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ON A GENERALIZED YORKE CONDITION FOR SCALAR DELAYED POPULATION MODELS

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(Dedicated to Professor István Győri on the occasion of his 60th birthday)

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Abstract. For a scalar delayed differential equation $\dot{x}(t) = f(t, x_t)$, we give sufficient conditions for the global attractivity of its zero solution. Some technical assumptions are imposed to insure boundedness of solutions and attractivity of non-oscillatory solutions. For controlling the behaviour of oscillatory solutions, we require a very general condition of Yorke type, together with a 3/2-condition. The results are particularly interesting when applied to scalar differential equations with delays which have served as models in populations dynamics, and can be written in the general form $\dot{x}(t) = (1 + x(t))F(t, x_t)$. Applications to several models are presented, improving known results in the literature.

1. **Introduction.**

Let $C := C([-h, 0]; \mathbb{R})$ be the space of continuous functions from $[-h, 0]$ to $\mathbb{R}$, $h > 0$, equipped with the sup norm $\|\varphi\| = \max_{-h\le\theta\le 0} |\varphi(\theta)|$. In the present work, we consider scalar functional differential equations (FDEs)

$$\dot{x}(t) = f(t, x_t), \quad t \geq 0, \tag{1.1}$$

where $f : [0, \infty) \times C \to \mathbb{R}$ is continuous. As usual, x_t denotes the function in C defined by $x_t(\theta) = x(t + \theta), -h \leq \theta \leq 0$. Clearly, the requirement of f continuous can be weakened (see [5, Chapter 2]); however, existence and continuity of solutions for (1.1) must be assumed.

Our research is mainly motivated by the applications of the so-called 3/2 stability results (see e.g. [6, Section 4.5]) to scalar population models which can be written in the form

$$\dot{x}(t) = (1 + x(t))F(t, x_t), \quad t \geq 0. \tag{1.2}$$

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Recently, the global attractivity of the zero solution of (1.2) was investigated in [1] assuming that F satisfies the following generalization of the well-known Yorke condition ([15], [6, p. 141]):

$$-\lambda(t)M(\varphi) \le F(t,\varphi) \le \lambda(t)M(-\varphi), \quad \text{for} \quad t \ge 0,\ \varphi \in C, \tag{1.3}$$

where $\lambda : [0,\infty) \to [0,\infty)$ is a piecewise continuous function, and the functional $M(\varphi) := \max\{0, \sup_{\theta\in[-h,0]} \varphi(\theta)\}$ was introduced by Yorke [15]. We remark that condition (1.3) with $\lambda(t) \equiv a > 0$ was first introduced in [15], not in the setting of equation (1.2), but to study the stability of the zero solution of (1.1), for $F = f$; later, Yoneyama [13] generalized Yorke's hypothesis by replacing the constant a with a continuous function $\lambda(t) \ge 0$.

Connecting and unifying the approaches in [1] and [7] (in which another generalization of the Yorke condition was introduced, see Theorem 1.1 below), in the present paper we improve the results in the referred works: we establish a global attractivity result for (1.1), from which the global attractivity of (1.2) is obtained under a Yorke condition (see assumption (A3) in Section 3) more general than the ones considered in both [1] for (1.2) and [7] for (1.1). Our result is easy to apply, and allows us to improve some results in the literature for a number of concrete examples.

We set some notation. If $x(t)$ is defined for $t \ge 0$, we say that $x(t)$ is *oscillatory* if it is not eventually zero and it has arbitrarily large zeros; otherwise, it is called *non-oscillatory*. An equilibrium E_* of (1.1) is said to be *globally attractive* if all solutions of the equation tend to E_* as $t \to \infty$. In C, we consider the usual partial order

$$\varphi \ge \psi \quad \text{if and only if} \quad \varphi(\theta) \ge \psi(\theta),\ \theta \in [-h,0];$$

in particular, for $\varphi \in C$ and $c \in \mathbb{R}$, $\varphi \ge c$ (respectively $\varphi \le c$) means that $\varphi(\theta) \ge c$ (respectively $\varphi(\theta) \le c$) for all $\theta \in [-h,0]$. Analogously, for $\varphi, \psi \in C$ we define $\varphi > \psi$ if and only if $\varphi(\theta) > \psi(\theta)$, $\theta \in [-h,0]$.

In the following, the next hypotheses will be considered for f as in (1.1):

(H1) there exists a piecewise continuous function $\beta : [0,\infty) \to [0,\infty)$ for which

$$\beta_0 := \sup_{t\ge 0} \int_{t-h}^{t} \beta(s)ds < \infty,$$

and such that for each $q \in \mathbb{R}$ there is $\eta(q) \in \mathbb{R}$ such that for $t \ge 0$ and $\varphi \in C, \varphi \ge q$, then

$$f(t,\varphi) \le \beta(t)\eta(q);$$

(H2) if $w : [-h,\infty) \to \mathbb{R}$ is continuous and there is $\lim_{t\to\infty} w(t) = w^* \ne 0$, then $\int_0^\infty f(s, w_s)\, ds$ diverges;

(H3) there are a piecewise continuous function $\lambda : [0,\infty) \to [0,\infty)$ and a constant $b \ge 0$ such that, for $r(x) := \dfrac{-x}{1+bx}$, $x > -1/b$,

$$\lambda(t)r(M(\varphi)) \le f(t,\varphi) \le \lambda(t)r(-M(-\varphi)), \quad \text{for} \quad t \ge 0, \tag{1.4}$$

where the first inequality holds for all $\varphi \in C$ and the second one for $\varphi \in C$ such that $\varphi > -1/b \in [-\infty, 0)$, and $M(\varphi)$ is the Yorke functional

$$M(\varphi) := \max\{0, \sup_{\theta\in[-h,0]} \varphi(\theta)\};$$

(H4) for $\lambda(t)$ as in (H3), there is $T \geq h$ such that, for

$$\alpha := \alpha(T) = \sup_{t \geq T} \int_{t-h}^{t} \lambda(s)ds,$$

$\alpha \leq 3/2$ if $b > 0$, and $\alpha < 3/2$ if $b = 0$.

Without loss of generality, by a time scaling, we may assume $h = 1$. Also, if $b > 0$, for b as in (H3), the scaling $x \mapsto bx$ allows us to consider $b = 1$. By the change of variables $x \mapsto y = -x$, we may as well consider a function $f(t, \varphi)$ such that $g(t, \varphi) = -f(t, -\varphi)$ satisfies (H1)–(H4).

In [7] the following result was proven:

Theorem 1.1. [7] *Assume (H1)–(H4) with $\beta(t) \equiv 1$ and $\lambda(t) \equiv a$, a being a positive constant. Then, all solutions of (1.1) are defined on $[0, \infty)$, and converge to zero as $t \to \infty$.*

Our first purpose is to prove Theorem 1.1 under hypotheses (H1)–(H4) with general $\beta(t), \lambda(t)$. Actually, by a change of variables introduced in [7] (see (2.2)–(2.3) below), it turns out that the framework in (H3) can be reduced to the situation of (H3) with $\lambda(t) \equiv a > 0$, if the additional condition $\lambda(t) > 0$ for large t is imposed (cf. Lemma 2.3). As we shall show in Section 3, the application of this result to general delayed scalar population models (1.2) provides a generalization of the criterion for global attractivity established in [1] (see Theorem 3.1 below). In Section 4, some particular models that have been considered in the literature are addressed within the present framework, and weaker sufficient conditions for the global attractivity of equilibria or periodic solutions of such models are obtained. Also some open problems and counter-examples will be presented.

2. Global stability for (1.1).

We start this section with some preliminary lemmas.

Lemma 2.1. *Assume (H1)–(H3) with $b > 0$, and*

$$\alpha_0 = \sup_{t \geq 0} \int_{t-h}^{t} \lambda(s)ds < \infty \,. \tag{2.1}$$

Then, every solution $x(t) = x(\varphi)(t)$ of (1.1) with initial condition $x_0 = \varphi \in C$ is defined and bounded on $[0, \infty)$. Furthermore, if $x(t)$ is non-oscillatory, then $x(t) \to 0$ as $t \to \infty$.

Proof. Assume $h = 1$ and $b = 1$, and consider $x(t)$ a solution of (1.1). From (H3), $f(t, \varphi) \geq -\lambda(t)$ for all $t \geq 0, \varphi \in C$, hence $x(t) \geq x(0) - \int_0^t \lambda(s)ds$ is bounded from below on each interval $[0, a), a > 0$. Fix any $a > 0$, and let $x(t) \geq q$ on $[-1, a)$. From (H1),

$$f(t, x_t) \leq \beta(t)\eta(q), \quad t \in [0, a),$$

for some $\eta(q) \geq 0$. This implies that

$$x(t) \leq x(0) + \eta(q) \int_0^t \beta(s)ds, \quad t \in [0, a),$$

and $x(t)$ is bounded from above on $[0, a)$. On the other hand, conditions (H1) and (H3) imply that f maps bounded sets of $[0, \infty) \times C$ into bounded sets of $\mathbb{R}$. This shows that $x(t)$ is extensible to $[0, \infty)$.

We now prove that $x(t)$ is bounded on $[0,\infty)$. (For a similar proof, see [1, Theorem 3.3].) First, we observe that from (H3) we have, for $t \geq 0, \varphi \in C$,

$$f(t,\varphi) \leq 0 \text{ if } \varphi \geq 0 \quad \text{and} \quad f(t,\varphi) \geq 0 \text{ if } \varphi \leq 0.$$

Let $x(t)$ be a non-oscillatory solution of (1.1). If $x(t)$ is eventually positive, then $f(t,x_t) \leq 0$ for t large, and $x(t)$ is eventually non-increasing, thus $x(t)$ is bounded. Analogously, if $x(t)$ is eventually negative, then $\dot{x}(t) \geq 0$ for t large, and $x(t)$ is bounded.

Now consider the case of $x(t)$ oscillatory. Let t_0 be a local minimum point of $x(t)$, $x(t_0) < 0$. We may assume that $x(t) > x(t_0)$ for $t_0 - t > 0$ small. Then (cf. [7, Remark 3] and [1, Lemma 3.2]) there is $\xi_0 \in [t_0 - 1, t_0)$ such that $x(\xi_0) = 0$. Otherwise, since $x(t)$ is continuous, then $x(t) < 0$ on $[t_0 - 1 - \delta, t_0]$, hence $\dot{x}(t) = f(t,x_t) \geq 0$ for $t \in [t_0 - \delta, t_0]$, contradicting the definition of t_0. From (H3), we obtain

$$x(t_0) = \int_{\xi_0}^{t_0} f(t,x_t)\,dt \geq -\int_{t_0-1}^{t_0} \lambda(t)\,dt \geq -\alpha_0.$$

We therefore deduce that $x(t)$ is bounded from below on $[0,\infty)$.

Let $q \in \mathbb{R}$ be such that $x(t) \geq q,\ t \geq 0$. If t_1 is a local maximum point of $x(t)$ with $0 < x(t) < x(t_1)$ for $t_1 - t > 0$ small, in a similar fashion we deduce that there is $\xi_1 \in [t_1 - 1, t_1)$ such that $x(\xi_1) = 0$. From (H1), there is $\eta = \eta(q) \geq 0$ such that

$$x(t_1) = \int_{\xi_1}^{t_1} f(t,x_t)\,dt \leq \eta \int_{t_1-1}^{t_1} \beta(t)\,dt \leq \eta\beta_0\,.$$

Then there is an upper bound of $x(t)$. The proof of the boundedness of $x(t)$ is complete.

If $x(t)$ is a non-oscillatory solution of (1.1), we have already shown that $x(t)$ is eventually monotone, therefore there is $\lim_{t\to\infty} x(t) := c$. If $c \neq 0$, from (H2) we obtain a contradiction, thus $x(t) \to 0$ as $t \to \infty$. □

Remark 2.2. If $b = 0$, clearly (H3) and (2.1) imply (H1) with $\beta(t) = \lambda(t)$. Note that if $b = 0$ the inequality

$$f(t,\varphi) \geq -\lambda(t), \quad t \geq 0,\, \varphi \in C$$

does not hold, and we cannot deduce the boundedness of all solutions of (1.1). However, for the situation $b = 0$ Yoneyama [14] proved that all solutions of (1.1) converge to zero as $t \to \infty$ under hypotheses (H2), (H3) and (H4). On the other hand, for $b > 0$, in the above proof hypothesis (H1) was used only to prove that all solutions of (1.1) are bounded from above on $[0,\infty)$. Note that if $b > 0$ this result does not follow from (H3), since the second inequality in (H3) holds solely for $\varphi \in C$ such that $\varphi > -1/b$.

Lemma 2.3. *Assume (H1)–(H4). If* $\lambda(t) > 0$ *for* $t \geq 0$ *large, then the zero solution of (1.1) is globally attractive.*

Proof. Set $h = 1$. Under (H1)–(H4) and $b = 0$ in (H3), the result follows from [14], so we assume $b = 1$. We first observe that (H2), (H3) imply that $\int_0^\infty \lambda(t)\,dt = \infty$. In fact, choose $w \in (-1/b, 0)$. From (H2), (H3), $0 \leq f(t,w) \leq \lambda(t)r(w)$, with $\int_0^\infty f(t,w)\,dt = \infty$, thus also $\int_0^\infty \lambda(t)\,dt = \infty$.

Suppose that $\lambda(t) > 0$ for $t \geq T_0$, and let $t_0 = \max\{T, T_0\}$, where T is as in (H4). Similarly to what was done in [7], we define $s : [t_0, \infty) \to [s(t_0), \infty)$,

$$s(t) = \frac{1}{\alpha} \int_0^t \lambda(u)\, du, \quad t \geq t_0, \tag{2.2}$$

for α as in (H4). The function $s(t)$ is one-to-one and onto. Denote by $t = t(s)$ its inverse, and consider the change of variables

$$y(s) = x(t(s)), \quad s \geq s_0 := s(t_0). \tag{2.3}$$

For $s \geq s_0$, Eq. (1.1) is transformed into

$$\dot{y}(s) = g(s, y_s), \tag{2.4}$$

where

$$g(s, \varphi) = \frac{\alpha}{\lambda(t(s))} f\big(t(s), \varphi(-\sigma(s, \cdot))\big),$$

and

$$\sigma(s, \theta) = \frac{1}{\alpha} \int_{t(s)+\theta}^{t(s)} \lambda(\tau)\, d\tau, \quad \theta \in [-1, 0].$$

(Note that $\varphi(-\sigma(s, \cdot)) \in C$ if $\varphi \in C$.) From (H3), it is easy to verify that the function g in (2.4) satisfies

$$\alpha r(M(\varphi)) \leq g(s, \varphi) \leq \alpha r(-M(-\varphi)), \quad \text{for} \quad s \geq s_0,$$

where the first inequality holds for all $\varphi \in C$ and the second one for $\varphi \in C$ such that $\varphi > -1/b \in [-\infty, 0)$. Thus, $g(s, \varphi)$ satisfies (H3)–(H4) with $\lambda(s) \equiv \alpha$. In order to apply Theorem 1.1, condition (H1) of [7], i.e., condition (H1) above with $\beta(s) \equiv 1$, and (H2) should be fulfilled. However, it is clear that g may not satisfy condition (H1) of [7], nor (H2). On the other hand, conditions (H1) of [7] and (H2) above were used only to prove, respectively, that all solutions of (2.4) are defined and bounded on $[0, \infty)$, and that non-oscillatory solutions go to zero as $t \to \infty$. For the present situation, the existence and boundedness of all solutions of (2.4) follow from Lemma 2.1. Therefore, invoking Theorem 1.1, we deduce that $y(s) \to 0$ as $t \to \infty$, for all oscillatory solutions $y(s)$ of (2.4). Since $s(t) \to \infty$ as $t \to \infty$, then all oscillatory solutions $x(t)$ of (1.1) satisfy $x(t) \to 0$ as $t \to \infty$. For non-oscillatory solutions, the same is true from Lemma 2.1. □

Note that the change of variables (2.3) (cf. [7]) is very powerful, since it allows to reduce Eq. (1.1), with f satisfying (H3)–(H4), to Eq. (2.4), with g satisfying (H3)–(H4) for $\lambda(s) \equiv a > 0$.

Lemma 2.4. *Assume (H1)–(H4) with $\alpha < 3/2$, for α defined as in (H4). Then the zero solution of (1.1) is globally attractive.*

Proof. If (H4) holds with $\alpha < 3/2$, we can find $\varepsilon > 0$ such that (H3) and (H4) are fulfilled with $\lambda(t)$ replaced by $\lambda_1(t) := \lambda(t) + \varepsilon$. The result follows now from Lemma 2.3. □

We are now in position to state the following improvement of Theorem 1.1.

Theorem 2.5. *Assume (H1)–(H4). If $b > 0$, assume in addition that either $\lambda(t) > 0$ for t large, or $\alpha := \alpha(T) < 3/2$ for some $T \geq h$. Then the zero solution of (1.1) is globally attractive.*

Proof. The result follows immediately from Lemmas 2.3 and 2.4. □

Remark 2.6. For the case $b = 0$, it was proven in [8] that, for the particular case of equation

$$\dot{x}(t) = \lambda(t) g(x(t-h)),$$

Theorem 2.5 is valid if we have the equality $\alpha = 3/2$ in (H4), and (H3) holds with the strict inequality for $\varphi(-h) \neq 0$. Therefore, an interesting open question is whether Theorem 2.5 is still valid if in the case $b = 0$ we assume (H1)–(H3), allow $\alpha = 3/2$ in (H4), and further impose

$$-\lambda(t) M(\varphi) < f(t, \varphi) < \lambda(t) M(-\varphi)$$

for all $t \geq 0$ and $\varphi \in C$ with $\varphi(\theta) \neq 0$, $\theta \in [-h, 0]$. For the general case of distributed delays, some additional conditions on the behaviour of $f(t, \varphi)$ might be required.

Another interesting question is whether it is possible to replace (H3) in Theorem 2.5 by the following weaker condition:

(H3') there is a piecewise continuous function $\lambda : [0, \infty) \to [0, \infty)$ and there are constants $b_1, b_2 \geq 0, b_1 \leq b_2$, such that, for $r_i(x) := \dfrac{-x}{1 + b_i x}$,

$$\lambda(t) r_1(M(\varphi)) \leq f(t, \varphi) \leq \lambda(t) r_2(-M(-\varphi)), \quad \text{for} \quad t \geq 0, \tag{2.5}$$

where the first inequality holds for all $\varphi \in C$ and the second one for $\varphi \in C$ such that $\varphi > -1/b_2 \in [-\infty, 0)$, and $M(\varphi) = \max\{0, \sup_{\theta \in [-h,0]} \varphi(\theta)\}$.

Note that the case $b_1 \geq b_2$ in (2.5) is not of interest, in the sense that it reduces (2.5) to (1.4). In fact, if $b_1 \geq b_2$ we have $r_1(x) \geq r_2(x)$ for all $x > -1/b_1$; thus, condition (2.5) implies (1.4) with $r(x) = r_2(x)$.

The following result shows that, under additional restrictions on (H4) and on the size of b_1/b_2, we can replace (H3) by (H3') in Theorem 2.5.

Theorem 2.7. *Assume (H1), (H2), (H3') and that, for $\lambda(t)$ as in (H3'), there is $T \geq h$ such that*

$$\alpha := \alpha(T) = \sup_{t \geq T} \int_{t-h}^{t} \lambda(s) ds \leq \sqrt[3]{2}. \tag{2.6}$$

Assume also that

$$\frac{\alpha^2}{2 + \alpha} < \frac{b_1}{b_2}. \tag{2.7}$$

Then all solutions $x(t)$ of (1.1) satisfy $x(t) \to 0$ as $t \to \infty$.

Proof. As already mentioned, we may consider $h = 1$. Arguing as in the proof of Lemma 2.1, one deduces that all solutions of (1.1) are defined and bounded on $[0, \infty)$, and that non-oscillatory solutions of (1.1) go to zero as $t \to \infty$. Hence, only oscillatory solutions $x(t)$ will be considered.

Define $\lambda_0(t) = \alpha^{-1} \lambda(t)$ and $R_i(x) = \alpha r_i(x)$, for $r_i(x)$, $i = 1, 2$, as in (H3'). Recall that $r_i(x)$, hence also $R_i(x)$, $i = 1, 2$, are decreasing functions on their domains. We now apply (H3'), and argue by adapting the proofs in [7, Lemma 4] and [1, Lemma 3.5]. Some details are omitted.

Let $x(t)$ be an oscillatory solution of (1.1), and define

$$u = \limsup_{t \to \infty} x(t), \quad -v = \liminf_{t \to \infty} x(t).$$

Fix $\varepsilon > 0$, and for T as in (2.6) choose $T_0 \geq T$ such that

$$-(v + \varepsilon) \leq x(t) \leq u + \varepsilon, \ t \geq T_0 - 2.$$

Now consider a sequence $(x(s_n))$ of local minima, $x(s_n) < 0, s_n \to \infty, s_n - 2 \geq T_0, x(s_n) \to -v$ as $n \to \infty$. As in [7, Remark 3] and [1, Lemma 3.2], we deduce that, if s_n are chosen so that $x(t) > x(s_n)$ for $s_n - t > 0$ small, then there exists $\eta_n \in [s_n - 1, s_n)$ such that $x(\eta_n) = 0$ and $x(t) < 0$ for $t \in (\eta_n, s_n]$.

For $t \in [T_0, \eta_n]$, then $x_t \leq u + \varepsilon$ and

$$-x(t) = \int_t^{\eta_n} f(s, x_s)ds \geq \int_t^{\eta_n} \lambda_0(s)R_1(M(x_s)) \geq R_1(u+\varepsilon)\int_t^{\eta_n} \lambda_0(s)ds.$$

Thus, for $t \in [\eta_n, s_n]$, we have $x_t \leq -R_1(u+\varepsilon)\int_{t-1}^{\eta_n} \lambda_0(s)ds$. Hence,

$$\begin{aligned}
x(s_n) &= \int_{\eta_n}^{s_n} f(t, x_t)dt \\
&\geq \int_{\eta_n}^{s_n} \lambda_0(t)R_1\Big(-R_1(u+\varepsilon)\int_{t-1}^{\eta_n} \lambda_0(s)ds\Big)dt \\
&\geq \int_{\eta_n}^{s_n} \lambda_0(t)R_1\Big(-R_1(u+\varepsilon)[1 - \int_{\eta_n}^{t} \lambda_0(s)ds]\Big)dt \\
&= -\frac{1}{R_1(u+\varepsilon)}\int_{\psi(s_n)}^{\psi(\eta_n)} R_1(x)\,dx,
\end{aligned}$$

where $\psi(t) = -R_1(u+\varepsilon)[1 - \int_{\eta_n}^t \lambda_0(s)ds]$. Since $\psi(s_n) \geq 0$ and $R_1(x)$ is a convex function, from Jensen's inequality we have

$$\begin{aligned}
x(s_n) &\geq -\frac{1}{R_1(u+\varepsilon)}\int_0^{-R_1(u+\varepsilon)} R_1(x)\,dx \\
&\geq R_1\Big(-\frac{1}{R_1(u+\varepsilon)}\int_0^{-R_1(u+\varepsilon)} x\,dx\Big) = R_1\Big(-\frac{R_1(u+\varepsilon)}{2}\Big).
\end{aligned}$$

By letting $n \to \infty$ and $\varepsilon \to 0^+$, we obtain

$$-v \geq R_1(-R_1(u)/2). \tag{2.8}$$

From (2.6) and (2.7), we have $R_1(-R_1(u)/2) = \frac{-\alpha^2 u}{2+(2+\alpha)b_1 u} > -1/b_2$, and the estimate above implies that $-v > -1/b_2$, thus $R_2(-v)$ and $R_2 \circ R_1(-R_1(u)/2)$ are well defined.

Consider now a sequence $(x(t_n))$ of local maxima, $x(t_n) > 0, t_n \to \infty, t_n - 2 \geq T_0, x(t_n) \to u$ as $n \to \infty$. We may assume that $x(t) < x(t_n)$ for $t_n - t > 0$ but small. In a similar way, we deduce that there exists $\xi_n \in [t_n - 1, t_n)$ such that $x(\xi_n) = 0$ and $x(t) > 0$ for $t \in (\xi_n, t_n]$. For $t \in [T_0, \xi_n]$, then $x_t \geq -(v+\varepsilon)$, hence $M(-x_t) \leq v + \varepsilon$ and, for ε small such that $-(v+\varepsilon) > -1/b_2$,

$$\begin{aligned}
x(t_n) &= \int_{\xi_n}^{t_n} f(t, x_t)\,dt \leq \int_{\xi_n}^{t_n} \lambda_0(t)R_2(-M(-x_t))\,dt \\
&\leq R_2(-(v+\varepsilon))\int_{t_n-1}^{t_n} \lambda_0(t)\,dt \leq R_2(-(v+\varepsilon)).
\end{aligned}$$

By letting $n \to \infty$ and $\varepsilon \to 0^+$, we obtain

$$u \leq R_2(-v). \tag{2.9}$$

From (2.8) and (2.9), we get

$$u \leq R_2 \circ R_1\Big(-\frac{R_1(u)}{2}\Big),$$

i.e., $u \leq \frac{\alpha^3 u}{2+(2b_1+\alpha b_1-b_2\alpha^2)u}$, and (2.7) implies that $2b_1 + \alpha b_1 - b_2\alpha^2 > 0$ for $\alpha \in (0, \sqrt[3]{2}]$. If $u > 0$, for $\alpha \in (0, \sqrt[3]{2}]$ we obtain a contradiction. Thus, $u = 0$. From (2.8), it follows that also $v = 0$. □

From the above proof, it is easy to see that the result holds if, instead of (2.6) and (2.7), we have

$$\alpha < \sqrt[3]{2} \quad \text{and} \quad \frac{\alpha^2}{2+\alpha} \leq \frac{b_1}{b_2}.$$

We remark that the above framework is valid for equations (1.1) with time-dependent bounded delays $h(t)$ ($h(t) \leq h$ for $t \geq 0$), provided existence and smoothness of solutions are assumed, since the arguments used here or in [1, 7] can be adapted to this situation (cf. [1, Remark 3.11] and Remark 3.6 below).

Remark 2.8. As the next example shows, it is impossible to extend the result in Theorem 2.7 up to $\alpha = 3/2$ even if $b_1/b_2 > 1 - \delta$, for $\delta > 0$ arbitrarily small.

Let us consider the following $T(M)$–periodic delay differential equation

$$x'(t) = f(x(t - h(t))), \tag{2.10}$$

where, for $a \in (-3/2, -1)$, $b > 0$ and $\beta \in (-1/b, 0)$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function defined by

$$f(x) = \begin{cases} r_1(x) = \frac{ax}{1+x} & \text{if } x \geq 0, \\ r_2(x) = \frac{ax}{1+bx} & \text{if } x \in (\beta, 0), \\ r_2(\beta) & \text{if } x \leq \beta. \end{cases}$$

Clearly, (H3') is satisfied with $\lambda(t) \equiv -a, b_1 = 1$ and $b_2 = b$. For an arbitrary $b > 1$, the ratio b_1/b_2 is arbitrarily close to 1.

On the other hand, for $M \in (0, -a-1)$, let $h(t)$ be the positive piecewise linear $T(M)$-periodic function, defined on the period interval $[0, T]$ as

$$h(t) = \begin{cases} t & \text{if } t \in (0, 1], \\ 1 & \text{if } t \in [1, \lambda(M)], \\ t - \lambda(M) & \text{if } t \in (\lambda(M), \lambda(M) + 1], \\ 1 & \text{if } t \in [1 + \lambda(M), T(M)], \end{cases}$$

and then extended over all $\mathbb{R}$ in a periodic way. Here,

$$\lambda(M) = 1 - \frac{1+M}{a} \in (1, 2), \tag{2.11}$$

and $T(M)$ will be defined in the proof of the next theorem, which, for convenience of the reader, will be presented in Appendix A at the end of the paper.

Theorem 2.9. *For every real number $b > 1$, there exists $\delta > 0$ such that for each $a \in (-3/2, -3/2 + \delta)$ there is $M = M(a) > 0$, such that Eq. (2.10) has at least one $T(M)$–periodic solution $x(t)$ with* $\max_{\mathbb{R}} x(t) = M$.

3. **Scalar population models.** We apply now Theorem 2.5 to scalar population models that can be written in the form

$$\dot{x}(t) = (1 + x(t))F(t, x_t), \quad t \geq 0, \tag{3.1}$$

where $F : [0, \infty) \times C \to \mathbb{R}$ is a continuous function. Due to biological reasons (cf. Section 4), we only consider *admissible* initial conditions

$$x_0 = \varphi, \quad \varphi \in C_{-1}, \tag{3.2}$$

where $C_{-1} := \{\varphi \in C : \varphi(\theta) \geq -1 \text{ for } \theta \in [-h, 0) \text{ and } \varphi(0) > -1\}$. Solutions of the initial value problem (IVP) (3.1)–(3.2) are denoted by $x(t) = x(\varphi)(t)$ or simply $x(t)$, and are supposed to be defined on their maximal interval of existence.

In this section, the following hypotheses will be considered:

(A1) there exists a piecewise continuous function $\beta : [0, \infty) \to [0, \infty)$ for which

$$\beta_0 := \sup_{t \geq 0} \int_{t-h}^{t} \beta(s) ds < \infty,$$

and such that for each $q \geq -1$ there is $\eta(q) \in \mathbb{R}$ such that for $t \geq 0$ and $\varphi \in C_{-1}, \varphi \geq q$, then

$$F(t, \varphi) \leq \beta(t)\eta(q);$$

(A2) if $w : [-h, \infty) \to \mathbb{R}$ is continuous and there is $\lim_{t\to\infty} w(t) = w^* > -1, w^* \neq 0$, then $\int_0^\infty F(s, w_s)\, ds$ diverges;

(A3) there are a piecewise continuous function $\lambda : [0, \infty) \to [0, \infty)$ and a constant $b \geq 0$ such that, for $r(x) := \dfrac{-x}{1 + bx}$, $x > -1/b$,

$$\lambda(t)r(M(\varphi)) \leq F(t, \varphi) \leq \lambda(t)r(-M(-\varphi)), \quad \text{for} \quad t \geq 0,$$

where the first inequality holds for all $\varphi \in C_{-1}$ and the second one for $\varphi \in C$ such that $\varphi > \max\{-1/b, -1\}$, and $M(\varphi) = \max\{0, \sup_{\theta\in[-h,0]} \varphi(\theta)\}$;

(A4) for $\lambda(t)$ as in (A3), there is $T \geq h$ such that, for

$$\alpha := \alpha(T) = \sup_{t \geq T} \int_{t-h}^{t} \lambda(s)\, ds,$$

$\alpha \leq 3/2$ if $b \neq 1/2$, and $\alpha < 3/2$ if $b = 1/2$.

As in Section 2, we again remark that, for case $b = 0$, (A3) and (A4) imply (A1).

Clearly, hypothesis (A2) is imposed to force non-oscillatory solutions to go to zero, as t goes to infinity. The case $b = 0$ was studied in [1], where a hypothesis (conditions (H1)–(H2) in [1]) slightly stronger than (A2) was assumed. However, it was noticed that conditions (H1)–(H2) in [1] could be replaced by the above assumption (A2), similarly to what was considered in [12] for Eq. (1.1), hence the following result follows from [1].

Theorem 3.1. [1] *Assume (A2)–(A4) with $b = 0$. Then all solutions $x(t)$ of (3.1) with admissible initial conditions (3.2) are defined on $[0, \infty)$ and satisfy $x(t) \to 0$ as $t \to \infty$.*

Under (A1)-(A4), we now prove that the statement in Theorem 3.1 is still true for any $b > 0$, where an additional condition similar to the one in Theorem 2.5 may be required.

Theorem 3.2. *Assume (A1)–(A4). If $b \neq 1/2$, assume in addition that either $\lambda(t) > 0$ for t large, or $\alpha < 3/2$ for some $T \geq h$ and $\alpha := \alpha(T)$ as in (A4). Then all solutions $x(t)$ of (3.1) with admissible initial conditions (3.2) satisfy $x(t) \to 0$ as $t \to \infty$.*

Proof. The case $b = 0$ was addressed in Theorem 3.1, so we assume $b > 0$. From [1], it follows that all solutions $x(t)$ of (3.1) with admissible initial conditions (3.2) are defined, bounded and bounded below away from -1 on $[0, \infty)$.

The change of variables $y(t) = \log(1 + x(t)), t \geq 0$, transforms (3.1) into

$$\dot{y}(t) = f(t, y_t), \tag{3.3}$$

where $f(t, \varphi) = F(t, e^{\varphi} - 1)$. For $\varphi \in C$, then $\psi = e^{\varphi} - 1 \in C_{-1}$, and from (A3) we have

$$\begin{aligned} &f(t, \varphi) \geq \lambda(t) r(M(e^{\varphi} - 1)), \quad t \geq 0, \ \varphi \in C \\ &f(t, \varphi) \leq \lambda(t) r(-M(-e^{\varphi} + 1)), \quad t \geq 0, \ \varphi \in C \text{ with } \varphi > -1/b. \end{aligned} \tag{3.4}$$

On the other hand, clearly

$$M(e^{\varphi} - 1) = e^{M(\varphi)} - 1, \quad M(-e^{\varphi} + 1) = -e^{-M(-\varphi)} + 1, \quad \text{for all } \varphi \in C.$$

We first consider the case $b \geq 1/2$. Define

$$r_1(x) := \frac{-x}{1 + (b - 1/2)x}, \ x > -1/(b - 1/2).$$

We deduce that

$$r_1(x) \leq r(e^x - 1) \quad \text{for all } x \geq 0, \tag{3.5}$$

$$r_1(x) \geq r(e^x - 1) \quad \text{for all } x \in (-1/(b - 1/2), 0]. \tag{3.6}$$

In fact, (3.5) is equivalent to the inequality $w(x) := 1 + \frac{1}{2}x + e^x\left(\frac{1}{2}x - 1\right) \geq 0, \ x \geq 0$, which can be proven easily by studying the signs of $w'(x)$ and $w''(x)$. Analogously, we prove (3.6). From (3.4), (3.5) and (3.6), we obtain for $t \geq 0$

$$\lambda(t) r_1(M(\varphi)) \leq f(t, \varphi) \leq \lambda(t) r_1(-M(-\varphi)),$$

where the first inequality holds for every $\varphi \in C$ and the second one for $\varphi \in C$ such that $\varphi > -1/(b - 1/2)$. Thus, the function f in (3.3) satisfies (H3) with b replaced by $b - 1/2$. Assumptions (A1), (A2) and (A4) imply that (H1), (H2) and (H4) hold for f. Consequently, the result follows from Theorem 2.5.

If $0 < b < 1/2$, we effect the change of variables $z(t) = -\log(1 + x(t))$, and Eq. (3.1) becomes

$$\dot{z}(t) = g(t, z_t),$$

for $g(t, \varphi) = -F(t, e^{-\varphi} - 1)$. We can check that hypotheses (H1)–(H4) hold for g, with $r(x)$ replaced by $r_2(x) := \dfrac{-x}{1 + (1/2 - b)x}$, $x > -1/(1/2 - b)$. Again, the result follows from Theorem 2.5. □

The next result follows immediately from Theorem 2.7 and the proof of Theorem 3.2.

Corollary 3.3. *Assume (A1), (A2). Moreover, assume that:*

(i) *there is a piecewise continuous function $\lambda : [0, \infty) \to [0, \infty)$ and there are constants $b_1, b_2 \geq 0, b_1 \leq b_2$, such that, for $r_i(x) := \dfrac{-x}{1 + b_i x}$,*

$$\lambda(t) r_1(M(\varphi)) \leq F(t, \varphi) \leq \lambda(t) r_2(-M(-\varphi)), \quad \text{for} \quad t \geq 0,$$

where the first inequality holds for all $\varphi \in C_{-1}$ and the second one for $\varphi \in C$ such that $\varphi > \max\{-1/b_2, -1\}$;

(ii) *either $b_1 > 1/2$ or $b_2 < 1/2$;*

(iii) *for $\lambda(t)$ as in (i), there is $T \geq h$ such that*

$$\alpha := \alpha(T) = \sup_{t \geq T} \int_{t-h}^{t} \lambda(s)ds \leq \sqrt[3]{2}; \tag{3.7}$$

(iv) *the constants b_1, b_2 satisfy*

$$\frac{\alpha^2}{2+\alpha} < \frac{b_1 - 1/2}{b_2 - 1/2}\,; \tag{3.8}$$

Then all solutions $x(t)$ of (3.1) satisfy $x(t) \to 0$ as $t \to \infty$.

Remark 3.4. By the translation $x \mapsto 1+x$, the scalar FDE (7) in [7] can be seen as a particular case of Eq. (3.1). Thus, Theorem 3 in [7] is a consequence of Theorem 3.2 above. Condition $\lambda(t) > 0$ for all $t \geq 0$ in [7] can be weakened according to the statement in Theorem 3.2. On the other hand, it seems that, in the statement of [7, Theorem 3], the authors have forgotten to include a hypothesis similar to (A1) (or (H1)), to assure that all solutions are bounded on $[0, \infty)$. That is, for model (7) in [7], if $b > 1$, it seems necessary to further impose that, for all $q \in (-1, 0)$, there is $\eta(q) \in \mathbb{R}$ such that $f(t, x+1) \leq \eta(q)$ for all $t \geq 0, x \in [q, 0)$.

Remark 3.5. For the case $b = 1/2$, Remark 2.6 suggests that Theorem 3.2 should be true under (A1)–(A3), (A4) with $\alpha \leq 3/2$, and if the strict inequality in (A3) is required for $\varphi \in C_{-1}$ such that $\varphi(\theta) \neq 0$ for $\theta \in [-h, 0]$, i.e., if it is further assumed that for $r(x) := \dfrac{-x}{1+x/2}$, $x > -2$,

$$\lambda(t)r(M(\varphi)) < F(t,\varphi) < \lambda(t)r(-M(-\varphi)), \quad \text{for} \quad t \geq 0,$$

where the first inequality holds for all $\varphi \in C_{-1}$ and $\varphi(\theta) \neq 0, \theta \in [-h, 0]$, and the second one for $\varphi \in C$ such that $\varphi(\theta) \neq 0, \theta \in [-h, 0]$, and $\varphi > -1$.

Remark 3.6. As remarked in [1, Remark 3.11], the present setting can be applied to Eq. (3.1) with time-dependent bounded discrete delays of the form

$$\dot{x}(t) = (1 + x(t))F_0\big(t, x(t - h_1(t)), \dots, x(t - h_n(t))\big), \tag{3.9}$$

where $h_i : [0, \infty) \to [0, \infty)$ are continuous and bounded. For $h(t) = \max\{h_i(t) : 1 \leq i \leq n\} > 0$, $t \geq 0$, Theorem 3.2 is valid for (3.9) if we replace $\int_{t-h}^{t} \beta(s)ds$, $\int_{t-h}^{t} \lambda(s)ds$ in (A1), (A4) by $\int_{t-h(t)}^{t} \beta(s)ds$, $\int_{t-h(t)}^{t} \lambda(s)ds$, respectively.

4. **Applications.** In this section, we consider some scalar FDEs which have served as models for the growth of a single population, and improve some results in the literature.

Example 4.1. Consider the following IVP, studied in [1, 8]:

$$\dot{y}(t) = -p(t)y(t-h)(a - y(t))(b + y(t)), \quad t \geq 0, \tag{4.1}$$
$$y_0 = \psi, \tag{4.2}$$

for

$$\psi \in C = C([-h, 0]; \mathbb{R}),\ \psi(\theta) \in [-b, a) \text{ for } \theta \in [-h, 0),\ \psi(0) \in (-b, a), \tag{4.3}$$

where $a, b, h > 0$ and $p : [0, \infty) \to (0, \infty)$ is continuous. By the change of variables

$$x(t) = \frac{1 + y(t)/b}{1 - y(t)/a} - 1,$$

(4.1)–(4.2) is transformed into

$$\dot{x}(t) = (1 + x(t))F(t, x_t), \quad t \geq 0$$
$$x_0 = \varphi \in C_{-1}$$

where

$$F(t, \varphi) = -p(t)\frac{ab\varphi(-h)}{1 + \frac{b}{a+b}\varphi(-h)}.$$

Thus, F is given by $F(t, \varphi) = abp(t)r(\varphi(-h))$, where

$$r(x) = \frac{-x}{1 + \frac{b}{a+b}x},$$

and (A3) is satisfied with $\lambda(t) = abp(t)$.

Theorem 4.2. *Suppose that*

$$\int_0^\infty p(t)dt = \infty$$

and there is $T \geq h$ with

$$ab \sup_{t \geq T} \int_{t-h}^t p(s)ds \leq \frac{3}{2}.$$

Then, the solutions of the IVPs (4.1)–(4.2) with ψ as in (4.3) are defined for $t \geq 0$ and satisfy $y(t) \to 0$ as $t \to \infty$.

Proof. For $a = b$, the result was proven in [8]. For $a \neq b$, one can verify (see [1] for details) that (A1)–(A2) hold with $\beta(t) = abp(t)$ if $\int_0^\infty p(t)dt = \infty$. Thus, the result is an immediate consequence of Theorem 3.2 $\square$

We note that Theorem 4.2 improves both Corollary 4.1 in [8], and Theorem 4.4 in [1]: in [8] the same result was proven under the stronger condition

$$\frac{(a+b)^2}{4} \sup_{t \geq T} \int_{t-h}^t p(s)\, ds \leq 3/2,$$

and in [1] this condition was replaced by

$$b(a+b) \sup_{t \geq T} \int_{t-h}^t p(s)\, ds \leq 3/2.$$

Example 4.3. Consider the delayed "food-limited" population model with environmental periodicity proposed by Gopalsamy et al. [3],

$$\dot{N}(t) = s(t)N(t)\frac{K(t) - N(t - m\omega)}{K(t) + c(t)s(t)N(t - m\omega)}, \tag{4.4}$$

where s, c, K are continuous, positive and ω-periodic functions and m is a positive integer. We only consider positive solutions of (4.4). In other words, we consider solutions of (4.4) with initial conditions

$$N_0 = \varphi, \tag{4.5}$$

for $\varphi \in C$ and $\varphi(\theta) \geq 0$ for $-m\omega \leq \theta < 0$ and $\varphi(0) > 0$.

From [3], it is known that there exists a unique positive ω-periodic solution $N^*(t)$, that satisfies $K_* \leq N^*(t) \leq K^*, t \in [0, \omega]$, for $K_* = \min_{0 \leq t \leq \omega} K(t), K^* = \max_{0 \leq t \leq \omega} K(t)$. Furthermore, the solution of any IVP (4.4)–(4.5) is bounded and bounded below away from zero on $[0,\infty)$. Effect the change $x(t)=N(t)/N^*(t)-1$. Then, (4.4) reads as

$$\dot{x}(t)=(1+x(t))F(t,x_t),\quad t\ge 0, \tag{4.6}$$

where

$$F(t,\varphi)=F_0(t,\varphi(-m\omega))$$

for

$$F_0(t,x)=-\frac{s(t)K(t)N^*(t)[1+s(t)c(t)]x}{[K(t)+s(t)c(t)N^*(t)][K(t)+s(t)c(t)N^*(t)(1+x)]}.$$

Clearly, initial conditions (4.5) for (4.4) are equivalent to initial conditions $x_0=\varphi$ with $\varphi\in C_{-1}$ for (4.6).

Define

$$b(t)=\frac{s(t)N^*(t)[1+s(t)c(t)]}{K(t)+s(t)c(t)N^*(t)}. \tag{4.7}$$

Thus (4.6) is written as

$$\dot{x}(t)=-(1+x(t))b(t)\frac{K(t)x(t-m\omega)}{K(t)+s(t)c(t)N^*(t)(1+x(t-m\omega))}.$$

Since the functions s,c,K have positive lower and upper bounds, it is easy to check that (A1) and (A2) hold for (4.6). On the other hand, for

$$v(t):=\frac{s(t)c(t)N^*(t)}{K(t)},$$

let

$$v_0=\min_{t\ge 0}v(t), \tag{4.8}$$

and define

$$r(x)=\frac{-x}{1+\frac{v_0}{1+v_0}x}.$$

For $x\ge 0$, we have

$$\frac{K(t)x}{K(t)+s(t)c(t)N^*(t)(1+x)}=\frac{x}{1+v(t)(1+x)}\le\frac{x}{1+v_0(1+x)}=-\frac{1}{1+v_0}r(x).$$

Analogously, for $-1\le x<0$,

$$\frac{K(t)x}{K(t)+s(t)c(t)N^*(t)(1+x)}\ge\frac{x}{1+v_0(1+x)}=-\frac{1}{1+v_0}r(x).$$

This implies that (4.6) satisfies (A3), for $r(x)$ defined as above, i.e., with $b=\frac{v_0}{1+v_0}$, and

$$\lambda(t)=\frac{b(t)}{1+v_0}.$$

Applying Theorem 3.2, we obtain the following criterion for the global attractivity of the periodic solution $N^*(t)$ of (4.4), which improves the result in [1, Theorem 4.8].

Theorem 4.4. *For $b(t)$ as in (4.7) and v_0 as in (4.8), assume that*

$$v_0\ne 1\quad\text{and}\quad\frac{1}{1+v_0}\int_0^{m\omega}b(t)\,dt\le\frac{3}{2},\qquad\text{or}$$

$$v_0=1\quad\text{and}\quad\int_0^{m\omega}b(t)\,dt<3.$$

Then every positive solution $N(t)$ of (4.4) satisfies

$$\lim_{t\to\infty}(N(t)-N^*(t))=0.$$

Example 4.5. Consider the Michaelis-Menton single-species growth equation with several delays

$$\dot{y}(t)=\gamma(t)y(t)\left[1-\sum_{i=1}^{n}\frac{a_iy(t-h_i(t))}{1+c_iy(t-h_i(t))}\right], \tag{4.9}$$

where $a_i>0$, $c_i>0$, $\gamma(t),h_i(t)$ are continuous functions, $0\le h_i(t)\le h$ for $i=1,\dots,n$, $\gamma(t)\ge 0, t\ge 0$, and $\gamma(t)>0$ for large t, $h(t)=\max\{h_i(t):i=1,2,\dots,n\}>0$, and

$$\sum_{i=1}^{n}\frac{a_i}{1+c_i}=1. \tag{4.10}$$

Equation (4.9) was investigated in [6, 11] in the particular case $\gamma(t)\equiv\gamma>0$ and $h_i(t)\equiv h_i\ge 0$, $i=1,\dots,n$. As shown in [6, Corollary 4.3.2], the change of variables $x(t)=y(t)-1$ reduces (4.9) to

$$\dot{x}(t)=(1+x(t))F(t,x_t), \tag{4.11}$$

where

$$F(t,\varphi)=-\gamma(t)\sum_{i=1}^{n}\frac{a_i\varphi(-h_i(t))}{(1+c_i)[1+c_i+c_i\varphi(-h_i(t))]}.$$

Set $a=\sum_{i=1}^{n}a_i$, and $c=\min\{c_i:i=1,2,\dots,n\}$.

Theorem 4.6. *Assume that*

$$\int_0^\infty\gamma(t)\,dt=\infty. \tag{4.12}$$

If $c\neq 1$ and

$$\frac{a}{(1+c)^2}\int_{t-h(t)}^{t}\gamma(s)\,ds\le\frac{3}{2},\quad\text{for large } t\ge 0, \tag{4.13}$$

then all positive solutions of (4.9) tend to the positive equilibrium $y_*=1$ as $t\to\infty$. If $c=1$, the same result holds if we replace (4.13) by

$$a\int_{t-h(t)}^{t}\gamma(s)\,ds<6,\quad\text{for large } t\ge 0.$$

Proof. We prove that all solutions of (4.11) with initial condition $x_0=\varphi\in C_{-1}$ go to zero as $t\to\infty$. Set

$$r(x)=\frac{-x}{1+(c/(1+c))x}\quad,\quad f_i(x)=\frac{a_ix}{(1+c_i)(1+c_i+c_ix)},\ i=1,2,\dots,n.$$

Since $\varphi\in C_{-1}$ we have, for $i=1,2,\dots,n$,

$$-f_i(\varphi(-h_i(t)))\ge -f_i(M(\varphi))\ge\frac{a_i}{(1+c)^2}r(M(\varphi)).$$

Hence,

$$\begin{aligned}F(t,\varphi)&=-\gamma(t)\sum_{i=1}^{n}f_i(\varphi(-h_i(t)))\ge\gamma(t)\sum_{i=1}^{n}\frac{a_i}{(1+c)^2}r(M(\varphi))\\&=\gamma(t)\frac{a}{(1+c)^2}\,r(M(\varphi)).\end{aligned} \tag{4.14}$$

Analogously,
$$-f_i(\varphi(-h_i(t))) \le \frac{a_i}{(1+c)^2} r(-M(-\varphi)),$$
and therefore
$$F(t,\varphi) \le \gamma(t)\frac{a}{(1+c)^2}\, r(-M(-\varphi)). \tag{4.15}$$
From (4.14) and (4.15), we conclude that (A3) is satisfied with $\lambda(t) = a\gamma(t)/(1+c)^2$, and $b = c/(1+c)$.

Condition (A2) follows from (4.12), and (A1) holds with $\beta(t) = \gamma(t)$, $\eta(q) = -\sum_{i=1}^n f_i(q)$, $q > -1$. Hence the result is now a consequence of Theorem 3.2. See also Remark 3.6. □

For the particular case $\gamma(t) \equiv \gamma > 0$ and $h_i(t) \equiv h_i \ge 0$, $i = 1,\dots,n$, $h = \max_{1\le i\le n} h_i > 0$, Theorem 4.6 was proven in [11] under condition $\gamma h \le 3/2$. Thus our result improves the one in [11] whenever $a < (1+c)^2$. This is always the case if $c_i = c$ for all $i = 1, 2, \dots, n$, since in this situation, from (4.10), $a = (1+c) < (1+c)^2$. In particular, our result is always better for $n = 1$.

Example 4.7. Consider the delay differential equation
$$\dot N(t) = \rho(t)N(t)\frac{K - \sum_{i=1}^n a_i N^p(t-\tau_i(t))}{K + \sum_{i=1}^n s_i(t)N^p(t-\tau_i(t))}, \quad t \ge 0, \tag{4.16}$$
where $a_i > 0, K > 0, p > 0$, $\rho(t), s_i(t), \tau_i(t)$ are continuous functions, $0 \le \tau_i(t) \le \tau$, $\rho(t), s_i(t) \ge 0, t \ge 0$, for $i = 1,\dots,n$, and $\rho(t) > 0$ for large t. Eq. (4.16) was studied in [1, 9]. In [9], possible unbounded delays were allowed. In [1], Theorem 3.1 (i.e., Theorem 3.2 with $b = 0$) was applied to the study of the global attractivity of the positive equilibrium of (4.16). However, our method with $b > 0$ is not easy to apply to certain models with more than one discrete delay. Therefore, here we only consider the case $n = 1$.

Let $n = 1$, $\tau_1(t) = \tau(t)$, $a = a_1$ and $S(t) = s_1(t)$, so that (4.16) reads as
$$\dot N(t) = \rho(t)N(t)\frac{K - aN^p(t-\tau(t))}{K + S(t)N^p(t-\tau(t))}, \quad t \ge 0. \tag{4.17}$$
Eq. (4.17) has been considered by many authors, since it has been proposed as an alternative model to the delayed logistic equation for a food limited single population model (see [1, 2, 4, 7] and references therein).

As usual, because of its biological context, we only consider positive solutions of (4.17). Following the approach in [1], we effect the change of variables
$$1 + x(t) = \Big(\frac{N(t)}{N_*}\Big)^p,$$
where
$$N_* = \Big(\frac{K}{a}\Big)^{1/p}$$
is the unique positive equilibrium of (4.17). Thus, Eq. (4.17) is written as
$$\dot x(t) = -p\rho(t)(1+x(t))\frac{ax(t-\tau(t))}{a + S(t)[1 + x(t-\tau(t))]}, \quad t \ge 0. \tag{4.18}$$

The following result was proven in [7, Corollary 3]. However, here we present a simpler proof.

Theorem 4.8. [7] *Assume* $S_0 := \inf_{t\geq 0} S(t) \neq a$, *and*

$$\int_0^\infty \frac{\rho(t)}{1+S(t)}\, dt = \infty, \tag{4.19}$$

$$\frac{p}{1+a^{-1}S_0} \int_{t-\tau(t)}^t \rho(s)\, ds \leq \frac{3}{2}, \quad \text{for large } t \geq 0. \tag{4.20}$$

Then all positive solutions of (4.17) tend to the positive equilibrium N_* *as* $t \to \infty$. *If* $S_0 = a$, *the same result holds if we assume (4.19) and*

$$p \int_{t-\tau(t)}^t \rho(s)\, ds < 3, \quad \text{for large } t \geq 0. \tag{4.21}$$

Proof. Note that (4.18) has the form (3.1), for F defined by

$$F(t,\varphi) = p\rho(t) F_0\big(t, \varphi(-\tau(t))\big), \quad t \geq 0, \varphi \in C_{-1}, \tag{4.22}$$

where

$$F_0 : [0,\infty) \times [-1,\infty) \to \mathbb{R}, \qquad F_0(t,x) = -\frac{x}{1+a^{-1}S(t)(1+x)}.$$

Assuming (4.19), then (A1) is fulfilled with $\beta(t) = \frac{\rho(t)}{1+a^{-1}S(t)}$, as well as assumptions (H1)–(H2) in [1] (cf. [1]). On the other hand, as we have already noticed, (H1)–(H2) in [1] imply (A2). Thus (4.19) implies that F given by (4.22) satisfies conditions (A1)–(A2).

We now argue as in Example 4.3 Define

$$r(x) = \frac{-x}{1 + \frac{S_0}{a+S_0} x}. \tag{4.23}$$

Then,

$$\begin{aligned} F_0(t,x) &\geq \frac{-x}{1+a^{-1}S_0(1+x)} = \frac{1}{1+a^{-1}S_0}\, r(x), \quad x \geq 0, \\ F_0(t,x) &\leq \frac{-x}{1+a^{-1}S_0(1+x)} = \frac{1}{1+a^{-1}S_0}\, r(x), \quad -1 \leq x < 0. \end{aligned} \tag{4.24}$$

From (4.22) and (4.24), we have

$$F(t,\varphi) \geq \frac{p\rho(t)}{1+a^{-1}S_0}\, r\big(\varphi(-\tau(t))\big) \geq \frac{p\rho(t)}{1+a^{-1}S_0}\, r\big(M(\varphi)\big)$$

if $\varphi(-\tau(t)) \geq 0$, and

$$F(t,\varphi) \leq \frac{p\rho(t)}{1+a^{-1}S_0}\, r\big(\varphi(-\tau(t))\big) \leq \frac{p\rho(t)}{1+a^{-1}S_0}\, r\big(-M(-\varphi)\big)$$

if $-1 \leq \varphi(-\tau(t)) \leq 0$. We deduce then that F satisfies (A3) where $r(x)$ is defined by (4.23) and

$$\lambda(t) = \frac{p\rho(t)}{1+a^{-1}S_0}.$$

Thus (4.20) and (4.21) imply (A4) in the form stated in Remark 3.6, and the result follows from Theorem 3.2. $\square$

For the particular case of Eq. (4.16) with $n = 1$, we remark that Theorem 4.8 improves the results in [1, 9].

From Corollary 3.3, the following results:

Theorem 4.9. *Let* $0 < S_0 := \inf_{t\geq 0} S(t)$, $S^0 := \sup_{t\geq 0} S(t) < \infty$, *and suppose that either* $S_0 > a$ *or* $S^0 < a$. *Assume (4.19), and that there is* $T \geq h$ *such that*

$$\alpha := \alpha(T) = \sup_{t\geq T}\Big(p\int_{t-\tau(t)}^{t}\frac{\rho(s)}{1+a^{-1}S(s)}\,ds\Big) \leq \sqrt[3]{2}\,, \tag{4.25}$$

with

$$\frac{\alpha^2}{2+\alpha} < \frac{(S_0-a)(a+S^0)}{(S^0-a)(a+S_0)}\,. \tag{4.26}$$

Then all positive solutions of (4.17) tend to the positive equilibrium N_* *as* $t \to \infty$.

Proof. Following the reasoning above, one deduces that F defined by (4.22) satisfies (i) of Corollary 3.3, i.e.,

$$\lambda(t)r_1(M(\varphi)) \leq F(t,\varphi) \leq \lambda(t)r_2(-M(-\varphi)), \quad \text{for} \quad t \geq 0,$$

where the first inequality holds for all $\varphi \in C_{-1}$ and the second one for $\varphi \in C_{-1}$ such that $\varphi > -1/b_2$, with $\lambda(t)$ and $r_i(x)$ defined by

$$\lambda(t) = \frac{p\rho(t)}{1+a^{-1}S(t)}, \quad r_i(x) = \frac{-x}{1+b_ix},\ i = 1,2,$$

where $0 < b_1 \leq b_2$ are given by

$$b_1 = \frac{S_0}{a+S_0}, \quad b_2 = \frac{S^0}{a+S^0}.$$

Also, $b_1 > 1/2$ if $S_0 > a$, and $b_2 < 1/2$ if $S^0 < a$. On the other hand, (4.25) and (4.26) imply (3.7) and (3.8), respectively. $\square$

Remark 4.10. For (4.17) with $a = 1$ and $\tau(t) \equiv \tau$, So and Yu [10] proved that N_* is uniformly and asymptotically stable assuming (4.19) and

$$p\sup_{t\geq\tau}\Big(\int_{t-\tau}^{t}\frac{\rho(s)}{1+S(s)}ds\Big) < \frac{3}{2}.$$

However, Example 4.11 below shows that this condition is not sufficient to insure the global attractivity of N_*, that is, the result in Theorem 4.8 does not hold if (4.20) is replaced by

$$p\int_{t-\tau(t)}^{t}\frac{\rho(s)}{1+a^{-1}S(s)}\,ds \leq \frac{3}{2}, \quad \text{for large } t \geq 0.$$

Example 4.11. Consider the following periodic delay differential equation with piecewise linear nonnegative coefficients $\rho(t+5) = \rho(t)$, $S(t) = S(t+5)$:

$$x'(t) = \rho(t)x(t)\frac{1-x(t-1)}{1+S(t)x(t-1)}, \tag{4.27}$$

where $S(t) > 0$ is continuous, and ρ, S are defined on the period interval $[-1,4)$ by

$$\rho(t) = \begin{cases} 0 & \text{if } t \in [-1,1)\cup[2,3); \\ a_1 > 0 & \text{if } x \in [1,2); \\ a_2 > 0 & \text{if } x \in [3,4); \end{cases} \qquad S(t) = \begin{cases} \text{is linear on } [-1,1) \text{ and on } [2,3); \\ b_1 > 0 \quad \text{if } x \in [1,2); \\ b_2 > 0 \quad \text{if } x \in [3,4). \end{cases}$$

For $z \in \mathbb{R}$, we denote $x = x(\cdot,z) : [0,+\infty) \to (0,+\infty)$ as the solution of the initial value problem $x(s,z) \equiv z,\ s \in [-1,0]$, for (4.27). Applying the above mentioned result from [10], we conclude that the inequality

$$\sup_{t\in\mathbb{R}}\int_{t-1}^{t}\frac{\rho(s)}{1+S(s)}ds = \max\left\{\frac{a_1}{1+b_1}, \frac{a_2}{1+b_2}\right\} < 3/2$$

is sufficient for the exponential stability of the equilibrium $x(t,1) \equiv 1$. Thus we will be within the stability zone for $x(t,1) \equiv 1$ with

$$a_1 = 1,\ b_1 = 0.1,\ b_2 = 10,\ a_2 = \frac{4.5(1 + 100\exp(-4.5))}{1 - 10\exp(-4.5))} = 10.68617...,$$

because in this case $\max\{a_1/(1+b_1), a_2/(1+b_2)\} < 1$.

Now, since every solution of (4.27) is eventually constant over intervals $[-1 + 5k, 5k]$, all 5-periodic solutions of (4.27) are determined by the algebraic equation $x(4,z) = z$. It is easy to find that

$$x(2,z) = z\exp\left(\frac{a_1(1-z)}{1+b_1 z}\right),\ x(4,z) = x(2,z)\exp\left(\frac{a_2(1-x(2,z))}{1+b_2 x(2,z)}\right),$$

and that $z_1 = 1,\ z_2 = 146.105...$ and $z_3 = 10$ satisfy $x(4,z) = z$. Thus Eq. (4.27) has at least three positive periodic solutions and cannot be globally attracting.

We remark that it is possible to replace the above $\rho(t)$ with a positive, 5-periodic and continuous $\rho_*(t)$ in such a way that the modified equation (4.27) will have a 5-periodic solution close to $x(t, z_2)$ in the sup-norm. Indeed, a direct analysis of the one-dimensional map $x(4,\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ shows that the fixed point z_2 is a hyperbolic attractor: $x'_z(4, z_2) \in (0,1)$. Thus we can find $\varepsilon_1 < \varepsilon_2$ such that

$$x(5, \bar{U}(z_2, \varepsilon_2)) = x(4, \bar{U}(z_2,\varepsilon_2)) \subset U(z_2, \varepsilon_1), \tag{4.28}$$

where $U(z_2, \varepsilon) = (z_2 - \varepsilon, z_2 + \varepsilon)$.

Now, let us consider the Poincaré map $\Pi_\rho : C_+ \to C_+,\ C_+ \stackrel{def}{=} C([-1,0);\mathbb{R}_+)$, defined as $\Pi_\rho(\psi) = x_5(\psi)$, where $x_t(\psi) : [0,+\infty) \to C_+$ solves the initial value problem $x(s,\psi) = \psi(s),\ s \in [-1,0]$ for (4.27). From (4.28), we get $\Pi_\rho \bar{U}(z_2,\varepsilon_2) \subset U(z_2, \varepsilon_1)$, where $U(\psi, \varepsilon) = \{\phi \in C_+ : |\phi - \psi|_C < \varepsilon\} \subset C_+$ denotes an open ball centered in $\psi \in C_+$ and with radius ε, while $\bar{X}$ stands for the closure of X.

Finally, using the step by step method, we easily find that $\Pi_\alpha(\phi)$ converges to $\Pi_\rho(\phi)$, uniformly on ϕ from bounded subsets of C_+, when $\alpha \to \rho$ in the space of all 5-periodic piecewise continuous functions equipped with the norm $|\alpha|_1 = \int_0^5 |\alpha(s)|ds$. This implies that we can perturb the coefficient ρ slightly (in the stated norm) to transform it into a positive continuous function ρ_* which satisfies $\Pi_{\rho_*}\bar{U}(z_2,\varepsilon_2) \subset U(z_2, \varepsilon_2)$. Since Π is compact, an application of the Schauder fixed point theorem assures the existence of a non-trivial 5-periodic solution of the perturbed equation.

Appendix A. **Proof of Theorem 2.9.** First we need the following lemma:

Lemma A.1. *Let $g : (-\varepsilon, \varepsilon) \times (\alpha, \beta) \to \mathbb{R}$ be a smooth function such that $g(0,a) = 0$ for every $a \in (\alpha, \beta)$, and there is $a_0 \in (\alpha, \beta)$ satisfying*

$$\frac{\partial g(0,a_0)}{\partial x} = 1, \qquad \frac{\partial^2 g(0,a_0)}{\partial a \partial x} < 0, \quad \text{and} \quad \frac{\partial^2 g(0,a_0)}{\partial x^2} > 0.$$

Then there is an open interval Γ about a_0 and a strictly increasing smooth function $x = x(a)$ such that $x(a_0) = 0$ and

$$g(x(a),a) = x(a),\ a \in \Gamma.$$

Proof. Set

$$H(x,a) = \begin{cases} g(x,a)/x - 1 & \text{if } x \neq 0, \\ \partial g(0,a)/\partial x - 1 & \text{if } x = 0. \end{cases}$$

Notice that $H(x,a)$ is smooth and

$$H(0,a_0) = \frac{\partial g(0,a_0)}{\partial x} - 1 = 0, \quad \frac{\partial H(0,a_0)}{\partial x} = \frac{1}{2}\frac{\partial^2 g(0,a_0)}{\partial x^2} > 0.$$

Thus we can apply the implicit function theorem to deduce the existence of a smooth function $x = x(a)$ with $x(a_0) = 0$ defined on some open interval $I \ni a_0$ and such that

$$H(x(a),a) \equiv 0, \ \ a \in I.$$

Since

$$x'(a_0) = -\frac{\partial H(0,a_0)/\partial a}{\partial H(0,a_0)/\partial x} = -2\frac{\partial^2 g(0,a_0)/\partial a \partial x}{\partial^2 g(0,a_0)/\partial x^2} > 0,$$

the function $x(a)$ is strictly increasing in some neighborhood $\Gamma \subset I$ of a_0. Recalling the definition of $H(x,a)$, we conclude that

$$g(x(a),a) = x(a), \ \ x'(a) > 0 \text{ for all } a \in \Gamma.$$

□

Proof of Theorem 2.9. For i=1,2, we introduce the functions (cf. [7])

$$A_i(x) = x + r_i(x) + \frac{1}{r_i(x)}\int_x^0 r_i(t)dt.$$

Notice that, for $\lambda(M)$ defined in (2.11), the solution $x(t) = x(t,M)$ of every initial value problem $x(s) = \psi(s), \ s \in [-1,0]$ satisfying $\psi(0) = M$ decreases on $[0,\lambda(M))$. Moreover,

$$x(\lambda(M)-1) = 0, \text{ and } x(\lambda(M)) = A_1(M).$$

Analogously, if we choose M sufficiently small to satisfy $A_1(M) > \beta$, and define $T(M)$ as

$$T(M) = \lambda(M) + 1 - \frac{1+bA_1(M)}{a},$$

we find that

$$x(T(M)) = A_2(A_1(M)).$$

This means that Eq. (2.10) has at least one nontrivial periodic solution if

$$A_2(A_1(M)) = M$$

for some $M > 0$. Notice that $A_2(A_1(x))$ depends on two parameters a, b. For b fixed, we vary a in order to obtain a positive solution M of $A_2(A_1(M)) = M$ sufficiently small to satisfy all the requirements. To this end, we apply Lemma A.1. Set $g(x,a) = A_2(A_1(x))$. A simple computation shows that

$$\frac{\partial g(0,-3/2)}{\partial x} = 1, \qquad \frac{\partial^2 g(0,-3/2)}{\partial a \partial x} = -2 < 0,$$

and

$$\frac{\partial^2 g(0,-3/2)}{\partial x^2} = \frac{8}{3}(b-1) > 0.$$

This implies, by Lemma A.1, that for every fixed $b > 1$ we can find a positive δ and a strictly increasing continuous function $M = M(a) : [-3/2, -3/2+\delta) \to [0,+\infty)$ such that $g(M(a),a) = M(a)$ for all $a \in (-3/2, -3/2+\delta)$. □

Remark A.2. It is possible to find a T-periodic, positive and continuous function h_*, close to h in the L_1-norm, and such that all conclusions of Theorem 2.9 remain valid if we replace h with h_* in (2.10). To do this, we can apply a perturbation argument similar to that used in Example 4.11.

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