Proceedings of the International Workshop
Future Directions in Difference Equations.
June 13-17, 2011, Vigo, Spain.
PAGES 157–161

A limit boundary value problem for nonlinear difference Equations

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Abstract

We give an asymptotic description of the monotone increasing solutions of a limit boundary value problem for a class of nonlinear difference equations with continuous arguments.

1 Introduction

In [1] Mallet-Paret studied the existence and some properties of the monotone increasing (nondecreasing) solutions $x : \mathbb{R} \to \mathbb{R}$ of the limit boundary value problem (LBVP for short)

$$-c\,x'(\xi) = F(x(\xi + r_1), x(\xi + r_2), \ldots, x(\xi + r_N)) \tag{1}$$

$$\lim_{\xi\to-\infty} x(\xi) = -1, \qquad \lim_{\xi\to\infty} x(\xi) = 1, \tag{2}$$

where $c \in \mathbb{R}$ is a parameter and the shifts $r_j \in \mathbb{R}$, $1 \le j \le N$, and the nonlinearity $F : \mathbb{R}^N \to \mathbb{R}$ satisfy the following standing assumptions:

(i) $N \ge 2$, $r_1 = 0$ and $r_j \ne r_k$ whenever $1 \le j < k \le N$,

(ii) $F : \mathbb{R}^N \to \mathbb{R}$ is a continuously differentiable function such that the partial derivatives D_jF, $1 \le j \le N$, are locally Lipschitz continuous,

(iii) $D_jF(u) > 0$ whenever $u \in \mathbb{R}^N$ and $2 \le j \le N$,

(iv) there exists $q \in (-1, 1)$ such that the function $\Phi : \mathbb{R} \to \mathbb{R}$ defined by

$$\Phi(x) = F(x, x, \ldots, x), \qquad x \in \mathbb{R}, \tag{3}$$

satisfies the following conditions:

- $\Phi(x) > 0$ for $x \in (-\infty, -1) \cup (q, 1)$,
- $\Phi(x) < 0$ for $x \in (-1, q) \cup (1, \infty)$,
- $\Phi(-1) = \Phi(q) = \Phi(1) = 0$,

(v) for the function Φ defined by (3), we have

$$\Phi'(-1) < 0, \qquad \Phi'(q) > 0, \qquad \text{and} \qquad \Phi'(1) < 0.$$

Note that assumption (iv) implies that the Eq. (1) has exactly three equilibria, $x = -1$, $x = q$ and $x = 1$. If $c \neq 0$ then Eq (1) is a functional differential equation of mixed type (including both delayed and advanced arguments), while in the case when $c = 0$ Eq. (1) reduces to a difference equation.

Under the above hypotheses, Mallet-Paret proved the existence of $c \in \mathbb{R}$ and a monotone increasing solution $x : \mathbb{R} \to \mathbb{R}$ of LBVP (1)-(2), and described the asymptotic behavior of $x(\xi)$ as $\xi \to \pm\infty$ as follows:

Theorem 1.1. ([1, Theorem 2.2]). *If $c \neq 0$ and $x : \mathbb{R} \to \mathbb{R}$ is a monotone increasing solution of LBVP (1)-(2) then there exist $C_\pm > 0$ and $\epsilon > 0$ such that*

$$x(\xi) = \begin{cases} -1 + C_- e^{\lambda_-^u \xi} + O(e^{(\lambda_-^u + \epsilon)\xi}), & \xi \to -\infty, \\ 1 - C_+ e^{\lambda_+^s \xi} + O(e^{(\lambda_+^s - \epsilon)\xi}), & \xi \to \infty, \end{cases}$$

where $\lambda_-^u \in (0, \infty)$ is the unique positive eigenvalue of the linearization of Eq. (1) about the equilibrium $x = -1$, and $\lambda_+^s \in (-\infty, 0)$ is the unique negative eigenvalue of the linearization of Eq. (1) about the equilibrium $x = 1$.

If $c = 0$ and $x : \mathbb{R} \to \mathbb{R}$ is a monotone increasing solution of LBVP (1)-(2) then

$$\lim_{\xi \to -\infty} \frac{1}{\xi} \log(1 + x(\xi)) = \lambda_-^u, \qquad \textit{if } r_{\max} > 0,$$
$$\lim_{\xi \to \infty} \frac{1}{\xi} \log(1 - x(\xi)) = \lambda_+^s, \qquad \textit{if } r_{\min} < 0,$$

where $r_{\max} = \max_{1 \le j \le N} r_j$, $r_{\min} = \min_{1 \le j \le N} r_j$ and λ_-^u, λ_+^s have the same meaning as before.

Clearly, if $c = 0$ (the case of difference equations) the asymptotic formulas for $x(\xi)$ are not as sharp as in the the case when $c \neq 0$. Our aim is to show that in the case $c = 0$ the asymptotic formulas for the monotone increasing solutions of LBVP (1)-(2) can be improved.

2 Main Results

If $c = 0$ then there is an important difference between the cases of rationally related shifts and rationally non-related shifts.

Definition 2.1. We say that the shifts r_j, $1 \le j \le N$, are *rationally related* if all ratios

$$\frac{r_j}{r_k}, \qquad 1 \le j < k \le N,$$

are rational. In this case there exists $\nu > 0$ such that all shifts r_j, $1 \le j \le N$, are integer multiples of ν, i.e.,

$$r_j = n_j \nu \qquad \text{for some } n_j \in \mathbb{Z}, \quad 1 \le j \le N.$$

Our main results are the following two theorems dealing separately with the cases of rationally related and rationally non-related shifts.

Theorem 2.1. *Suppose that $c = 0$ and the shifts r_j, $1 \leq j \leq N$, are rationally related. If $x : \mathbb{R} \to \mathbb{R}$ is a monotone increasing solution of LBVP (1)-(2) then for some $\epsilon > 0$ we have*

$$x(\xi) = \begin{cases} -1 + C_-(\xi)e^{\lambda_-^u \xi} + O(e^{(\lambda_-^u + \epsilon)\xi}), & \xi \to -\infty, \text{ if } r_{\max} > 0, \\ 1 - C_+(\xi)e^{\lambda_+^s \xi} + O(e^{(\lambda_+^s - \epsilon)\xi}), & \xi \to \infty, \quad \text{if } r_{\min} < 0, \end{cases}$$

where the functions $C_\pm : \mathbb{R} \to \mathbb{R}$ are ν-periodic (with ν as in Definition 2.1), uniformly positive and bounded, that is, for some $\alpha > 0$ and $\beta > 0$, we have

$$\alpha \leq C_\pm(\xi) \leq \beta, \qquad \xi \in \mathbb{R}.$$

Theorem 2.2. *Suppose that $c = 0$ and the shifts r_j, $1 \leq j \leq N$, are not rationally related. If $x : \mathbb{R} \to \mathbb{R}$ is a monotone increasing solution of LBVP (1)-(2) then there exist constants $C_\pm > 0$ such that*

$$x(\xi) = \begin{cases} -1 + C_- e^{\lambda_-^u \xi} + o(e^{\lambda_-^u \xi}), & \xi \to -\infty, \quad \text{if } r_{\max} > 0, \\ 1 - C_+ e^{\lambda_+^s \xi} + o(e^{\lambda_+^s \xi}), & \xi \to \infty, \quad \text{if } r_{\min} < 0. \end{cases}$$

In Theorems 2.1 and 2.2, the symbols λ_-^u and λ_+^s have the same meaning as in Theorem 1.1.

3 Sketch of the Proof

We will restrict ourselves to the case when $\xi \to \infty$. Suppose $c = 0$ and $x : \mathbb{R} \to \mathbb{R}$ is a monotone increasing solution of LBVP (1)-(2). Define

$$y(\xi) = 1 - x(\xi), \qquad \xi \in \mathbb{R}.$$

Then $y : \mathbb{R} \to [0, \infty)$ is a nonnegative, monotone decreasing solution of the linear equation

$$\sum_{j=1}^{N} A_j(\xi) y(\xi + r_j) = 0, \qquad \xi \in \mathbb{R},$$

where

$$A_j(\xi) = \int_0^1 D_j F(t\pi(x, \xi) + (1 - t)\kappa(1))\, dt, \quad 1 \leq j \leq N,$$

with

$$\pi(x, \xi) = (x(\xi + r_1), x(\xi + r_2), \ldots, x(\xi + r_N)) \in \mathbb{R}^N$$

and

$$\kappa(1) = (1, 1, \ldots, 1) \in \mathbb{R}^N.$$

The assumptions on F imply that the coefficients $A_j(\xi)$, $1 \leq j \leq N$, have the properties:

(P_1) There exist constants

$$\begin{aligned} \alpha_j,\ \beta_j \in \mathbb{R}, \qquad & 1 \leq j \leq N, \\ \alpha_j > 0, \qquad & 2 \leq j \leq N, \end{aligned}$$

such that

$$\alpha_j \le A_j(\xi) \le \beta_j, \qquad \xi \in \mathbb{R}, \quad 1 \le j \le N.$$

(P_2) The limits

$$A_{j,0} = \lim_{\xi \to \infty} A_j(\xi), \qquad 1 \le j \le N,$$

exist (in $\mathbb{R}$), and the convergence is exponentially fast, that is, for some $k > 0$, we have

$$A_j(\xi) = A_{j,0} + O(e^{-k\xi}), \qquad \xi \to \infty, \quad 1 \le j \le N.$$

(P_3) $A_{\Sigma+} = \sum_{j=1}^{N} A_{j,0} < 0$.

The desired asymptotic formulas for $\xi \to \infty$ follow from the following propositions.

Proposition 3.1. *Suppose that the coefficients $A_j : \mathbb{R} \to \mathbb{R}$, $1 \le j \le N$, of the linear equation*

$$\sum_{j=1}^{N} A_j(\xi) y(\xi + r_j) = 0 \tag{4}$$

are continuous and have properties (P_1)–(P_3). *Assume also that the shifts r_j, $1 \le j \le N$, are rationally related. If $y : \mathbb{R} \to (0, \infty)$ is a positive, monotone decreasing solution of (4) then for some $\epsilon > 0$, we have*

$$y(\xi) = C_+(\xi) e^{\lambda^s \xi} + O(e^{(\lambda^s - \epsilon)\xi}), \qquad \xi \to \infty,$$

where λ^s is the unique negative eigenvalue of the limiting equation

$$\sum_{j=1}^{N} A_{j,0}\, y(\xi + r_j) = 0, \tag{5}$$

and the function $C_+ : \mathbb{R} \to \mathbb{R}$ is ν-periodic (with ν as in Definition 2.1), uniformly positive and bounded.

Proposition 3.2. *Suppose that the coefficients $A_j : \mathbb{R} \to \mathbb{R}$, $1 \le j \le N$, of Eq. (4) are continuous and have properties* (P_1)–(P_3). *Assume also that the shifts r_j, $1 \le j \le N$, are not rationally related. If $y : \mathbb{R} \to (0, \infty)$ is a positive, monotone decreasing solution of (4) then there exists $C_+ > 0$ such that*

$$y(\xi) = C_+ e^{\lambda^s \xi} + o(e^{\lambda^s \xi}), \qquad \xi \to \infty,$$

where $\lambda^s \in (-\infty, 0)$ is the unique negative eigenvalue of the limiting equation (5).

The proofs of Propositions 3.1 and 3.2 will be given in our forthcoming paper. We only remark that if the shifts are rationally related then the linear difference equation (4) can be treated as a higher order linear recurrence equation. In fact, the proof of Proposition 3.1 uses only the Residue Theorem. The case of rationally non-related shifts is more difficult. The difficulty arises from the fact that the eigenvalues of the limiting constant coefficient equation (5) may be dense in a nontrivial interval. The proof of Proposition 3.2 will be based on a Tauberian theorem for the Laplace transform due to Ikehara [2], Chap. V, Theorem 17.

References

[1] J. Mallet-Paret, *The global structure of traveling waves in spatially discrete dynamical systems,* J. Dynam. Differential Equations **11** (1999), 49–127.

[2] D. V. Widder, *The Laplace Transform*, Princeton Mathematical Series v. 6, Princeton University Press, Princeton, 1946. (Second Printing)