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A geometric approach to some difference equations and systems, extending Lyness-type ones

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Abstract

We present a geometric approach to study some second order difference equations and systems of planar first order difference equations, whose orbits lie in a given family of algebraic curves. This approach allow us to introduce some tools to study the existence of periodic orbits, the corresponding set of periods, and a chaotic property as well.

1 Introduction

In the study of classical Lyness' equation $u_{n+2}u_n = a + u_{n+1}$, (see [1, 13]), there is a well-known geometric way to construct the point $M_{n+1} := (u_{n+2}, u_{n+1})$ from the point $M_n := (u_{n+1}, u_n)$, using the fact that any sequence $\{M_n\}$ lies in one of the invariant curves of the family Lyness' cubics,

$$(a + x + y)(x + 1)(y + 1) - Kxy = 0. \tag{1}$$

The geometrical construction is the following: Given the point M_n, we take the vertical line through M_n. This line cuts the Lyness' cubic containing M_n at certain point M'_n which is the symmetric one, with respect the diagonal $y = x$, of the point M_{n+1}. Notice that every cubic (1) is symmetric with respect $y = x$, and therefore the point M_{n+1} remains on same Lyness' cubic as M_n.

In the study of 2-periodic Lyness' equation $u_{n+2}u_n = a_n + u_{n+1}$, with $n \mapsto a_n$ 2-periodic (see [8, 10]), there is also an analogous geometric construction with an associated cubic, but using first the horizontal line and then the vertical one.

In [2, 3] and [5], together with Guy Bastien (Institut Mathématiques de Jussieu, Université Paris 6 and CNRS) we propose to generalize this geometric approach to other cases. In fact, this generalizations are particular cases of the *QRT maps* introduced in [12], and studied in the complex context in [11]. But perhaps our presentation is more natural for these particular cases. Moreover, using our approach, we can focus specially on the study of the periodic solutions and the corresponding set of periods, and also on a chaotic property of the dynamical system associated to the difference equations. To this end, we have developed some particular tools that we think are very useful in the context of order 2 difference equations or of systems of two order 1 difference equations. We present here this approach and these tools.

2 First geometrical approach

We reverse the problem studied in Lyness' equation, by starting with a family of algebraic curves $\mathcal{C}_K$ such that:

1. $\mathcal{C}_K$ is symmetric with respect to the diagonal line $y = x$;
2. Each vertical line cuts $\mathcal{C}_K$ at two points (real or complex, finite or infinite, distinct or not);
3. Through every point $M_0 \in U = \mathbb{R}_*^{+2}$ or $\mathbb{R}^2$, or some open set, it pass one and only one curve $\mathcal{C}_K$.

Now, we can define a map $F : U \to U$ by the following geometric construction: starting from a point M, we intersect the curve $\mathcal{C}_K$ which contains M by the vertical line passing through M, and then we take the symmetric of the obtained point M' with respect to the diagonal (which is on $\mathcal{C}_K$).

We can write the equation of $\mathcal{C}_K$ as:

$$y^2P(x) + yQ(x) + R(x) = 0, \text{ degrees of } P,\ Q,\ R \leq 2. \tag{2}$$

If $M = (x, y)$, then $M' = (x, z)$ and $F(M) = (z, x)$, with two rules for the calculation of z, namely the *multiplicative* or the *additive* case respectively:

$$yz = \frac{R(x)}{P(x)} \tag{$\mathcal{M}$}$$

with associated map

$$F_{\mathcal{M}}(x, y) = \Big(\frac{R(x)}{yP(x)}, x\Big),$$

which is the *multiplicative case*, and

$$y + z = -\frac{Q(x)}{P(x)} \tag{$\mathcal{A}$}$$

with associated map

$$F_{\mathcal{A}}(x, y) = \Big(-y - \frac{Q(x)}{P(x)}, x\Big),$$

which is the *additive case*.

To avoid a dependence on the parameter K in F, in the equation of $\mathcal{C}_K$ this parameter must appear only in the polynomial Q in the case $(\mathcal{M})$; and only in R in the case $(\mathcal{A})$. Observe now, that using the symmetry assumption, the equation (2) can be written as

$$ax^2y^2 + bxy(x + y) + c(x^2 + y^2) + dxy + e(x + y) + f = 0.$$

So, plugging $d = -K$ in the case $(\mathcal{M})$; and $f = -K$ in the case $(\mathcal{A})$, the corresponding maps F become

$$F_{\mathcal{M}}(x, y) = \Big(\frac{cx^2 + ex + f}{y(ax^2 + bx + c)}, x\Big), \text{ and } F_{\mathcal{A}}(x, y) = \Big(-y - \frac{bx^2 + dx + e}{ax^2 + bx + c}, x\Big),$$

and the associated difference equations are:

$$(\mathcal{M}) : u_{n+2}u_n = \frac{cu_{n+1}^2 + eu_{n+1} + f}{au_{n+1}^2 + bu_{n+1} + c}, \text{ and } (\mathcal{A}) : u_{n+2} + u_n = -\frac{bu_{n+1}^2 + du_{n+1} + e}{au_{n+1}^2 + bu_{n+1} + c}.$$

Observe that the equality of some coefficients is essential (the c's, the b's respectively); otherwise see for example the mysterious Sizer's equation $u_{n+2}u_n = u_{n+1} + Cu_{n+1}^2$ for $C < 1$.

In each case, K and the curves $\mathcal{C}_K$ are invariant, hence, the following are invariant functions for the dynamical system defined by F

$$G_{\mathcal{M}}(x,y) = \frac{ax^2y^2 + bxy(x+y) + c(x^2+y^2) + e(x+y) + f}{xy},$$

for $xy \neq 0$, in the multiplicative case, and

$$G_{\mathcal{A}}(x,y) = ax^2y^2 + bxy(x+y) + c(x^2+y^2) + dxy + e(x+y),$$

in the additive one.

We will denote the curves $\mathcal{C}_K^+$ as those level curves $\{G = K\}$ which are in $\mathbb{R}_*^{+2}$ in the case $(\mathcal{M})$, and in $\mathbb{R}^2$ in the case $(\mathcal{A})$. Observe that in the case $(\mathcal{M})$ the invariant curves are conics, cubic or quartic curves (*elliptic* in the two last cases). In the case $(\mathcal{A})$, the invariant curves are *elliptic quartics*, and the only interesting case can be transformed to a variant of Gumovski-Mira equation

$$u_{n+2} + u_n = \frac{b}{1 + u_{n+1}^2}, \quad |b| \leq 2.$$

3 Second geometrical approach

We start with a family of algebraic curves $\mathcal{C}_K$ such that:

1. Each vertical line has 2 points on $\mathcal{C}_K$ (in $\mathbb{P}^2(\mathbb{C})$);
2. Idem for horizontal lines;
3. Through each point $M \in U \subset \mathbb{R}^2$ it goes exactly one curve $\mathcal{C}_K$ (U is an open set to be precised in each particular case).

So we can define a map $F : (x,y) \mapsto (X,Y)$ in U by the same geometric method as in 2-periodic Lyness' case, [8]: We cut the curve $\mathcal{C}_K$ passing through M by the horizontal line going through M, and then by the vertical line passing through the obtained point.

If $\mathcal{C}_K$ has the equation $x^2P(y) + xQ(y) + R(y) = 0$, or $y^2T(x) + yU(x) + V(x) = 0$, it is easy to see that, with the multiplicative point of view, F is given by

$$\begin{cases} Xx &= \dfrac{R(y)}{P(y)}, \\ Yy &= \dfrac{V(X)}{T(X)}. \end{cases}$$

This gives a system of two order 1 difference equations:

$$\begin{cases} u_{n+1}u_n &= \dfrac{R(v_n)}{P(v_n)}, \\ v_{n+1}v_n &= \dfrac{V(u_{n+1})}{T(u_{n+1})} = \dfrac{V\left(\frac{R(v_n)}{u_n P(v_n)}\right)}{T\left(\frac{R(v_n)}{u_n P(v_n)}\right)}. \end{cases} \tag{3}$$

Setting $P = ay^2 + by + c$, $Q = a'y^2 + b'y + c'$, $R = a''y^2 + b''y + c''$; then $T = ax^2 + a'x + a''$, $U = bx^2 + b'x + b''$, and $V = cx^2 + c'x + c''$. To avoid the dependence on K in the expressions of system (3), the only possibility is to write $b' = -K$. So we have an invariant function $G(x,y) = H(x,y)/(xy)$, with $H(x,y) = ax^2y^2 + xy(bx + a'y) + cx^2 + a''y^2 + c'x + b''y + c''$, and the equation of $\mathcal{C}_K$ is $H(x,y) - Kxy = 0$.

4 Examples

Some examples of systems that fit within the settings of the previous sections are given below.

The "linear" system studied in [8] and [10]

$$\begin{cases} u_{n+1}u_n &= a + v_n, \\ v_{n+1}v_n &= b + u_{n+1} \end{cases} \tag{4}$$

is the one associated to 2-periodic Lyness' equation. The curves $\mathcal{C}_K$ are elliptic cubics.

The "biquadratic" system studied in [5]

$$\begin{cases} u_{n+1}u_n &= v_n^2 - bv_n + c, \\ v_{n+1}v_n &= u_{n+1}^2 - au_{n+1} + c \end{cases} \tag{5}$$

with $a, b, c > 0$, $\max(a,b) < 2\sqrt{c}$, that has ellipses as invariant curves, in the interior of a parabola $U = \{(x,y) \in {\mathbb{R}_*^+}^2 | (x-y)^2 - ax - by + c < 0\}$.

The "homographic" system studied in [6]

$$\begin{cases} u_{n+1}u_n &= 1 + \frac{d}{v_n},\ d > 0 \\ v_{n+1}v_n &= 1 + \frac{d}{u_{n+1}}, \end{cases} \tag{6}$$

whose corresponding curves $\mathcal{C}_K$ are elliptic cubics.

5 The questions to be studied in the two approaches

A typical scheme for an investigation of the dynamical behavior of the above type of equations and systems could be to try to give answers to the following questions:

1. Which is the behavior of the invariant G?

2. The topological and geometric nature of the invariant level curves of G: are they homeomorphic to circles?

3. Is the action of F on $\mathcal{C}_K$ conjugate to a rotation of angle $\theta(K)$ on $\mathbb{T}$?
4. Which is the behavior of the function $K \mapsto \theta(K)$? In particular, which is its image in $[0, 2\pi]$?
5. Which are the possible periods of the periodic solutions $\{u_n\}$ in the first approach, (u_n, v_n) in the second one?
6. If the parameters are rational, are there rational periodic solutions? Which are their periods?
7. What sort of “chaotic” behavior of the associated dynamical system may happen?

6 Some tools that are useful to get the answers

Here we give a brief account of some tools that are useful to obtain the answers to the above mentioned questions.

A. Nature of the level curves of an invariant. The following result can be found in [7], and allows to deal with the above questions 1 and 2, since it can be applied to the invariant functions $G_{\mathcal{M}}$ and $G_{\mathcal{A}}$, defined in Section 2.

Proposition 6.1. *Let be an open set $U \subset \mathbb{R}^2$, and $G : U \to \mathbb{R}$ a C^1 function satisfying*

(a) $G(M) \to G(\infty)$ when $M \to \infty$ in U, and $G < G(\infty)$ in U;

(b) G has a unique critical point $p \in U$.

Then U is a connected and simply connected open set; G attains its strict minimum in U at the point p; and the level curves $\{G = K\}$, for $\min_U G < K < G(\infty)$, are diffeomorphic to the circle $\mathbb{T}$.

B. The chord-tangent group law on an elliptic cubic. The goal is that one can interpret the action of the restriction of the dynamical system to an invariant elliptic curve $\mathcal{C}_K$ as the addition of a point H of $\mathcal{C}_K$: $M \mapsto M +_V H$, where $+_V$ is the group law with a point V (to be found in each case) as zero of the group law.

C. The use of Weierstrass' function $\wp$. If the original cubic $\mathcal{C}_K$ is transformed into the standard cubic Γ_K with equation $Y^2 = 4X^3 - g_2(K)X - g_3(K)$, then it can be parameterized by

$$\begin{cases} X &= \wp_K(z), \\ Y &= \wp'_K(z). \end{cases}$$

If V is the infinite vertical point, and $\widetilde{H}$ is the point we add, the action of F given by the addition $M \mapsto M +_V \widetilde{H}$ is conjugate to a rotation on $\mathbb{T}$ of angle $\theta(K)$. The inversion of Weierstrass function gives

$$\theta(K) = \pi \frac{\displaystyle\int_0^{\sqrt{\frac{e_1-e_3}{X(K)-e_1}}} \frac{du}{\sqrt{(1+u^2)(1+\varepsilon u^2)}}}{\displaystyle\int_0^{+\infty} \frac{du}{\sqrt{(1+u^2)(1+\varepsilon u^2)}}}, \tag{7}$$

where $X(K)$ is the abscissa of point $\widetilde{H}$, $\varepsilon = \frac{e_1 - e_2}{e_1 - e_3}$, and $e_3 < e_2 < e_1$ are the abscissas of the intersections of the standard cubic with the horizontal axis.

The equation (7) implies that in all the cases considered, the rotation number function is analytic in the interval $]\min_U G, +\infty[$. The asymptotic behavior of $\theta(K)$ when $K \to \min_U G$ can be studied by inspection of the eigenvalues of the Jacobian matrix at the fixed point, which corresponds to the minimum of G by Proposition 6.1, (see [4, Proposition 8] and [13]). Using again equation (7) it is possible to study the limit of $\theta(K)$ when $K \to +\infty$, taking into account the next tool.

D. Equivalents of some integrals and asymptotic behavior of the rotation number. The following result has been useful in various contexts (see [1, 6] and [8]), to obtain the asymptotic behavior of the rotation number at infinity.

Proposition 6.2. *Let $l > 0$, $\varepsilon > 0$ and $\alpha > 0$. For any map $\varepsilon \mapsto \phi(\varepsilon)$ tending to 0 together with ε, and satisfying $l + \phi(\varepsilon) > 0$ we put*

$$N(\varepsilon, l, \alpha) = \int_0^{\frac{l+\phi(\varepsilon)}{\varepsilon^\alpha}} \frac{\mathrm{d}u}{\sqrt{(1+u^2)(1+\varepsilon u^2)}} \quad \textit{and} \quad D(\varepsilon) = \int_0^{+\infty} \frac{\mathrm{d}u}{\sqrt{(1+u^2)(1+\varepsilon u^2)}}.$$

Then $D(\varepsilon) \underset{\varepsilon \to 0}{\sim} \frac{1}{2} \ln \frac{1}{\varepsilon}$, and if $\alpha < \frac{1}{2}$ we have $N(\varepsilon, l, \alpha) \underset{\varepsilon \to 0}{\sim} \alpha \ln \frac{1}{\varepsilon}$.

This result when applied to the expression of $\theta(K)$ given by (7) leads to

$$\lim_{K \to \infty} \theta(K) = \lim_{\varepsilon \to 0} \frac{N(\varepsilon, \lambda, \alpha)}{D(\varepsilon)} = 2\alpha.$$

So, in each case, to obtain the asymptotic behavior of $\theta(K)$ at infinity it is only necessary to compute the asymptotic equivalent of $\sqrt{\frac{e_1 - e_3}{X(K) - e_1}}$.

E. Rational rotation number and periodicity, irrational rotation numbers and density of solutions. If the two previous limits of θ are given by some numbers $\alpha < \beta$, then the image of θ contains the interval $I :=]\alpha, \beta[$. If $\pi \frac{q}{n} \in I$, being q and n coprime, then n is the minimal period of each point of the $\mathcal{C}_K$ for the K such that $\theta(K) = \pi \frac{q}{n}$: we have periodic orbits. If $\pi x \in I$, with x irrational, then the corresponding orbits are dense in the $\mathcal{C}_K$.

F. Arithmetical and computational tools. For finding minimal periods of solutions of the system of difference equations, once the rotation interval I is known, it is still necessary to determine which are the entire numbers n such that $q/n \in I$, being q and n coprime. To this end, one can use the following refinement of the *Prime Number Theorem*, which states that for $n \geq 52$, we have

$$\frac{n}{\ln n} \leq \pi(n) \leq \Big(1 + \frac{3}{2 \ln n}\Big) \frac{n}{\ln n},$$

where $\pi(n)$ is the *prime-counting function* that gives the number of primes less than or equal to n; and them a theorem which gives the following majorization of the number $\omega(n)$ of distinct prime factors of an integer n: $\omega(n) \leq 1.38402 \frac{\ln n}{\ln(\ln n)}$. Using these results, it can be computed a bound N, such that each natural number $n > N$ belongs to the set of periods of F, but it is also necessary to use a computer in order to determine the possible periods $n \leq N$ (for a, in general, big bound N, for example $N \approx 10^7$) such that there exists an integer q, coprime with n, such that q/n is in a given interval I (in [9] an alternative result is used, but it also requires a computer).

Moreover other arithmetic results may be useful for the study of periodic rational solutions (when the parameters are rational): for instance, in example (5) of the biquadratic system of [5] it is necessary to know under which condition an integer n can be written as $n = p^2 + 3q^2$ (p, q in $\mathbb{N}$).

G. Results on chaotic behavior. In all the cases of order 2 difference equations or of systems of two order 1 difference equations that we studied in this geometric approach, some globally chaotic behavior on every compact set not containing the fixed point can be proved with the following result.

Proposition 6.3. *Consider $0 < K_1 < K_2$, and let $T : [K_1, K_2] \times \mathbb{T} \to \mathbb{R}^2$ be a continuous and injective map. Set $H := \mathrm{Im}(T)$. Let $\theta : [K_1, K_2] \to \mathbb{T}$ be continuous and nonconstant on every nonempty open interval. Define $F : [K_1, K_2] \times \mathbb{T} \to [K_1, K_2] \times \mathbb{T} : (k, \phi) \mapsto (k, \phi + \theta(k))$. Then the map $\widetilde{F} := T \circ F \circ T^{-1} : H \to H$ is* globally chaotic*, in the sense that there exists $c > 0$ such that for every $M \in H$ and for every neighborhood V of M there exists $M' \in V$ such that $||\widetilde{F}^n(M) - \widetilde{F}^n(M')|| \geq c$ for infinitely many n.*

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