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Opinion dynamics – local and global

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Abstract

The paper presents analytical results on discrete dynamical systems modeling the formation of opinions for the case of local as well as global interaction among the agents. For the former, opinion dynamics under bounded confidence is considered and, for the latter, opinion dynamics by compromising. Beside some fundamental theorems the paper exhibits examples, open questions and links to the literature.

1 Introduction

Opinion dynamics is about the formation of opinions in small or large groups of interacting individuals or other decision units, called agents for short. Opinions could be assessments made by the agents of certain magnitudes as, for example, prices of goods or probabilities of events, in which case opinions can be represented by nonnegative real numbers. In more complex cases opinions might be better modeled by vectors or more general mathematical objects. The main focus in what follows will be on conditions which lead to a consensus among the agents, meaning the opinions of all the agents converge (asymptotically or in finite time) to a common value. This quest for consensus depends very much on whether the structure of interaction among the agents is a local or global one.

A local structure means that in forming his opinion an agent takes into account only his "nearest neighbours" in some specified sense. Local structures will be dealt with in Section 1 where confidence of agents to others is bounded and "nearest neighbours" are those an agent is confident in.

A global structure means that an agent takes into account the opinions of potentially all the other agents. Global structures are the topic of Section 2, where an agent makes up his opinion for the next period chosing a compromise with potentially any other agent. A particular case are compromises based on a mean, either a concrete mean as the arithmetic or geometric mean or a mean in a more abstract sense.

Opinion dynamics in the above sense is a rather new field with many fascinating questions which, however, are often difficult to solve analytically or are still open. In both cases computer simulations have proven to be very useful to generate conjectures or to illustrate difficult cases. Just to pick an example, an open question is the one for the dependence of the consensus (provided it exists) on the initial opinions of the agents. Beside the linear and some other particular cases this question seems to be a difficult one for global interaction (see Section 2) and an even harder one in case of local interaction.

The present paper gives a short survey of some fundamental analytical results in opinion dynamics from the point of view of local and global interaction. Concerning proofs the reader is referred to the corresponding literature at the end of the paper. In addition, for the interested reader references are supplied concerning certain topics mentioned in passing.

2 Opinion dynamics under bounded confidence

Consider n agents $i = 1, 2 \ldots, n$ and let $x^i(t) \in \mathbb{R}_+$ be the opinion of agent i at time t, where $t = 0, 1, 2, \ldots$. In making up his opinion in the next period, agent i takes into account from the previous period the opinions of those agents he has confidence in. More precisely, at a certain opinion profile $x = (x^1, \ldots, x^n)^T \in \mathbb{R}_+^n$ the *confidence set* of agent i is given by

$$I(i, x) = \{1 \leq j \leq n \mid |x^i - x^j| \leq \epsilon_i\}$$

where $\epsilon_i > 0$ is a certain confidence level of agent i. The *dynamics of opinion formation* then is given for $t = 0, 1, 2, \ldots$ and $i = 1, 2, \ldots, n$ by

$$x^i(t+1) = f_i(x^1(t), \ldots, x^n(t)) \text{ with } f_i(x) = |I(i, x)|^{-1} \sum_{j \in I(i,x)} x^j \tag{1}$$

and given initial opinions $x(0) \in \mathbb{R}_+^n$.

To analyze a convergence towards a consensus among the agents, the following concept plays a major role. A *chain of confidence* of agent i to agent j from period s to period $t > s$ is a sequence of agents $(i_0, i_1, \ldots, i_{t-s})$ such that $i_0 = i, i_{t-s} = j$ and $i_k \in I(i_{k-1}, x(s+k))$ for all $1 \leq k \leq t - s$.

The following result has been proven in different settings by various authors [5, 8, 10, 14].

Theorem 2.1. *For the model (1) of opinion dynamics under bounded confidence with $\epsilon_i = \epsilon$ for all $1 \leq i \leq n$ the following statements do hold.*

(i) There is opinion fragmentation in finite time, that is $\{1, \ldots, n\}$ is a disjunct union of nonempty subsets $A_j, 1 \leq j \leq k$ such that for some $T \in \mathbb{N}$

$$x^i(t) = c_j \textit{ for all } i \in A_j \textit{ and } t \geq T,$$

where c_j is a partial consensus of the agents in A_j which depends on the initial condition $x(0)$.

(ii) If for any two agents i and j there exists a third agent k such that confidence chains exist of i to k and j to k from s to $s + h$ for some fixed $h \geq 1$ and all s, then consensus will be reached in finite time, that is for some $T \in \mathbb{N}$

$$x^i(t) = c \textit{ for all } 1 \leq i \leq n \textit{ and } t \geq T,$$

where the overall consensus c depends on initial condition $x(0)$.

An immediate consequence of this result is the following statement.

Corollary 2.2. *Consensus will be reached in finite time if the principle of the third agent holds, that is for all* $i, j, t \geq T$

$$I(i, x(t)) \cap I(j, x(t)) \neq \emptyset.$$

Concerning the proof of Theorem 2.1, the nonlinear system given by (1) can be transformed into a linear but non-autonomous system as follows. For i, j and t given define $a_{ij} = |I(i, x(t))|^{-1}$ if $j \in I(i, x(t))$ and $a_{ij}(t) = 0$ if $j \notin I(i, x(t))$. The matrix $A(t) = (a_{ij}(t))_{1 \leq i,j \leq n}$ is a nonnegative matrix with row–sums all equal to 1. Furthermore, $x(t+1) = A(t)A(t-1)\ldots A(1)A(0)x(0)$ and the dynamics can be investigated using tools for heterogeneous products of row–stochastic matrices. (For the details of the proof see the references already mentioned as well as [11] for a more general setting.)

The first extensive study of model (1) was made in [8] following [10], which also outlines the historical background and illustrates the dynamics of the model by computer simulations. A different but related model has been analyzed in [3]. For a survey on opinion dynamics under bounded confidence which also compares the models in [3] and [8], see [15]. In a wider context, algorithmic aspects of model (1) and bounds for the rate of convergence have been analyzed in [1] and [2].

In model (1) the aggregation of opinions given by f_i is assumed to be an arithmetic mean over the confidence set of agent i. It is desirable, of course, to admit more general aggregation rules, as, for example, in the following model. Let agents and their confidence sets as before and assume for $t = 0, 1, 2\ldots$ and $i = 1, \ldots, n$

$x^i(t+1) = f_i(x^1(t), \ldots, x^n(t))$ where $f_i\colon \mathbb{R}^n_+ \longrightarrow \mathbb{R}_+$ is any mapping with

$$\min\{x^j \mid j \in I(i, x)\} \leq f_i(x) \leq \max\{x^j \mid j \in I(i, x)\} \text{ for all } x \in \mathbb{R}^n_+ \tag{2}$$

with equality only if the x^j for $j \in I(i, x)$ are all equal. In [9] the following result is proved.

Theorem 2.3. *For the model (2) with* $\epsilon_i = \epsilon$ *for all* $1 \leq i \leq n$ *the following statements do hold.*

(i) There is opinion fragmentation, that is

$$\lim_{t\to\infty} x^i(t) = c_j \text{ for all } i \in A_j,$$

where the nonempty sets A_j *form a disjoint union of* $\{1, \ldots, n\}$ *and* c_j *depends on initial condition* $x(0)$.

(ii) There exists a consensus brink $\epsilon^* = \epsilon^*(f, x(0))$ *such that consensus is approached, that is* $\lim_{t\to\infty} x^i(t) = c$ *for all i, if and only if* $\epsilon \geq \epsilon^*$ *(the consensus being dependent on* $x(0)$*).*

Model (1), of course, is a special case of model (2). From Theorem 2.1 we know that in this special case fragmentation and consensus happen to occur in finite time (if consensus occurs at all). This, however, is not longer true for model (2). Even in case of weighted arithmetic means it may happen that consensus is asymptotically reached but not in finite time.

There are a lot of open questions considering the dynamics of models (1) and (2), respectively. Among others the *open questions*:

(a) What can be proven for $n \geq 3$ if not all ϵ_i are equal? (The case $n = 2$ is simple.)

(b) How can the consensus brink ϵ^* determined in dependence of rule f and/or initial condition $x(0)$?

3 Opinion dynamics by compromising

The models (1) and (2) of the previous section are local in that agents take into account only other agents from a neighborhood given by the confidence set for a certain parameter $\epsilon > 0$. The models become global ones if one sets $\epsilon = \infty$. Model (1) is not very interesting in that respect since $f_i(x) = \frac{1}{n}\sum_{j=1}^{n} x^j$ for all i and, hence, consensus is obtained for $t = 1$. For model (2), however, for $\epsilon = \infty$ one obtains the interesting condition

$$\min_{1\le j\le n} x^j \le f_i(x) \le \max_{1\le j\le n} x^j \tag{3}$$

with equality only if the x^j are all equal. This condition is, for example, satisfied for means like the weighted arithmetic mean $\sum_{j=1}^{n} p_j x_j$, the geometric mean $\prod_{j=1}^{n} x_j^{p_j}$ or the power mean $\left(\sum_{j=1}^{n} p_j x_j^p\right)^{\frac{1}{p}}$ for $p \neq 0$, whereby the weights p_j are positive with $\sum_{j=1}^{n} p_j = 1$. Indeed, condition (3) captures the essence of the meaning of a mean. Alternatively, condition (3) can be considered a rule of compromising on the side of agent i, namely to act between the extremes. This idea can be generalized to more involved opinions, modeled by vectors. More precisely, let S be a non–empty convex subset of $\mathbb{R}^d$ and define a self–mapping $f = (f_1, \ldots, f_n)$ of S^n to be a *compromise map* on S for n agents if

$$\operatorname{conv}\{f_1(x), \ldots, f_n(x)\} \subseteq \operatorname{conv}\{x^1, \ldots, x^n\} \tag{4}$$

holds for all $x = (x^1, \ldots, x^n) \in S^n$ with equality only if the x^j are all equal. Thus, making a compromise means for an agent to pick a point in $\operatorname{conv}\{x^1, \ldots, x^n\}$, the convex hull of the opinions of all agents, in such a way that in case of non–consensus the convex hull of the new opinions of all agents is strictly smaller.

The next result is proved in [12].

Theorem 3.1. *For a continuous compromise map $f\colon S^n \longrightarrow S^n$ the discrete dynamical system given by*

$$x^i(t+1) = f_i(x^1(t), \ldots, x^n(t)), 1 \le i \le n, t = 0, 1, 2, \ldots$$

converges for each $x(0) \in S^n$ to a consensus, that is $\lim_{t\to\infty} x^i(t) = c$ for all i, where c depends on $x(0)$.

Since in $\mathbb{R}^d$ for $d = 1$ convex sets are simply intervals, a compromise map in this case is characterized by the earlier condition (3). For this case Theorem 3.1 yields the following result.

Corollary 3.2. *Let $f\colon \mathbb{R}_+^n \longrightarrow \mathbb{R}_+^n, f(x) = (f_1(x), \ldots, f_n(x))$ be continuous with property (3) and such that $f_i(x) = \min_k x^k$ and $f_j(x) = \max_k x^k$ for two indices $i \neq j$ implies that all components of x are equal. Then $\lim_{t\to\infty} x^i(t) = c$ for all i, where c depends on $x(0) > 0$.*

An example of a compromise map as in Corollary 3.2 is given by a so called Gauss soup, the components of which are either a weighted arithmetic mean or a weighted geometric mean. More precisely, for a given row–stochastic matrix $P = (p_{ij})_{1\le i,j\le n}$, a self-mapping f of $G = \{x \in \mathbb{R}^n_+ \mid x_i > 0 \text{ for } 1 \le i \le n\}$ is called a **Gauss soup** if for each $1 \le i \le n$

$$f_i(x) = \sum_{j=1}^{n} p_{ij}x^j \text{ or } f_i(x) = \prod_{j=1}^{n} (x^j)^{p_{ij}}.$$

As already seen those maps possess property (3) and it is easily seen that the additional conditions in Corollary 3.2 are satisfied if P is scrambled, that is, given any two rows i and j there exists a column k such that $p_{ik} > 0$ and $p_{jk} > 0$. Another application of Corollary 3.2 is obtained if the arithmetic mean in the Gauss soup is replaced by a power mean. (A different proof for convergence in this case is given in [6] using stronger assumptions.) Particular cases of a Gauss soup are the ones where each component is a weighted arithmetic mean or where each component is a geometric mean. In the former case $f(x) = Px, f$ is a Markov chain given by P, and the convergence to a consensus in Corollary 3.2 is well–known as the "Basic Limit Theorem for Markov Chains". (Actually, the Markov chain is given by the transposed matrix P^T; usually P^T is assumed to be primitive, which is a stronger assumption than assuming P to be scrambling.) Furthermore, in this case the value of consensus c can be easily computed in dependence of the initial value. From $x^i(t) = (P^t x(0))_i$ we have that $\lim_{t\to\infty} (P^t x(0)_i) = c$ for all i. Since P is scrambled, P^T has a unique eigenvector $\bar{x} > 0$ with $\sum_{i=1}^{n} \bar{x}_i = 1$ for the eigenvalue 1. It follows that $c = c(x(0)) = \sum_{i=1}^{n} x^i(0)\bar{x}^i$. Thus, the consensus is given by a weighted arithmetic mean of the initial opinions with weights given by $\bar{x}$.

Concerning the other extreme of a Gauss soup where each component is given by a geometric mean one finds similarly for the consensus $c = c(x(0)) = \sum_{i=1}^{n} (x^i(0))^{\bar{x}^i}$. That is, the consensus is a weighted geometric mean, with weights given by $\bar{x}$.

Now, the gist of a Gauss soup is that arithmetic and geometric means are mixed. The simplest case of a mixture is for $n = 2$, $f_1(x_1, x_2) = \frac{x_1+x_2}{2}$ and $f_2(x_1, x_2) = \sqrt{x_1 x_2}$. In this very special case $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ and Corollary 3.2 implies $\lim_{t\to\infty} x^i(t) = c$ for $i = 1, 2$. How then depends the consensus on initial conditions $x(0)$?

The answer is quite surprising and was found by Gauss in 1799, namely

$$c(x(0)) = \frac{\pi}{2} \left[\int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{x_1(0)^2 \cos^2\varphi + x_2(0)^2 \sin^2\varphi}} \right]^{-1}.$$

Thus, the consensus is neither an arithmetic nor a geometric mean in terms of initial conditions — but something completely different, an elliptic integral. As this indicates, the dependence of the consensus on initial conditions is quite an involved question. It is a difficult and completely open problem how for a Gauss soup the consensus, which exists for P scrambled, does depend on initial conditions. Some interesting variants of the arithmetic–geometric mean are analyzed in [7] by means of a first integral for discrete systems, that is a mapping H such that $H(f(x)) = H(x)$ on the domain of definition. If H is a continuous first integral for a continuous compromise map f then it follows from Theorem 3.1 that $H(x) = H(c(x)e)$ (e the vector $(1, \ldots, 1)^T$). Thus, $c(x)$

can be computed from a first integral. Conversely, in case of Theorem 3.1, the consensus yields a first integral since $c(f(x)) = c(x)$. For example, for the linear map $f(x) = Px$ discussed above, a first integral is given by $H(x) = c(x) = \sum_{i=1}^{n} x^i \bar{x}^i$. (In [7] the linear case is dealt with quite differently.) Similarly, for the Gauss soup consisting of geometric means only, as discussed above, $H(x) = c(x) = \prod_{i=1}^{n} (x^i)^{\bar{x}^i}$ is a first integral. For the variant of the arithmetic–geometric mean given by

$$f_1(x_1, x_2) = \sqrt{x_1 \frac{x_1 + x_2}{2}}, f_2(x_1, x_2) = \sqrt{x_2 \frac{x_1 + x_2}{2}}$$

a first integral is in [7] constructed as $H(x_1, x_2) = \frac{x_2^2 - x_1^2}{\log x_2 - \log x_1}$. Since this variant satisfies condition (3) from Theorem 3.1 it follows that $H(x) = H(c(x)e)$ and, hence, $c(x) = \sqrt{\frac{1}{2} \frac{x_2^2 - x_1^2}{\log x_2 - \log x_1}}$. After Gauss' discovery in 1799 mathematicians, in particular Borchardt, have found for some variants of the arithmetic–geometric mean formulas for what is called a consensus in this paper. It seems, however, that till today formulas for the function $c(x)$ have not been found for a comprehensive class of variants, much less for a general Gauss soup — not to mention compromise maps.

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