslant x_{-1}$, $x_0 = 0$, it is clear that $x_n = 0$ for all $n \geqslant 0$.

Now, assume that $x_0 \geqslant x_{-1}$, $x_0 < 0$. Then $x_1 = (1 - a + b)x_0$. If $1 < a - b < 2$ then $1 - a + b < 0$ and therefore $x_1 > 0 > x_0$. Thus we can argue as in the first case. Otherwise, $0 < 1 - a + b < 1$ and $0 > x_1 > x_0$. By induction, $x_{n+1} = (1 - a + b)^n x_0$ for all $n \geqslant 0$, and therefore $\lim_{n\to+\infty} x_n = 0$.

In the case when the initial string x_{-1}, x_0 satisfies $x_0 < x_{-1}$ the proof can be carried out exactly as above if the solution x_n is such that $x_{n+1} \geqslant x_n$ for some $n \geqslant 1$. If this is not the case, $\{x_n\}$ is a strictly decreasing sequence, and therefore Eq. (22) reduces to the second-order linear difference equation

$$x_{n+1} = (1 - a)x_n + bx_{n-1}. \tag{23}$$

The characteristic polynomial of (23) is $p(\lambda) = \lambda^2 + (a-1)\lambda - b$. Under our assumption, $0 < b < a$, $a > 1$, $p(\lambda) = 0$ has two real solutions $\lambda_1 \in (0,1)$ and $\lambda_2 < -\lambda_1 < 0$. Since we are only interested in the strictly decreasing solutions of (23) (which are also solutions of (22)), the unique possibility is $x_n = x_0\lambda_1^n$, with $x_0 > 0$. Thus, such solution converges exponentially to zero, and the proof is complete. □

Remark. Numerical experiments show that the conclusion of Theorem 6 also holds for the general case $r > 1$. However, with r arbitrary our method leads to consider many cases, and we think that other approach is necessary.

5. The Lozi map

The piecewise linear Lozi map

$$L(x, y) = (1 - a|x| + by, x) \tag{24}$$

is a well-known two-dimensional map introduced and studied by Lozi in 1978, and later by many other authors (see [6,9] and references therein).

Since the chaotic behavior appears for some values of the real parameters a, b, it is interesting to determine the values of the parameters for which a steady state (x_0, y_0) is the global attractor, that is, $\lim_{n\to\infty} L^n(x, y) = (x_0, y_0)$ for all $(x, y) \in \mathbb{R}^2$.

We show that Theorem 2 can be used to obtain some results in this direction.

Define $L_1(x, y) = 1 - a|x| + by$. Then the iterations of the Lozi mapping (24) can be written as

$$\begin{aligned} x_{n+1} &= L_1(x_n, y_n), \\ y_{n+1} &= x_n, \quad n \geqslant 0, \end{aligned} \tag{25}$$

where $(x_0, y_0) \in \mathbb{R}^2$ is a given initial condition.

Thus, (24) is equivalent to the second-order difference equation

$$x_{n+1} = L_1(x_n, x_{n-1}) = 1 - a|x_n| + bx_{n-1}. \tag{26}$$

We only consider the case of a positive equilibrium, the other case being analogous.

It is immediate to check that the positive equilibrium $x_* = (1 + a - b)^{-1}$ exists if and only if $a + 1 > b$. In this case, the change of variable $z_n = x_n - x_*$ transforms (26) into

$$z_{n+1} = -a|z_n + x_*| + ax_* + bz_{n-1},$$

that is,

$$\Delta z_n = -z_n + f(z_n, z_{n-1}), \tag{27}$$

where $f(z_n, z_{n-1}) = -a|z_n + x_*| + ax_* + bz_{n-1}$.

Now, it is obvious that the convergence of x_n to x_* is equivalent to the convergence of z_n to 0, and it is straightforward to check that

$$|f(z_n, z_{n-1})| \leqslant (|a| + |b|) \max\{|z_n|, |z_{n-1}|\}.$$

Hence, an application of Theorem 2 provides the following.

Corollary 7. *If* $|a| + |b| < 1$ *then*

$$|x_n - x_*| \leqslant \max\{|x_0 - x_*|, |x_1 - x_*|\}(|a| + |b|)^{n/2}, \quad n \geqslant 0,$$

for all solutions $\{x_n\}$ *to Eq.* (26).

Remark. In particular, if $|a| + |b| < 1$, (x_*, x_*) is the global attractor of the two-dimensional dynamical system generated by (25). This result is sharp for $a > 0$, $b > 0$, since in this case the equilibrium (x_*, x_*) is a saddle point for $a > 1 - b$.

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