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Attractivity, multistability, and bifurcation in delayed Hopfield's model with non-monotonic feedback

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ABSTRACT

For a system of delayed neural networks of Hopfield type, we deal with the study of global attractivity, multistability, and bifurcations. In general, we do not assume monotonicity conditions in the activation functions. For some architectures of the network and for some families of activation functions, we get optimal results on global attractivity. Our approach relies on a link between a system of functional differential equations and a finite-dimensional discrete dynamical system. For it, we introduce the notion of strong attractor for a discrete dynamical system, which is more restrictive than the usual concept of attractor when the dimension of the system is higher than one. Our principal result shows that a strong attractor of a discrete map gives a globally attractive equilibrium of a corresponding system of delay differential equations. Our abstract setting is not limited to applications in systems of neural networks; we illustrate its use in an equation with distributed delay motivated by biological models. We also obtain some results for neural systems with variable coefficients.

1. Introduction

In the last decades, there has been an increasing interest in the theory of artificial networks. One of the most famous model is given by the system of delay differential equations

$$u_i'(t) = -u_i(t) + \sum_{j=1}^{s} w_{ij} f_j\big(u_j(t-\tau_{ij})\big), \quad 1 \leqslant i \leqslant s, \tag{1.1}$$

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where $u_i(t)$ represents the voltage on the input of neuron i at time t, τ_{ij} denote the synaptic delays and $f_j : \mathbb{R} \to \mathbb{R}$ are the neuron activation functions. The matrix $W = (w_{ij})$ measures the connection strengths between the neurons. If the output from neuron j excites neuron i then $w_{ij} \geqslant 0$ and in the case of inhibitory interaction $w_{ij} \leqslant 0$. System (1.1) without delays was first proposed by Hopfield in [12,13] and later modified by Marcus and Westervelt in [26] incorporating time delays. Such time delays arise from axonal conduction time, distance between the neurons, or finite switching speeds of neurons.

The previous system has been applied in different areas such as classification, associative memory, pattern recognition, parallel computations, and optimization. When a neural circuit is used as an associative memory, stable equilibria correspond with static memory (retrievable) and stable periodic orbits are temporally patterned spike trains. In potential applications of storage and retrieval of information, it is desirable for (1.1) to posses multiple stable patterns (multistability). Unfortunately, most of the network architectures with the usual sigmoid activation functions present a low memory capacity [27]. Motivated by this fact, Morita in [27] and Yoshizawa, Morita, and Amari in [44] have proved that the introduction of a non-monotonic activation function in (1.1) considerably increases its memory capacity. This improvement has deep practical implications in many real situations.

The main goal for system (1.1) is to understand the dynamical behaviors of all trajectories. Due to its complexity, an analysis of bifurcation or stability is still a difficult problem. Moreover, it remains poorly understood how the dynamical behavior depends on the network architecture since the technical difficulties in the study of (1.1) usually impose the restriction to networks of either small size or simple connection architectures. These facts have inspired a large amount of papers; most of them focus on the global convergence to an equilibrium by using Lyapunov functions, the effect of the signal transmission delay, some interesting local phenomena, and multistability by using monotone arguments.

The purpose of this paper is to provide an analytic study of global stability, multistability, and bifurcation in Hopfield's model without assuming monotonicity on f. Our strategy is to link some dynamical behaviors of (1.1) with a discrete system in finite dimension. As we will see, some of our criteria of multistability and global stability are optimal. On the other hand, with our approach we derive criteria of bifurcation without using hard computations of characteristic values or center manifolds. Note that we are dealing with non-monotone systems; comparing with the little progress of the global dynamics in this setting, remarkable developments have been done in the case of monotone networks where generic convergence is guaranteed by the theory of monotone systems [33]. It is also important to recall that the connection of some dynamical behaviors of a discrete equation with some properties of a scalar delay differential equation (DDE) is not new; a systematic approach was initiated by early papers of Mallet-Paret and Nussbaum [24,25], and Ivanov and Sharkovsky [16]; for further generalizations and applications, see, e.g., [11,15,21,22,30,42,43]. The novelty of our method is the connection between systems of DDEs and discrete dynamical systems of dimension higher than one, and the link of the latter with one-dimensional maps. For it, we introduce the notion of strong attractor for a discrete dynamical system, which is more restrictive than the usual concept of attractor when the dimension of the system is higher than one.

This paper is structured as follows. In Section 2, we present the general framework of this paper; we recall some basic notions of discrete systems and show how to derive global convergence of the solutions of (1.1) to an equilibrium from attraction properties of a finite-dimensional map. In Section 3, we apply the results of the previous section to get sufficient conditions for absolute attraction (independent of the size of the delays τ_{ij}) in system (1.1). We also discuss some properties of global bifurcation in (1.1) for a ring of neurons with the activation function considered by Morita in [27]. As a consequence of this discussion, we are able to further describe some steady-state bifurcations studied by Ma and Wu in [23]. Finally, in Section 4, we give an application of our approach to models with distributed delay; we illustrate these results with a functional differential equation recently studied by Yuan and Zhao [45]. We show that it is possible to weaken the conditions for global stability required in the main result of [45].

2. Abstract framework

In this section we present the necessary framework for this paper. We split it into three subsections: in the first one, we introduce the notion of strong attractor for a discrete dynamical system; then, in the second one we provide some useful examples. Finally, the last subsection is devoted to show the link between a strong attractor of a map and an attracting equilibrium of a related system of differential equations with delay.

2.1. Notion of strong attractor

We begin by recalling some notions of attraction in one dimension. Consider equation

$$x(N+1) = f\bigl(x(N)\bigr), \quad N = 0, 1, \ldots, \tag{2.1}$$

where $f : I \subset \mathbb{R} \to f(I) \subset I$ is a continuous function defined on an open interval I. We say that an equilibrium x_* of (2.1) is an attractor in I if for every initial condition $x(0) \in I$ the solution of (2.1) starting at $x(0)$ converges to x_*. The notion of attraction in one dimension has deep implications; below we comment two of them that are especially important for our purposes.

On the one hand, there are no unstable attractors (see, e.g., [4] or [5, Theorem 4.7, p. 182]). Moreover, if x_* is an attractor for (2.1) in I then for every compact interval $K \subset I$,

$$\lim_{n\to\infty} f^n(K) = \{x_*\} \tag{2.2}$$

under the Hausdorff distance (see, e.g., [24, Proposition 1.2]). Remark 1.5 in [24] shows that this result is no longer true in dimensions higher than one.

We need the following specific property of an attractor x^* of (2.1):

(A) for each compact set $K \subset I$, it is possible to find a family of compact intervals $\{J_n\}_{n\in\mathbb{N}}$ with the following properties:
  (A1) $K \subset \mathrm{Int}(J_1) \subset I$,
  (A2) $f(J_n) \subset J_{n+1} \subset \mathrm{Int}(J_n)$ for all $n \in \mathbb{N}$,
  (A3) $x_* \in \mathrm{Int}(J_n)$ for all $n \in \mathbb{N}$, and $\bigcap_{n=1}^{+\infty} J_n = \{x_*\}$.

Note that this property is more restrictive than those commonly used in the literature in regard to the characterization of attraction via a nested family of invariant sets converging to the attractor. In Appendix A, we prove that if x^* is an attractor of (2.1) in I, then property (A) holds.

On the other hand, the global attraction of a fixed point x_* can be characterized by excluding the possibility of fixed points of $f^2 = f \circ f$ different from x_*. This result goes back at least to a paper of Coppel [2], and has been rediscovered many times. We shall use a formulation from Thieme's book [36, Section 9.3].

Proposition 2.1. *Assume that the following condition holds*:

(C) *For $b \in (0,\infty]$, $f : (0,b) \to (0,b)$ is continuous, has a unique fixed point $x_* \in (0,b)$, and is bounded on $(0,x_*)$. Moreover, there exist x_1, x_2, $0 < x_1 < x_* < x_2 < b$, such that $f(x_1) > x_1$ and $f(x_2) < x_2$.*

Then:

(1) *There is a compact interval $[m,M] \subset (0,b)$ such that every solution of (2.1) enters $[m,M]$ and remains in it.*
(2) *x_* is a global attractor of (2.1) in $(0,b)$ if and only if there is no fixed point of f^2 different from x_*.*

From this proposition it is possible to deduce many results on attraction in one dimension for maps f satisfying assumption (C). See, for example, [3,18] and their references.

Next we introduce the notion of strong attractor for a discrete finite-dimensional system. Roughly speaking, a strong attractor is an attractor satisfying property (A).

Definition 2.1. Let $F: D \subset \mathbb{R}^s \to D$ be a continuous map defined on $D = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_s, b_s)$. An equilibrium $z_* \in D$ of the system

$$x(N+1) = F\big(x(N)\big), \quad N = 0, 1, \ldots, \tag{2.3}$$

is a strong attractor in D if for every compact set $K \subset D$ there exists a family of sets $\{I_n\}_{n \in \mathbb{N}}$, where I_n is the product of s nonempty compact intervals, satisfying that

(B1) $K \subset \mathrm{Int}(I_1) \subset D$,
(B2) $F(I_n) \subset I_{n+1} \subset \mathrm{Int}(I_n)$ for all $n \in \mathbb{N}$,
(B3) $z_* \in \mathrm{Int}(I_n)$ for all $n \in \mathbb{N}$, and $\bigcap_{n=1}^{\infty} I_n = \{z_*\}$.

A key feature of the notion of strong attractor is the geometry of the sets I_n. We stress that, for all n, they are the product of s nonempty compact intervals. While for one-dimensional maps the definitions of attractor, stable attractor and strong attractor coincide, in dimensions higher than one the three concepts are different; in Appendix A we give an example showing that a 2-dimensional map can have a stable attractor which is not a strong attractor in the sense of Definition 2.1.

2.2. Examples of strong attractors

The purpose of this subsection is to discuss several examples of strong attractors. These examples are chosen by their applicability in Hopfield's model (1.1).

Example 1. Let $F = (F_1, F_2, \ldots, F_s): \mathbb{R}^s \to \mathbb{R}^s$ be a map satisfying that $F(0) = 0$ and, for all $i, j \in \{1, 2, \ldots, s\}$, F_i is globally Lipschitz-continuous in the j-th coordinate with Lipschitz constant L_{ij}. An elementary argument shows that 0 is a strong attractor in $\mathbb{R}^s$ for the system associated with F provided

$$\sum_{j=1}^{s} L_{ij} < 1$$

for all $i = 1, 2, \ldots, s$. Note that, for each set of the type $[-a, a]^s$, we have that

$$F\big([-a, a]^s\big) \subset [-La, La]^s,$$

where $L = \max\{\sum_{j=1}^{s} L_{ij}\colon\ i = 1, 2, \ldots, s\}$.

Example 2. Consider a map $H: D \subset \mathbb{R}^s \to \mathbb{R}^s$ defined by

$$H(x_1, x_2, \ldots, x_s) = \left(\sum_{j=1}^{s} a_{1j} f_{1j}(x_j), \ldots, \sum_{j=1}^{s} a_{sj} f_{sj}(x_j)\right),$$

where $a_{ij} \in \mathbb{R}$ for all $i, j \in \{1, 2, \ldots, s\}$. We prove the following result:

Proposition 2.2. *Assume that, for each $i, j \in \{1, 2, \ldots, s\}$, $f_{ij} : \mathbb{R} \to \mathbb{R}$ is continuous and odd (i.e. $f_{ij}(x) = -f_{ij}(-x)$ for all $x \in \mathbb{R}$), and define*

$$g_{ij}(x) = \left(\sum_{k=1}^{s} |a_{ik}|\right) f_{ij}(x).$$

If there is $\Delta \in (0, \infty]$ such that $0 \in \mathbb{R}$ is an attractor for the one-dimensional discrete system

$$x(N+1) = g_{ij}\big(x(N)\big), \quad N = 0, 1, \ldots, \tag{2.4}$$

in $(-\Delta, \Delta)$ for every $i, j \in \{1, \ldots, s\}$, then $0 \in \mathbb{R}^s$ is a strong attractor for the dynamical system associated with H in $(-\Delta, \Delta)^s$.

Proof. Take a real number $a \in (0, \Delta)$ and consider the interval $[-a, a]$. Our aim is to prove that for $I = [-a, a]^s$ there exists a family of sets $\{I_n\}_{n \in \mathbb{N}}$ such that $I_n = [-a_n, a_n]^s$, where $\{a_n\}$ is a strictly decreasing sequence converging to zero, $a_1 = a$, and $H(I_n) \subset I_{n+1}$ for all $n \in \mathbb{N}$.

Take a pair $(i, j) \in (\{1, 2, \ldots, s\})^2$. Since g_{ij} is odd, $g_{ij}([-a, a])$ is a compact interval centered at 0. Moreover, using that $0 \in \mathbb{R}$ is an attractor for (2.4) in $(-\Delta, \Delta)$, we deduce from (2.2) that $g_{ij}([-a, a]) \subset (-a, a)$. Thus, there exists $\varepsilon_{ij} \in (0, a]$ such that

$$g_{ij}\big([-a, a]\big) = [-a + \varepsilon_{ij}, a - \varepsilon_{ij}].$$

After that we define

$$E^1 = \min\big\{\varepsilon_{ij}\colon\ i, j \in \{1, 2, \ldots, s\}\big\}.$$

Clearly $E^1 > 0$ and, by its definition,

$$H\big([-a, a]^s\big) \subset \big[-a + E^1, a - E^1\big]^s.$$

To prove this claim, observe that, for each $i = 1, 2, \ldots, s$,

$$\begin{aligned} &\max\left\{\sum_{j=1}^{s} a_{ij} f_{ij}(x)\colon\ x \in [-a, a]\right\} \\ &\quad \leqslant \left(\sum_{k=1}^{s} |a_{ik}|\right) \max\big\{f_{ij}(x)\colon\ x \in [-a, a],\ j = 1, 2, \ldots, s\big\} \\ &\quad = \max\big\{g_{ij}(x)\colon\ x \in [-a, a],\ j = 1, \ldots, s\big\}. \end{aligned}$$

Here we have used that f_{ij} is an odd function and $[-a, a]$ is a symmetric interval to deduce that

$$\max\big\{f_{ij}(x)\colon\ x \in [-a, a],\ j = 1, 2, \ldots, s\big\} \geqslant 0.$$

The special case $E^1 = a$ implies that $H([-a, a]^s) = \{0\}$ and thus the construction of a strictly decreasing sequence with the desired properties can be done taking $a_n = a/n$. If $E^1 < a$ then we define $a_2 = a - E^1$. After that we repeat the same procedure with $[-a_2, a_2]$. Again we can have two possible situations: either $E^2 = a_2$ or $E^2 < a_2$. In the first case, we have that $H([-a_2, a_2]^s) = \{0\}$ and the desired sequence is $a_n = a_2/(n-1)$, with $n \geqslant 2$. In the second case, define $a_3 = a_2 - E^2$. In an inductive way, we construct a strictly decreasing sequence $\{a_n\}_{n\in\mathbb{N}}$. Our aim is to prove that $\{a_n\}$ converges to zero. By contradiction, assume that $\{a_n\} \searrow \beta > 0$. In this case, arguing as before, we deduce that, for each $i, j \in \{1, 2, \dots, n\}$, there exists $\delta_{ij} > 0$ such that

$$g_{ij}\big([-\beta, \beta]\big) = [-\beta + \delta_{ij}, \beta - \delta_{ij}].$$

Define $\widetilde{E} := \min\{\delta_{ij}\colon\ i, j = 1, 2, \dots, s\} > 0$. By continuity, there exists $\gamma > 0$ such that

$$g_{ij}\big([-\beta - \gamma, \beta + \gamma]\big) \subset \left[-\beta + \frac{\widetilde{E}}{2}, \beta - \frac{\widetilde{E}}{2}\right]. \tag{2.5}$$

It follows from the definition of β that, for all $n \geqslant N$ with N large,

$$[-a_n, a_n] \subset [-\beta - \gamma, \beta + \gamma]. \tag{2.6}$$

On the other hand, using the construction of E^N, there exist two indices i, j such that

$$g_{ij}\big([-a_N, a_N]\big) = \big[-a_N + E^N, a_N - E^N\big] = [-a_{N+1}, a_{N+1}] \supset [-\beta, \beta],$$

a contradiction with (2.5) and (2.6). $\square$

The previous result is not true when functions f_{ij} are not odd. For instance we can consider

$$H(x, y) = \big(f_1(y), f_2(x)\big),$$

where

$$f_1(x) = \begin{cases} -x & \text{if } x \geqslant 0, \\ 0 & \text{if } x < 0, \end{cases}$$

and

$$f_2(x) = \begin{cases} 0 & \text{if } x \geqslant 0, \\ -x & \text{if } x < 0. \end{cases}$$

Clearly, for $i \in \{1, 2\}$, $0 \in \mathbb{R}$ is an attractor for

$$x(N+1) = f_i\big(x(N)\big)$$

in $\mathbb{R}$. However, $(-x_0, x_0)$ is a fixed point of H for all $x_0 \geqslant 0$, and therefore $(0, 0)$ is not an attractor of H in $\mathbb{R}^2$.

Example 3. Now we consider the map $G : \mathbb{R}^s \to \mathbb{R}^s$ defined by

$$G(x_1, x_2, \dots, x_s) = \left(\sum_{j=1}^{s} a_{1j} f(x_j), \dots, \sum_{j=1}^{s} a_{sj} f(x_j)\right),$$

where $f : \mathbb{R} \to \mathbb{R}$. We have the following result:

Proposition 2.3. *Assume that*

(i) *f is continuous, and $f((0,\beta)) \subset (0,\beta)$ for some $\beta \in (0,+\infty]$,*
(ii) *all coefficients a_{ij} are nonnegative and*

$$\sum_{j=1}^{s} a_{ij} = 1,$$

for all $i \in \{1,2,\dots,s\}$.

If $x_* \in (0,\beta)$ is an attractor for

$$x(N+1) = f\bigl(x(N)\bigr), \quad N = 0,1,\dots, \tag{2.7}$$

in $(0,\beta)$, then $(x_*, x_*, \dots, x_*) \in \mathbb{R}^s$ is a strong attractor for the system associated with G in $(0,\beta)^s$.

Proof. Given a compact set $K \subset (0,\beta)^s$, take a compact interval $K_0 \subset (0,\beta)$ such that $K \subset (K_0)^s$. By using that x_* is an attractor for (2.7) in $(0,\beta)$, we can find a family of sets $\{J_n\}_{n\in\mathbb{N}}$ satisfying conditions (A1)–(A3) for K_0. In a direct way we can check that the family $\{I_n\}_{n\in\mathbb{N}}$, with $I_n = (J_n)^s$, meets conditions (B1)–(B3) for the system associated with G. $\square$

Remark 2.1. The case of a negative fixed point of (2.7) attracting an interval $(-\beta,0)$ can be treated in the same way.

Example 4. The aim of this example is to study properties of attraction when (2.7) has a cycle of period two. For it, we consider the function $T : \mathbb{R}^{2s} \to \mathbb{R}^{2s}$ given by

$$T(x_1,x_2,\dots,x_{2s}) = \bigl(f(x_2), f(x_3), \dots, f(x_{2s}), f(x_1)\bigr).$$

Proposition 2.4. *Assume that $f : (a,b) \to f((a,b)) \subset (a,b)$ is a continuous function so that x_* is an attractor for*

$$x(N+1) = f^2\bigl(x(N)\bigr), \quad N = 0,1,\dots, \tag{2.8}$$

in the interval (a,b) and f is strictly decreasing in this interval. Then

$$\bigl(x_*, f(x_*), x_*, f(x_*), \dots, x_*, f(x_*)\bigr) \in \mathbb{R}^{2s}$$

is a strong attractor for the system associated with T in the set

$$\bigl((a,b) \times f\bigl((a,b)\bigr)\bigr)^s.$$

Proof. Given a compact set $K_0 \subset ((a,b) \times f((a,b)))^s$, it is possible to find a compact interval K and numbers $\tilde{a}, \tilde{b}$ such that $a < \tilde{a} < \tilde{b} < b$, $K \subset (\tilde{a}, \tilde{b})$ and

$$K_0 \subset \bigl(K \times f(K)\bigr)^s.$$

Clearly, we can find $\varepsilon > 0$ so that

$$f^2\big([\tilde{a},\tilde{b}]\big) \subset [\tilde{a}+\varepsilon, \tilde{b}-\varepsilon] \tag{2.9}$$

and

$$K \subset (\tilde{a}+\varepsilon, \tilde{b}-\varepsilon).$$

After that, we consider

$$I_1 = \left(\left[\tilde{a}+\frac{\varepsilon}{2}, \tilde{b}-\frac{\varepsilon}{2}\right] \times f\big([\tilde{a},\tilde{b}]\big)\right)^s$$

and $I_{n+1} = T^n(I_1)$, $\forall n \geqslant 1$.

It is clear that $T(I_n) = I_{n+1}$. To prove that $I_{n+1} \subset \operatorname{Int}(I_n)$, observe that

$$f^2\big([\tilde{a},\tilde{b}]\big) \subset \left(\tilde{a}+\frac{\varepsilon}{2}, \tilde{b}-\frac{\varepsilon}{2}\right),$$

and, using that f is strictly decreasing,

$$f\left(\left[\tilde{a}+\frac{\varepsilon}{2}, \tilde{b}-\frac{\varepsilon}{2}\right]\right) \subset \operatorname{Int}\big(f\big([\tilde{a},\tilde{b}]\big)\big) = \big(f(\tilde{b}), f(\tilde{a})\big).$$

Finally, we notice that

$$I_{2n+1} \subset \big(f^{2n}\big([\tilde{a},\tilde{b}]\big) \times f^{2n+1}\big([\tilde{a},\tilde{b}]\big)\big)^s,$$

and therefore

$$\bigcap_{n=1}^{\infty} I_n = \big\{\big(x_*, f(x_*), \ldots, x_*, f(x_*)\big)\big\}. \qquad \square$$

2.3. Delay differential equations and strong attractors

In this subsection we consider the system of delay differential equations

$$x_i'(t) = -x_i(t) + F_i\big(x_1(t-\tau_{i1}), x_2(t-\tau_{i2}), \ldots, x_s(t-\tau_{is})\big), \quad 1 \leqslant i \leqslant s, \tag{2.10}$$

where $F_i : \mathbb{R}^s \to \mathbb{R}$ is locally Lipschitz-continuous and $\tau_{ij} \geqslant 0$ for all $i, j = 1, 2, \ldots, s$. Our goal is to deduce convergence properties in (2.10) via the notion of strong attractor for the discrete system

$$x(N+1) = F\big(x(N)\big), \quad N = 0, 1, \ldots, \tag{2.11}$$

where $x(N) = (x_1(N), x_2(N), \ldots, x_s(N)) \in \mathbb{R}^s$ and $F = (F_1, F_2, \ldots, F_s)$. The natural phase space for (2.10) is $X = \mathcal{C}([-\tau, 0], \mathbb{R}^s)$, equipped with the max-norm

$$|\phi|_\infty = \max\big\{\big|\phi(t)\big|: t \in [-\tau, 0]\big\},$$

where $\phi(t) = (\phi_1(t), \phi_2(t), \ldots, \phi_s(t))$, $|\phi(t)| = \|\phi(t)\|_\infty = \max\{|\phi_i(t)|: 1 \leqslant i \leqslant s\}$, and

$$\tau = \max\{\tau_{ij}\colon i, j = 1, \dots, s\}.$$

For each $\phi \in X$, we employ the notation

$$x(t, \phi) = \big(x_1(t, \phi), x_2(t, \phi), \dots, x_s(t, \phi)\big)$$

for the solution of (2.10) with initial condition ϕ.

Theorem 2.5. *Assume that $F : D \subset \mathbb{R}^s \to D$ is a continuous map defined on $D = (a_1, b_1) \times \dots \times (a_s, b_s)$ and $z_* \in \mathbb{R}^s$ is a strong attractor for*

$$x(N+1) = F\big(x(N)\big), \quad N = 0, 1, \dots, \tag{2.12}$$

in D. Then, for each $\phi \in \{x \in \mathcal{C}([-\tau, 0], \mathbb{R}^s)\colon x(t) \in D$ for all $t \in [-\tau, 0]\}$, $\lim_{t\to\infty} x(t, \phi) = z_*$.

Proof. Fix a function ϕ as in the statement of the theorem. By using that z_* is a strong attractor for (2.12) in D, we can obtain a family $\{I_n\}_{n\in\mathbb{N}}$ satisfying conditions (B1)–(B3) for the compact set

$$K = \big\{\phi(t)\colon t \in [-\tau, 0]\big\}.$$

First we prove that

$$x(t, \phi) \in I_1, \quad \forall t \geqslant 0. \tag{2.13}$$

Notice that $x(t, \phi) = \phi(t) \in K \subset \operatorname{Int}(I_1)$ for all $t \in [-\tau, 0]$. Assume, by contradiction, that $I_1 = [c_1, d_1] \times [c_2, d_2] \times \dots \times [c_s, d_s]$ and there exists a first time $t_0 > 0$ so that $x(t_0, \phi) \in \partial I_1$. This claim implies that there is an index i so that $x_i(t_0, \phi) = d_i$ or $x_i(t_0, \phi) = c_i$. In the first case, $x_i'(t_0, \phi) \geqslant 0$ but, by the i-th equation in (2.10) and the inclusion $F(I_1) \subset \operatorname{Int}(I_1)$, we deduce that $x_i'(t_0) < 0$. The analogous argument works when $x_i(t_0, \phi) = c_i$. Next we prove that there is a time $t_1 > 0$ such that

$$x(t, \phi) \in I_2, \quad \forall t \geqslant t_1, \tag{2.14}$$

where we suppose that

$$I_2 = [\tilde{c}_1, \tilde{d}_1] \times [\tilde{c}_2, \tilde{d}_2] \times \dots \times [\tilde{c}_s, \tilde{d}_s]. \tag{2.15}$$

First we note that if there is an index j such that

$$x_j(t_2, \phi) \in [\tilde{c}_j, \tilde{d}_j]$$

for some $t_2 > 0$ then

$$x_j(t, \phi) \in [\tilde{c}_j, \tilde{d}_j]$$

for all $t \geqslant t_2$. By contradiction, assume that there exist $t_* > t_2$ and $\delta > 0$ satisfying that

$$x_j'(t_*, \phi) > 0,$$
$$x_j(t_*, \phi) = \tilde{d}_j + \delta.$$

Using (2.13), the j-th equation in (2.10) and the inclusions

$$F(I_1) \subset I_2 \subset \operatorname{Int}(I_1) \tag{2.16}$$

(see (B2)), we obtain the contradiction

$$0 < -(\tilde{b}_j + \delta) + F_j\big(x_1(t_* - \tau_{j1}, \phi), \dots, x_s(t_* - \tau_{js}, \phi)\big) \leqslant -\delta.$$

An analogous argument works if

$$x_j'(t_*, \phi) < 0,$$
$$x_j(t_*, \phi) = \tilde{c}_j - \delta.$$

Finally, we prove that there is $t_1 > 0$ such that (2.14) holds. To prove this claim we reason by contradiction. By the previous argument, we can assume that there exists a family of indices P such that

$$x_j(t, \phi) \notin I_2$$

for all $t > 0$ and $j \in P$. Then, either

$$x_j(t, \phi) > \tilde{d}_j$$

or

$$x_j(t, \phi) < \tilde{c}_j.$$

In the first case, $x_j(t, \phi)$ is strictly decreasing and in the second one, strictly increasing (these properties follow easily from (2.13), (2.16) and the j-th equation in (2.10)). On the other hand, it is possible to find a sequence $\{t_n\}_{n \in \mathbb{N}}$ so that $x_j'(t_n, \phi) \to 0$. By this property together with (2.16) we can deduce that $x_j(t, \phi) \to \tilde{d}_j$ (in the first case) or $x_j(t, \phi) \to \tilde{c}_j$ (in the second one). Collecting this information, we have that given an index $j \in P$, $x_j(t, \phi)$ converges to $\tilde{d}_j$ or $\tilde{c}_j$, and given an index $j \notin P$, $x_j(t, \phi) \in [\tilde{c}_j, \tilde{d}_j]$ for all $t \geqslant t_j$ for a suitable t_j. This is a contradiction since

$$F(I_2) \subset \operatorname{Int}(I_2)$$

and given $j \in P$, taking the limit as $t \to \infty$ in both sides of the j-th equation in (2.10), we obtain that

$$0 = -\tilde{d}_j + F_j(x_0)$$

or

$$0 = -\tilde{c}_j + F_j(x_0),$$

with $x_0 \in \partial I_2$.

We can repeat the previous procedure replacing I_1 and I_2 with I_2 and I_3, respectively. An inductive argument shows that, for each $n \geqslant 1$, there is a number $t_n > 0$ such that $x(t, \phi) \in I_n$ for all $t \geqslant t_n$. Since, by (B3), $\bigcap_{n=1}^{\infty} I_n = \{z^*\}$, it follows that $\lim_{t \to \infty} x(t, \phi) = z^*$. $\square$

We can generalize Theorem 2.5 to systems with distributed delay. To illustrate this application, and for the sake of simplicity, we study the following system:

$$u_i'(t) = -u_i(t) + \sum_{j=1}^{s} w_{ij} \int_{-\infty}^{0} \theta_{ij}(s) f\big(u_j(t+s)\big)\, ds, \quad 1 \leqslant i \leqslant s, \tag{2.17}$$

where $\theta_{ij} : (-\infty, 0] \to [0, +\infty)$ are positive integrable functions satisfying that $\theta_{ij}(s) = 0$ when $s \leqslant \tau_{ij} \in (-\infty, 0)$, and $\int_{-\infty}^{0} \theta_{ij}(s)\, ds = 1$.

In order to get some properties of global stability for system (2.17) via the discrete system

$$F(x_1, x_2, \dots, x_s) = \left(\sum_{j=1}^{s} w_{1j} f(x_j), \dots, \sum_{j=1}^{s} w_{sj} f(x_j)\right),$$

we use the following elementary property:

(H) If $k : \mathbb{R} \to (0, +\infty)$ is a positive integrable function and $g : \mathbb{R} \to \mathbb{R}$ is a continuous function, then for each interval $[a, b]$ there is $\eta \in [a, b]$ so that

$$g(\eta) \int_a^b k(s)\, ds = \int_a^b g(s) k(s)\, ds.$$

Theorem 2.6. *Assume that f is continuous and $z_* \in \mathbb{R}^s$ is a strong attractor for*

$$x(N+1) = F\big(x(N)\big), \quad N = 0, 1, \dots, \tag{2.18}$$

on $D = (a_1, b_1) \times \cdots \times (a_s, b_s)$. Then, for each

$$\phi \in \big\{x \in \mathcal{C}\big([-\tau, 0], \mathbb{R}^s\big)\colon x(t) \in D \text{ for all } t \in [-\tau, 0]\big\},$$

the solution $u(t, \phi)$ of (2.17) with initial condition ϕ satisfies that

$$\lim_{t \to \infty} u(t, \phi) = z_*.$$

Proof. The proof consists of adapting the arguments used for proving Theorem 2.5, having in mind property (H). Indeed, for instance, if $F(I_1) \subset \mathrm{Int}(I_1)$ with $I_1 = [c_1, d_1] \times [c_2, d_2] \times \cdots \times [c_s, d_s]$ then, for every initial condition in

$$\phi \in \big\{x \in \mathcal{C}\big([-\tau, 0], \mathbb{R}^s\big)\colon x(t) \in \mathrm{Int}(I_1) \text{ for all } t \in [-\tau, 0]\big\}$$

with $\tau = \max\{\tau_{ij}\}$, the solution $u(t, \phi)$ of (2.17) satisfies that $u(t, \phi) \in \mathrm{Int}(I_1)$ for all $t \geqslant 0$. To prove this property, assume, by contradiction, that there exists a first time $t_0 > 0$ so that $u(t_0, \phi) \in \partial I_1$. This claim implies that there is an index i so that $u_i(t_0, \phi) = d_i$ or $u_i(t_0, \phi) = c_i$. In the first case, $u_i'(t_0, \phi) \geqslant 0$ but, by the i-th equation in (2.10) and the property (H), we find nonnegative numbers $m_j \geqslant 0$ so that

$$u_i'(t_0, \phi) = -u_i(t_0, \phi) + \sum_{j=1}^{s} w_{ij} f\big(u_j(t_0 - m_j, \phi)\big).$$

Now, since $F(I_1) \subset \mathrm{Int}(I_1)$, we deduce that $u_i'(t_0, \phi) < 0$. It is easy to check that an analogous argument works when $u_i(t_0, \phi) = c_i$, and the remainder steps of the proof of Theorem 2.5 can be adapted in the same way. □

Remark 2.2. It is not difficult to verify that the arguments in the proofs of Theorems 2.5 and 2.6 can also be adapted to consider systems of the type

$$x_i'(t) = -g_i\big(x_i(t)\big) + F_i\big(x_1(t-\tau_{i1}), x_2(t-\tau_{i2}), \ldots, x_s(t-\tau_{is})\big), \quad 1 \leqslant i \leqslant s,$$

provided $g_i : \mathbb{R} \to \mathbb{R}$ is a homeomorphism for every $i = 1, 2, \ldots, s$. In this case we have to assume that z^* is a strong attractor for the discrete system

$$x(N+1) = G\big(x(N)\big), \quad N = 0, 1, \ldots,$$

where $G(x_1, x_2, \ldots, x_s) = (g_1^{-1}(F_1(x_1, x_2, \ldots, x_s)), \ldots, g_s^{-1}(F_s(x_1, x_2, \ldots, x_s)))$.

3. Global results in Hopfield's model

In this section we apply the results of Section 2 to system (1.1). First we present some results on global attractivity independent of the delays, which is usually referred to as *absolute global attractivity*. To find criteria of absolute global attractivity for (1.1) when 0 is the unique equilibrium is one of the open problems suggested by J. Wu in [41, p. 98]. This problem has been approached in many interesting papers; see, e.g., [28,32,35,37,38,46] and their references.

In a second section, we get some results of global bifurcation for (1.1) with a particular architecture of the network.

3.1. Global attractivity results

For our first result, we assume that, for each $j \in \{1, 2, \ldots, n\}$, $f_j : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz with constant L_j, that is,

$$\big|f_j(u) - f_j(v)\big| \leqslant L_j|u - v|, \quad \text{for all } u, v \in \mathbb{R}. \tag{3.1}$$

A direct application of Theorem 2.5 and Example 1 in Section 2.2 provides the following result:

Theorem 3.1. *If condition* (3.1) *holds and*

$$\max_{1 \leqslant i \leqslant s} \left\{ \sum_{j=1}^{s} |a_{ij}| L_j \right\} < 1, \tag{3.2}$$

then every solution $u(t)$ *of* (1.1) *converges to* 0*, regardless the value of the delays.*

Similar conditions to (3.2) have been used in the literature for proving global attraction in system (1.1), using Lyapunov functions, when the activation functions satisfy (3.1); see, e.g., [9,38,41].

Next we list some properties that we will assume for the activation functions:

(Ac1) f is of class $\mathcal{C}^1$,
(Ac2) $f(0) = 0$ with $f'(0) > 0$,
(Ac3) f is odd (i.e. $f(x) = -f(-x)$) and bounded.

Most of the activation functions considered in the related literature meet these conditions (sometimes after a simple change of variables); for example, the usual sigmoid functions (e.g., $f(x) = \tanh(x)$), and the non-monotone Morita's activation function [23,27] given by

$$f(x) = \frac{1-\exp(-\alpha x)}{1+\exp(-\alpha x)} \times \frac{1+k\exp(\beta(|x|-h))}{1+\exp(\beta(|x|-h))}, \tag{3.3}$$

where h, k, α, β are real parameters, $\alpha > 0$, $\beta > 0$.

Our next result is based on Proposition 2.2, and provides a global attractivity result for a system more general than (1.1); specifically, we consider system

$$x_i'(t) = -x_i(t) + \sum_{j=1}^{s} a_{ij}(t) f_{ij}\big(x_j(t-\tau_{ij})\big), \quad 1 \leqslant i \leqslant s. \tag{3.4}$$

Theorem 3.2. *Suppose that functions $f_{ij} : \mathbb{R} \to \mathbb{R}$ satisfy conditions* (Ac1)–(Ac3) *for all $i, j \in \{1, 2, \dots, s\}$, and the coefficients $a_{ij} : \mathbb{R} \to \mathbb{R}$ are continuous and bounded.*

If, for every $i, j \in \{1, 2, \dots, s\}$, 0 is an attractor in $\mathbb{R}$ for the discrete equation

$$y(N+1) = \left(\sum_{k=1}^{s} |a_{ik}|_\infty\right) f_{ij}\big(y(N)\big), \quad N = 0, 1, \dots, \tag{3.5}$$

then all solutions $x(t)$ of (3.4) satisfy that $\lim_{t\to\infty} x_i(t) = 0$, for all $i = 1, 2, \dots, s$.

Proof. The method of proof consists of combining the ideas of Proposition 2.2 and Theorem 2.5. First, it follows from Proposition 2.2 that $0 \in \mathbb{R}^s$ is a strong attractor for

$$x(N+1) = \widetilde{F}\big(x(N)\big), \quad N = 0, 1, \dots, \tag{3.6}$$

in $\mathbb{R}^s$, where

$$\widetilde{F}(x_1, x_2, \dots, x_s) = \left(\sum_{j=1}^{s} |a_{1j}|_\infty f_{1j}(x_j), \dots, \sum_{j=1}^{s} |a_{sj}|_\infty f_{sj}(x_j)\right).$$

In fact, by the proof of this result we know that for each set of the type $[-a_1, a_1]^s$ with $a_1 > 0$, there is a strictly decreasing sequence $\{a_n\} \searrow 0$ so that

$$\widetilde{F}\big([-a_n, a_n]^s\big) \subset [-a_{n+1}, a_{n+1}]^s \quad \text{for all } n \geqslant 1.$$

Now we take an initial condition $\phi \in \mathcal{C}([-\tau, 0], \mathbb{R}^s)$, where $\tau = \max\{\tau_{ij}\colon i, j = 1, \dots, s\}$. Fix $a_1 > 0$ so that

$$\phi(t) \in (-a_1, a_1)^s = \operatorname{Int}(I_1)$$

for all $t \in [-\tau, 0]$. At this moment we have to repeat exactly the same arguments in the proof of Theorem 2.5 with the family $I_n = [-a_n, a_n]^s$. For the reader's convenience, we prove, for instance, that

$$x(t, \phi) \in [-a_1, a_1]^s, \quad \text{for all } t \geqslant 0.$$

By contradiction, assume that there is a first time $t_0 > 0$ so that

$$x(t_0,\phi)\in\partial\big([-a_1,a_1]^s\big).$$

In this situation, there is an index j so that either

$$x_j(t_0,\phi)=a_1$$

or

$$x_j(t_0,\phi)=-a_1.$$

Assume that we are in the first case (the other case is analogous). In this setting, clearly

$$x_j'(t_0,\phi)\geqslant 0.$$

However,

$$\begin{aligned}
x_j'(t_0,\phi)&=-x_j(t_0,\phi)+\sum_{k=1}^{s}a_{jk}(t)f_{jk}\big(x_k(t_0-\tau_{jk})\big)\\
&\leqslant -x_j(t_0,\phi)+\sum_{k=1}^{s}|a_{jk}|_\infty\max\big\{f_{jk}(x)\colon x\in[-a_1,a_1]\big\}\\
&=-x_j(t_0,\phi)+\max\big\{\widetilde{F}_j(x_1,x_2,\dots,x_s)\colon x_i\in[-a_1,a_1]\big\}<0.
\end{aligned}$$

In these inequalities we use that f_{ij} is an odd function to deduce that $\max\{f_{jk}(x)\colon x\in[-a_1,a_1]\}\geqslant 0$. $\square$

The problem of global stability for neural systems of Hopfield type with variable coefficients has been addressed in several papers, usually assuming some property of recurrence on the coefficients $a_{ij}(t)$, such as periodicity (e.g., [35]) or almost periodicity (e.g., [28]). Note that in Theorem 3.2 we do not impose any property of recurrence on the coefficients.

In the remainder of this section, to get sharper results and motivated by the classical papers of Hopfield and Marcus and Westervelt (see [26, p. 348]), we assume that the activation functions are equal, and the connection matrix is normalized as

$$\sum_{j=1}^{s}|w_{ij}|=1\quad\text{for } 1\leqslant i\leqslant s. \tag{3.7}$$

Thus we consider the system

$$u_i'(t)=-u_i(t)+\sum_{j=1}^{s}w_{ij}f\big(u_j(t-\tau_{ij})\big),\quad 1\leqslant i\leqslant s, \tag{3.8}$$

and our target is to derive global stability criteria for (3.8) based on the dynamics of the one-dimensional discrete equation

$$y(N+1)=f\big(y(N)\big),\quad N=0,1,\dots. \tag{3.9}$$

In the results below, we assume that f satisfies conditions (Ac1)–(Ac3), but in general we do not require any monotonicity properties. We recall that $y_* \in \mathbb{R}$ is an attractor for (3.9) in (a, b) if $f((a, b)) \subset (a, b)$ and $\lim_{N\to\infty} y(N) = y_*$, for all initial conditions $y(0) \in (a, b)$.

Our first result for system (3.8) deals with the stability of the trivial equilibrium.

Theorem 3.3. *Assume that* $0 \in \mathbb{R}$ *is an attractor for* (3.9) *in* $(-\Delta, \Delta)$ *for some* $\Delta > 0$*. Then every solution of* (3.8) *with initial condition in the set*

$$\left\{\phi \in \mathcal{C}\left([-\tau, 0], \mathbb{R}^s\right):\ \phi(t) \in (-\Delta, \Delta)^s \text{ for all } -\tau \leqslant t \leqslant 0\right\}$$

satisfies that $\lim_{t\to\infty} x_i(t, \phi) = 0$ *for all* $i = 1, 2, \ldots, s$.

Proof. This result is a direct consequence of Proposition 2.2 and Theorem 2.5. □

Theorem 3.3 is sharp in the following sense: assume that there is an interval $(0, b)$ such that $f((0, b)) \subset (0, b)$. If 0 is an attractor for f in $(-b, b)$ then 0 is an attractor for (3.8) in $(-b, b)^s$. If 0 does not attract $(-b, b)$, then there must be at least two fixed points of f, namely x_- and x_+, satisfying that $-x_+ = x_-$ and

$$-b < x_- < 0 < x_+ < b.$$

If we assume excitatory connections in (3.8), that is, $w_{ij} \geqslant 0$ for all $i, j = 1, 2, \ldots, s$, then $(x_+, x_+, \ldots, x_+)$ and $(x_-, x_-, \ldots, x_-)$ are nontrivial equilibria of (3.8) in $(-b, b)^s$, and therefore 0 is not an attractor for (3.8) in $(-b, b)^s$. In our next result, we state the stability properties of these nontrivial equilibria.

Theorem 3.4. *Assume excitatory connections in* (3.8)*, that is* $w_{ij} \geqslant 0$ *for every index. If* $x_* \in \mathbb{R}$ *is an attractor for* (3.9) *in* $(0, b)$ *with* $b \in (0, +\infty]$*, then every solution of* (3.8) *with initial condition in the set*

$$\left\{\phi \in \mathcal{C}\left([-\tau, 0], \mathbb{R}^s\right):\ \phi(t) \in (0, b)^s \text{ for all } -\tau \leqslant t \leqslant 0\right\}$$

satisfies that $\lim_{t\to\infty} x_i(t, \phi) = x_*$ *for* $1 \leqslant i \leqslant s$.

Proof. This result is a direct consequence of Proposition 2.3 and Theorem 2.5. □

Assuming excitatory connections, Theorems 3.3 and 3.4 provide a sharp criterion of global attraction for (3.8) when f is a sigmoid function, satisfying the usual conditions of monotonicity ($f'(x) > 0$ for all x) and strong convexity ($xf''(x) < 0$ for all $x \neq 0$). Indeed, in this case, either 0 is the only fixed point of f, and then all solutions of (3.8) converge to zero, or there are two nontrivial fixed points of f, $x_- < 0 < x_+$, and then $(x_+, x_+, \ldots, x_+)$ and $(x_-, x_-, \ldots, x_-)$ are equilibria of (3.8) attracting $(0, \infty)^s$ and $(-\infty, 0)^s$, respectively.

For more general activation functions, Theorem 3.4 is also sharp in some architectures of the network, namely in the ring of an even number of neurons (see [1]). Indeed, consider the system

$$x_i'(t) = -x_i(t) + f\left(x_{i+1}(t - \tau_i)\right), \quad 1 \leqslant i \leqslant 2s \tag{3.10}$$

(with the convention mod $(2s)$). Assume that f satisfies assumption (C) in Proposition 2.1, and x_* is not an attractor of (3.9) in $(0, b)$. Then, this proposition guarantees that there is a two cycle $\{y_1, y_2\}$ for (3.9) with $y_1 < x_* < y_2 = f(y_1)$. This 2-cycle produces two equilibria of (3.10) different from $(x_*, x_*, \ldots, x_*)$, namely, $(y_1, y_2, \ldots, y_1, y_2)$ and $(y_2, y_1, \ldots, y_2, y_1)$.

Our following result deals with their stability properties.

Theorem 3.5. *Assume that $y_1 \in \mathbb{R}$ is an attractor for*

$$y(N+1) = f^2\big(y(N)\big) \tag{3.11}$$

in (a,b) and f is strictly decreasing in this interval. Then every solution of (3.10) with initial condition in

$$\big\{\phi \in \mathcal{C}\big([-\tau,0],\mathbb{R}^s\big)\colon\ \phi(t) \in \big((a,b) \times f\big((a,b)\big)\big)^s \text{ for all } -\tau \leqslant t \leqslant 0\big\}$$

satisfies that $\lim_{t\to\infty} x(t,\phi) = (y_1, f(y_1), \ldots, y_1, f(y_1))$.

Proof. This result is a direct consequence of Proposition 2.4 and Theorem 2.5. □

We notice that it is difficult to apply our method to get global attractivity results based on Examples 3 and 4 when there is dependence on time, as in system (3.4). We plan to study this problem in the future.

3.2. Global bifurcation results

Theorems 3.3–3.5 show how the changes in the dynamics of (3.9) lead to the creation of new stable patterns in (3.10). In an easy way, these results can be used to provide criteria of global bifurcation. To illustrate this point of view, we consider system (3.10) with Morita's activation function (3.3). This example is motivated by the recent paper of Ma and Wu [23], which considers the system (3.10) with $s=1$ and f given by (3.3). Notice that for $s=1$ the equilibria of (3.10) are completely determined by the periodic points of f with periods 1 and 2.

First, we list some elementary properties of the function f defined by (3.3); see [23] for more details.

- f is odd.
- If $k>0$, then $x_0=0$ is the only zero of f. If $k<0$, then f has three zeros given by

$$x_0 = 0, \qquad x_0^+ = h + \frac{\ln(\frac{-1}{k})}{\beta}, \qquad x_0^- = -h - \frac{\ln(\frac{-1}{k})}{\beta}.$$

- When $k<0$, there are two critical points $x_M^- \in (x_0^-, 0)$ and $x_M^+ \in (0, x_0^+)$. Moreover, f is strictly decreasing in $(-\infty, x_M^-) \cup (x_M^+, +\infty)$ and f is strictly increasing in (x_M^-, x_M^+) (see Fig. 1).

We consider the numerical example studied in [23], that corresponds to $k=-0.8$, $h=0.5$, and $\alpha=7.5$, and use β as the bifurcation parameter. The bifurcation diagram of (3.9) for $\beta \in (0, 5.4)$ is shown in Fig. 2.

From this bifurcation diagram, we can study bifurcation properties for (3.10). Let $\beta_1 = 0.749387$ be the unique value of β for which $f'(0)=1$. For $\beta \in (0,\beta_1)$, $f'(0)<1$ and 0 is the unique solution of equation $f^2(x)=x$. Therefore, by Proposition 2.1, 0 is a global attractor for (3.9) in $\mathbb{R}$, and it follows from Theorem 3.3 that $0 \in \mathbb{R}^{2s}$ is a global attractor of (3.10).

At $\beta=\beta_1$, there is a pitchfork bifurcation in (3.9) producing two nontrivial fixed points $x_+ \in (0, x_0^+)$ and $x_- \in (x_0^-, 0)$, while 0 becomes unstable.

We can check that f maps $(0, x_0^+)$ into itself for $\beta \in (0, 6.85562)$. Moreover, for this range of values, the interval $J = [f^2(x_M^+), f(x_M^+)]$ is invariant and attracting. Recall that x_M^+ is the only local maximum of f.

Next, define $\beta_2 = 4.12413$ as the value of β for which $f'(x_+) = -1$. For $\beta \in (\beta_1, \beta_2)$, there are exactly three fixed points of f (namely, 0, x_+, x_-), and no other 2-periodic point. Thus, x_+ attracts $(0, x_0^+)$, and Theorem 3.4 ensures that $(x_+, x_+, \ldots, x_+)$ attracts all solutions of (3.10) with initial condition in the set

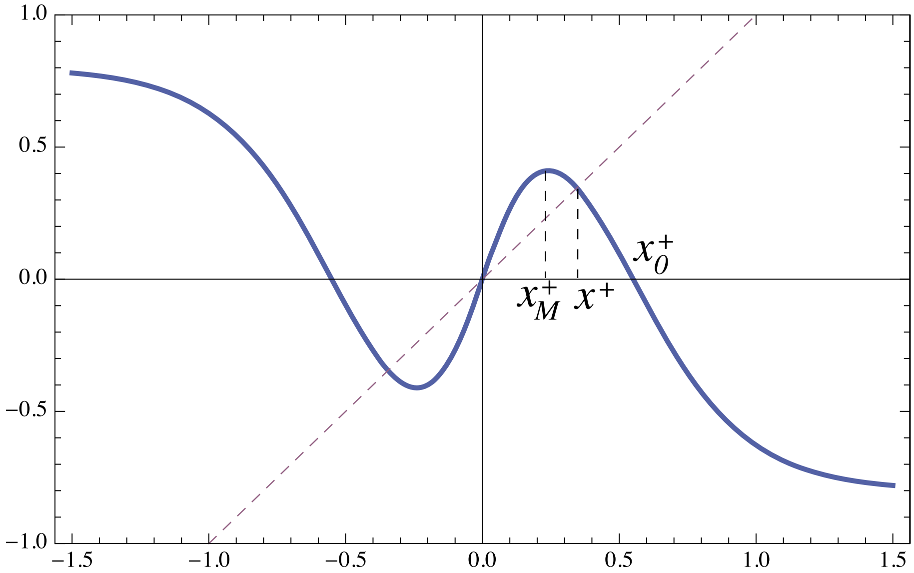

Fig. 1. Graph of the map f defined in (3.3) for $k = -0.8$, $h = 0.5$, $\alpha = 7.5$, and $\beta = 3$. The equilibria are the intersections of the solid curve $y = f(x)$ and the dashed line $y = x$.

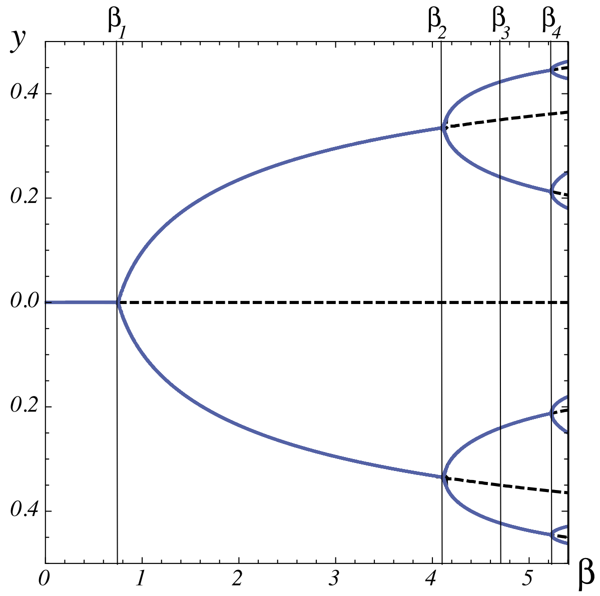

Fig. 2. Bifurcation diagram of (3.9), with f defined by (3.3) for $k = -0.8$, $h = 0.5$, and $\alpha = 7.5$. For each value of $\beta \in (0, 5.4)$ (with step 0.001), we produce 300 iterations of (3.9) with a random initial condition $x_0 \in [-1, 1]$, and plot the last 50, to let the transients die out. The dashed lines correspond to unstable equilibria. For the meaning of β_1–β_4, see the text.

$$\left\{\phi \in \mathcal{C}\left([-\tau, 0], \mathbb{R}^{2s}\right): \phi(t) \in \left(0, x_0^+\right)^{2s} \text{ for all } -\tau \leqslant t \leqslant 0\right\}.$$

An analogous result is obtained for the equilibrium $(x_-, x_-, \ldots, x_-)$ with respect to the set of initial data

$$\left\{\phi \in \mathcal{C}\left([-\tau, 0], \mathbb{R}^{2s}\right): \phi(t) \in \left(x_0^-, 0\right)^{2s} \text{ for all } -\tau \leqslant t \leqslant 0\right\}.$$

At $\beta = \beta_2$, the fixed points x_+ and x_- of (3.9) lose their stability in a period-doubling bifurcation, giving rise to a pair of cycles of period 2, which we denote by $\{y_1^+, y_2^+\}$ and $\{y_1^-, y_2^-\}$. They satisfy the inequalities

$$y_2^- < x_- < y_1^- < 0 < y_1^+ < x_+ < y_2^+.$$

We can verify that f is decreasing in the invariant interval J if $\beta \in (\beta_2, \beta_3)$, where $\beta_3 = 4.69233$. For $\beta \in (\beta_2, \beta_3)$, let us consider the 2-periodic orbit $\{y_1^+, y_2^+\}$. The point y_1^+ is an attractor of f^2 in $(f^2(x_M^+), x_+)$, y_2^+ is an attractor of f^2 in $(x_+, f(x_M^+))$, and f is decreasing in both intervals. Then, Theorem 3.5 applies to establish that the equilibrium $(y_1^+, y_2^+, \ldots, y_1^+, y_2^+)$ attracts all solutions of (3.10) with initial condition in

$$\bigl\{\phi \in \mathcal{C}\bigl([-\tau, 0], \mathbb{R}^{2s}\bigr)\colon \phi(t) \in \bigl(f^2\bigl(x_M^+\bigr), x_+\bigr)^s \times \bigl(x_+, f^3\bigl(x_M^+\bigr)\bigr)^s \text{ for all } -\tau \leqslant t \leqslant 0\bigr\},$$

and the equilibrium $(y_2^+, y_1^+, \ldots, y_2^+, y_1^+)$ attracts all solutions of (3.10) with initial condition in

$$\bigl\{\phi \in \mathcal{C}\bigl([-\tau, 0], \mathbb{R}^{2s}\bigr)\colon \phi(t) \in \bigl(x_+, f\bigl(x_M^+\bigr)\bigr)^s \times \bigl(f^2\bigl(x_M^+\bigr), x_+\bigr)^s \text{ for all } -\tau \leqslant t \leqslant 0\bigr\}.$$

Note that if the number of neurons in (3.10) is $4s$ with $s \in \mathbb{N}$, then the period-doubling bifurcation of a 2-cycle also produces a pitchfork bifurcation in (3.10) since if z_1 is a fixed point of f^4 then

$$\bigl(f^3(z_1), f^2(z_1), f(z_1), z_1, \ldots, f^3(z_1), f^2(z_1), f(z_1), z_1\bigr)$$

is an equilibrium of (3.10). In our example, the second period-doubling bifurcation occurs for $\beta = \beta_4 = 5.2286$ (see Fig. 2). Actually, a typical route of period-doubling bifurcations to chaos is observed for larger values of β, which would lead to a series of pitchfork bifurcations in system (3.10) with a number of $(2^n s)$ neurons, where $n, s \in \mathbb{N}$, n large.

The problem of bifurcation in system (1.1) has been extensively studied and several tools such as the theory of equivariant bifurcations [8], or some reductions to a center manifold [7] have been employed in interesting papers; see, e.g., [6,10,14,29,39,40]. In connection with these works, our approach has mainly two advantages: its simplicity and its global character. As the previous example shows, without involving hard computations, we can derive criteria of bifurcation providing a good knowledge of the bases of attraction of the stable equilibria. A drawback of our approach is that it is not possible to study bifurcations different from the pitchfork one.

4. Models with distributed delay

In some biological neural networks, the time delays in propagation processes are not fixed. This can be caused, for instance, by diffusion processes across the synapses and along the dendrites. Hence, biologically realistic models (see [34,41]) can be represented as a system of type (3.8) with distributed delay, specifically

$$u_i'(t) = -u_i(t) + \sum_{j=1}^{s} w_{ij} \int_{-\infty}^{0} \theta_{ij}(s) f\bigl(u_j(t+s)\bigr)\, ds, \quad 1 \leqslant i \leqslant s, \tag{4.1}$$

where, for each $i, j \in \{1, 2, \ldots, s\}$, $\theta_{ij} : (-\infty, 0] \to [0, +\infty)$ is a positive integrable function with $\int_{-\infty}^{0} \theta_{ij}(s)\, ds = 1$, and $\theta_{ij}(s) = 0$ when $s \leqslant \tau_{ij} \in (-\infty, 0)$.

As a direct consequence of Proposition 2.3 and Theorem 2.6, we get the following result:

Theorem 4.1. *Assume that $w_{ij} \geqslant 0$ for every index, and the normalization condition (3.7) holds. If $x_* \in \mathbb{R}$ is an attractor for*

$$x(N+1) = f\bigl(x(N)\bigr)$$

in $(0,b)$ *with* $b \in (0,+\infty]$, *then every solution of* (4.1) *with initial condition in the set*

$$\{\phi \in \mathcal{C}([-\tau,0],\mathbb{R}^s)\colon\ \phi(t) \in (0,b)^s \text{ for all } -\tau \leqslant t \leqslant 0\}$$

satisfies that $\lim_{t\to\infty} u_i(t,\phi) = x_*$ *for* $1 \leqslant i \leqslant s$.

Next we show that our approach can be used to deduce some results of absolute attractivity for other mathematical models. We consider the following equation recently studied in [45]:

$$u'(t) = -\alpha u(t) + f_1\big(u(t)\big) + \int_{\tau_0}^{\tau_1} \hat{k}(s) f_2\big(u(t-s)\big)\,ds, \tag{4.2}$$

where $\alpha > 0$, and $\hat{k}(s) \geqslant 0$ is an integrable function so that $\hat{k}(s) = 0$ when $s \leqslant \tau_0$ or $s \geqslant \tau_1$. If $f_1(u) = 0$ and $\hat{k}(s) = \delta(s-\tau)$ (Dirac's delta), Eq. (4.2) corresponds to well-known models with an instantaneous destruction rate, and a delayed production function, such as the celebrated Nicholson's blowflies and Mackey–Glass equations; see, e.g., [25,31] and references therein. Some global stability results for the case of $f_1 \equiv 0$ with a general distributed delay term can be found in [19,20].

An application of Theorem 2.6, combined with Remark 2.2, leads to the following result for Eq. (4.2):

Theorem 4.2. *Assume that* $g(x) = \alpha x - f_1(x)$ *is strictly increasing in* $(0,M)$ *for some* $M > 0$, *and* $x_* > 0$ *is an attractor for*

$$x(N+1) = h\big(x(N)\big) \tag{4.3}$$

in $(0,M)$, *where* $h(x) = g^{-1}(k^* f_2(x))$ *and* $k^* = \int_{\tau_0}^{\tau_1} \hat{k}(s)\,ds$. *Then*

$$\lim_{t\to\infty} u(t,\phi) = x_*$$

for all solutions $u(t,\phi)$ *of* (4.2) *with initial condition* ϕ *in the set*

$$\{\phi \in \mathcal{C}([-\tau,0],\mathbb{R})\colon\ \phi(t) \in (0,M) \text{ for all } -\tau \leqslant t \leqslant 0\}.$$

Proposition 2.1 ensures that x_* is an attractor for (4.3) in $(0,M)$ if the following conditions hold:

(a) $h((0,M)) \subset (0,M)$,
(b) x_* is the unique fixed point of h^2 in $(0,M)$,
(c) there are two points x_1, x_2, $0 < x_1 < x_* < x_2 \leqslant M$, such that $h(x_1) > x_1$ and $h(x_2) < x_2$.

Next we show that Theorem 4.2 allows us to remove some conditions from Theorem 2.3 in [45]. For convenience of the reader, we list the assumptions required in [45, Theorem 2.3].

(H1) f_1 and f_2 are Lipschitz-continuous with $f_1(0) + k^* f_2(0) = 0$ and $f_2'(0) > 0$, $k^* = \int_{\tau_0}^{\tau_1} \hat{k}(s)\,ds > 0$.
(H2) $f_2(u) \geqslant f_2(0)$ for all $u \geqslant 0$ and there exists a number $M > 0$ such that $f_1(u) + k^* \times \max_{v\in[0,u]} f_2(v) < \alpha u$ for all $u > M$.
(H3) $f_1'(0) + k^* f_2'(0) > \alpha$ and there exists a unique steady state u^* such that $-\alpha u^* + f_1(u^*) + k^* f_2(u^*) = 0$.
(H4) $\alpha u - f_1(u)$ is strictly increasing, $\frac{f_1(u)+k^* f_2(u)}{u}$ is strictly decreasing in $u \in (0,M]$ and both f_1 and f_2 have the following property:

(P) For any $v, w \in (0, M]$ with $v \leqslant u^* \leqslant w$, if $f_1(v) + k^* f_2(w) \leqslant \alpha v$ and $f_1(w) + k^* f_2(v) \geqslant \alpha w$, then $v = w$.

Assumptions (H1) and (H2) have been used in [45] to guarantee that the solution $u(t)$ of system (4.2) is positive whenever $u(0) > 0$, and ultimately bounded by M.

To apply our method we have to impose that $f_2(u) > f_2(0)$ for all $u \in (0, M]$. First, by this property and using that g is strictly increasing (see the first part of (H4)), we deduce that $h((0, M)) \subset (0, +\infty)$. Condition (H2) and the monotony of $\alpha u - f_1(u)$ imply (a).

Next, it is easy to check that the inequality $f_1'(0) + k^* f_2'(0) > \alpha$ from (H3) implies that

$$h'(0) > 1. \tag{4.4}$$

It is clear that property (c) follows from (4.4) and (H2).

Observe that (H4) is more restrictive than (b) because we do not use that $\frac{f_1(u)+k^* f_2(u)}{u}$ is strictly decreasing in $u \in (0, M]$, and (b) is weaker than property (P).

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Appendix A

In this appendix, we prove that every attractor of a one-dimensional map is a strong attractor, that is, it meets property (A) in Section 2.1. Then we give an example showing that this property is not true in general for stable attractors in dimension higher than one.

A.1. Strong attractors in one dimension

We prove the following result:

Proposition A.1. *Assume that $f : I \to I$ is a continuous function defined on an open interval and x_* is an attractor in I of the discrete dynamical system*

$$x(N+1) = f\big(x(N)\big), \quad N = 0, 1, \ldots. \tag{A.1}$$

Then condition (A) *holds.*

The key property to deduce Proposition A.1 is given in the following result.

Lemma A.2. *Consider $J = [a, b]$ a compact interval invariant under (A.1). Then for every $\varepsilon > 0$, there exist $0 < \epsilon_1, \epsilon_2 < \varepsilon$ such that*

$$f\big([a - \epsilon_1, b + \epsilon_2]\big) \subset (a - \epsilon_1, b + \epsilon_2).$$

Proof. It is not restrictive to assume that x_* does not belong to the border of J; otherwise we work with $J' = J \cup U$, with U an arbitrary small compact and invariant neighborhood of x_* (the existence of this interval is guaranteed by Lemma 2.3 in [17]). On the other hand, if $a < f(a), f(b) < b$, then our lemma is trivial. By using that x_* is an attractor, we observe that it is not possible that $f(a) = b$

and $f(b)=a$ simultaneously. Then, we have just to consider the case $f(a)=b$ and $f(b)<a$ (the case $f(b)=a$ and $f(a)>b$ can be treated in the same way). Clearly, we can take ϵ_2 small enough so that $f([b,b+\epsilon_2])\subset(a,b)$ and then ϵ_1 small enough so that $f([a-\epsilon_1,a])\subset(a,b+\epsilon_2)$. $\square$

Proof of Proposition A.1. By property (i) of Lemma 2.3 in [17], there is a compact and invariant interval J_1 such that $K\subset \operatorname{Int}(J_1)$. In fact, by Lemma A.2, we can assume that $f(J_1)\subset \operatorname{Int}(J_1)$. Applying again this lemma with $J=f(J_1)$, we can find J_2 such that $f(J_1)\subset J_2\subset \operatorname{Int}(J_1)$ and $f(J_2)\subset \operatorname{Int}(J_2)$. Reasoning by induction, we obtain a sequence $\{J_n\}_{n\in\mathbb{N}}$. Note that $x_*\in \operatorname{Int}(J_n)$ for all $n\in\mathbb{N}$ and

$$\bigcap_{n=1}^{+\infty} J_n=\{x_*\}.$$

To see this last claim, we argue as follows. To avoid trivial cases, we can assume that $f(J_n)\neq\{x_*\}$ for all n. We notice that our construction allows us to suppose that if $J_n=[a_n,b_n]$ then $f(J_n)$ contains an interval of the type $[a_{n+1}-\varepsilon_{n+1},b_{n+1}+\varepsilon_{n+1}]$, with $\lim_{n\to\infty}\varepsilon_n=0$. Therefore

$$\bigcap_{n=1}^{+\infty} J_n=\bigcap_{n=1}^{+\infty} f(J_n)=L.$$

Note that L is a compact interval satisfying that $f(L)=L$. To guarantee this property, we observe that if $x\in f(L)$ then there are $y_n\in J_n$ such that $f(y_n)=x$; if y is an accumulation point of $\{y_n\}_{n\in\mathbb{N}}$ then it is clear that $y\in L$ and so $f(y)=x$. Finally, we obtain from (2.2) that $L=\{x_*\}$. $\square$

A.2. Stable attraction does not imply strong attraction in dimensions higher than one

To show that the notion of strong attractor is more restrictive than the definition of stable attractor in a general discrete dynamical system, we give an example in $\mathbb{R}^2$.

Let us consider the rotation of angle $\frac{\pi}{4}$ with center at the origin $R_{\frac{\pi}{4}}:\mathbb{R}^2\to\mathbb{R}^2$, and the map $G:[0,+\infty)\times\mathbb{R}\to[0,+\infty)\times\mathbb{R}$ given in polar coordinates by

$$G(\rho,\theta)=\begin{cases}(\rho,\theta) & \text{if } \theta\in[\frac{-\pi}{2},\pi]+2\pi\mathbb{Z},\\ (f(\theta)\rho,\theta) & \text{if } \theta\in[-\pi,\frac{-\pi}{2}]+2\pi\mathbb{Z},\end{cases}$$

if $\rho>0$, and $G(0,0)=0$. Here $f:[-\pi,\frac{-\pi}{2}]\to\mathbb{R}$ is defined by

$$f(\theta)=\begin{cases}\frac{-2}{\pi}\theta-1 & \text{if } \theta\in[-\pi,\frac{-3\pi}{4}],\\ \frac{2}{\pi}\theta+2 & \text{if } \theta\in[\frac{-3\pi}{4},\frac{-\pi}{2}].\end{cases}$$

Define the map $F:\mathbb{R}^2\to\mathbb{R}^2$ by $F=G\circ R_{\frac{\pi}{4}}$. The geometric behavior of the map F is as follows: we rotate $\frac{\pi}{4}$ and after that we compress the modulus in the third quadrant. Observe that under F, the angular coordinate is periodic with period 8 and the radial coordinate is nonincreasing. Moreover, given $(x,y)=(\rho\cos\theta,\rho\sin\theta)\neq(0,0)$, it is easy to see that $F^8(\rho\cos\theta,\rho\sin\theta)=k_0(\rho\cos\theta,\rho\sin\theta)$ with $0<k_0\leqslant\frac{9}{16}$. It follows from these properties that $(0,0)$ is an asymptotically stable attractor for the system associated with the map F. On the other hand, we note that $(0,0)$ is not a strong attractor. Indeed, given any set $I=[a_1,a_2]\times[b_1,b_2]$ with $a_1<0<a_2$ and $b_1<0<b_2$ then, for

$$c=\min\{|a_1|,|a_2|,|b_1|,|b_2|\},$$

we have that $(-c,c),(c,c),(c,-c),(-c,-c)\in I$ and, by the definition of F,

$$F(-c,c)=(-\sqrt{2}c,0); \qquad F(c,c)=(0,\sqrt{2}c);$$
$$F(c,-c)=(\sqrt{2}c,0); \qquad F(-c,-c)=(0,-\sqrt{2}c).$$

Therefore, if $F(I)\subset[\tilde{a}_1,\tilde{a}_2]\times[\tilde{b}_1,\tilde{b}_2]$ then

$$B:=[-\sqrt{2}c,\sqrt{2}c]\times[-\sqrt{2}c,\sqrt{2}c]\subset[\tilde{a}_1,\tilde{a}_2]\times[\tilde{b}_1,\tilde{b}_2].$$

It is clear that $B\not\subset \mathrm{Int}(I)$ and therefore $(0,0)$ is not a strong attractor for F.

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