Proceedings of the International Workshop
Future Directions in Difference Equations.
June 13-17, 2011, Vigo, Spain.
PAGES 107–112

How to find invariant hypersufaces for birational maps

Kyounghee Kim

Florida State University, USA

kim@math.fsu.edu

Abstract

Let $f(z)$, $z = (z_1, \dots, z_d)$, be a rational map from $\mathbf{C}^d$ to itself. We describe a method for finding polynomials $p(z)$ which have the invariance property $p(f) = t \cdot p(z) \cdot j_f$, where t is a nonzero constant, and j_f is the Jacobian determinant of f. In particular, if $p(z)$ and $q(z)$ are two such invariant polynomials, then $r(z) = p(z)/q(z)$ is an *integral* of f in the sense that $r(f(z)) = r(z)$ for almost every z.

1 Birational maps

A mapping $f = (f_1, \dots, f_d) : \mathbb{C}^d \longrightarrow \mathbb{C}^d$ is said to be rational if each coordinate function is rational, that is, $f_i = p_i/q_i$ is a quotient of polynomials. A rational map f of $\mathbb{C}^d$ induces a map $\hat{f} = [\hat{f}_0 : \hat{f}_1 : \cdots : \hat{f}_d]$ of $\mathbb{P}^d$ through the identification $[1 : x_1 : \cdots : x_d] \leftrightarrow (x_1, \dots, x_d)$. By dividing out common factors, a rational map $\hat{f}$ can be written in the form where $\hat{f}_i$'s are homogenous polynomials of lowest possible degree. A rational map $f : \mathbb{P}^d \longrightarrow \mathbb{P}^d$ is said to be *birational* if there exists another rational map $g : \mathbb{P}^d \longrightarrow \mathbb{P}^d$ and an algebraic variety V such that $f \circ g = g \circ f = \mathrm{Id}$ on $\mathbb{P}^d \setminus V$.

Let $f = [f_0 : f_1 : \cdots : f_d] : \mathbb{P}^d \longrightarrow \mathbb{P}^d$ be birational. The *indeterminacy locus* is the set

$$\mathcal{I}(f) = \{x \in \mathbb{P}^d : f_0(x) = \cdots = f_d(x) = 0\},$$

and the birational map f defines a homolorphic map on $\mathbb{P}^d \setminus \mathcal{I}(f)$ to $\mathbb{P}^d$. For $a \in \mathbb{P}^d$, let

$$Cl_f(a) = \overline{\lim_{a' \to a} f(a')},$$

be the set of all limits of $f(a')$ for $a' \in \mathbb{P}^d \setminus \mathcal{I}(f)$ as $a' \to a$. The cluster set $Cl_f(a)$ contains more than one point exactly when $a \in \mathcal{I}$. In this case, the cluster set is a connected variety of dimension at least one. We say that an algebraic variety S is invariant if S is not contained in $\mathcal{I}(f)$ and if the closure of $f(S - \mathcal{I}(f))$ is equal to S. We define j_f to be the determinant of the $(d+1) \times (d+1)$ matrix $\left(\partial f_i / \partial x_j\right)_{0 \le i,j \le d}$. An irreducible subvariety V is called *exceptional* if $V - \mathcal{I}(f) \neq \emptyset$ and if $\dim f(V - \mathcal{I}(f)) < \dim V$. The *exceptional locus*, $\mathcal{E}(f)$, is the union of all irreducible exceptional varieties. It is evident that the exceptional locus is where the jacobian of f vanishes.

For a homogeneous polynomial h we consider the condition that there exists $t \in \mathbb{C}^*$ such that

$$h \circ f = t \cdot j_f \cdot h. \tag{1}$$

If $h_1, \dots, h_m$ satisfy (1), then their linear combination also satisfies (1) and the quotient of any two of these polynomials will give an invariant function ϕ in the sense that $\phi = \phi \circ f$. If there exists such a rational function ϕ, we say that the corresponding dynamical system is *integrable*. Each map on dimension d can have at most $d-1$ integrals. If a mapping f has $d-1$ integrals, we say that f is *completely integrable*. If f has an integral, we say that f is *integrable*. In case the dimension $d = 2$, f can preserve either a rational fibration if $\deg(f^n)$ grows linearly or an elliptic fibration if $\deg(f^n)$ grows quadratically. (See [10] and [12].) If $\deg(f^n)$ grows exponentially, the surface map can not preserve a fibration. For a surface automorphism (the case $d = 2$), exponential degree growth is equivalent to f having positive entropy. In [2] it is shown that certain surface automorphisms with positive entropy have an invariant cubic satisfying (1). In [9] and [16], it was shown that invariant curves have played a useful starting place in the construction of surface automorphisms with positive entropy. In the case of [9], the invariant curve is useful because it simplifies the problem of tracking the orbit of the exceptional curve and making it land on the point of indeterminacy. However, as was shown in [2], invariant curves do not always exist.

Here we also consider the case $d \geq 3$. Several authors have considered the Lyness map. (See for example [6], [7], [8], [13],[14], and [15]). In [4] it was shown that these maps have quadratic degree growth. In fact, the Lyness process is a special case of the linear fractional recurrences

$$z_{d+1} = \frac{a_0 + a_1 z_1 + a_2 z_2 + a_3 z_3 + \cdots + a_d z_d}{b_0 + b_1 z_1 + b_2 z_2 + b_3 z_3 + \cdots + b_d z_d}.$$

Certain of pseudo-automorphism with positive entropy in dimension ≥ 3 can have more than one invariant hypersurfaces satisfying (1) and therefore those maps have an invariant foliation. In contrast with the case $d = 2$, there are certain linear fractional recurrences which have both exponential degree growth and integrals. Thus the level sets $S_c := \{r(z) = c\}$ form an invariant fibration. In [5] we showed that the generic level sets S_c are $K3$ surfaces. There are also a finite number of invariant rational surfaces. Each level set S_c is invariant. From Lemma 1.2 below, we see that S_c contains the set $\mathcal{I}(f) \cup \mathcal{I}(f^{-1}) \cup \mathcal{I}(r)$ and thus all the leaves of this fibration intersect.

Lemma 1.1. *Let $f : \mathbb{P}^d \longrightarrow \mathbb{P}^d$ be a birational map. Suppose that $S = \{h = 0\}$ is an f invariant curve satisfying (1). Then the degree of $h = d + 1$.*

Proof. Suppose the degree of f is equal to k. It follows that the degree of j_f is equal to $(d + 1)(k-1)$ and the degree of $h \circ f$ is equal to $k \cdot (\text{degree of } h)$. If h satisfies (1) then we have $k \cdot (\text{degree of } h) = (d + 1)(k - 1) + (\text{degree of } h)$. □

For each exceptional irreducible hypersurface $V \in \mathcal{E}(f)$, let us set $\mathcal{O}_V := \{V_j = f^j V : f^{j-1} V \notin \mathcal{I}(f), j \geq 1\}$ the orbit of the critical image of V. Since V is exceptional, it follows that $\operatorname{codim} V_j \geq 2$ for all $V_j \in \mathcal{O}_V$.

Lemma 1.2. *Suppose that $S = \{h = 0\}$ is an f invariant curve satisfying (1). Then S contains $\mathcal{O}_V$ for all exceptional hypersurface V. Furthermore if j_f vanishes to order r at V then h will vanish to order at least r at $V_j \in \mathcal{O}_V$.*

Proof. Since V is exceptional, we have for all $x \in V$, $j_f(x) = 0$. Thus for all $x \in V \setminus \mathcal{I}(f)$, $h \circ f(x) = t \cdot j_f(x) \cdot h(x) = 0$. It follows that $V_1 = f(V) \in \{h = 0\}$. Since S is an f invariant, S also contains every $V_j \in \mathcal{O}_V$. □

2 Surface maps

For the purposes of illustrating our method, we work in dimension $d = 2$. Let us consider a family of birational maps $f_{a,b}$ on $\mathbb{P}^2$ of the form

$$f_{a,b} : [x_0 : x_1 : x_2] \mapsto [x_0(bx_0 + x_1) : x_2(bx_0 + x_1) : x_0(ax_0 + x_2)]$$
$$f_{a,b}^{-1} : [x_0 : x_1 : x_2] \mapsto [x_0x_2 : x_0(ax_0 + x_1 - bx_2) : x_1x_2].$$

The exceptional curves for the map f are given by the lines $\Sigma_0 = \{x_0 = 0\}, \Sigma_\beta = \{bx_0+x_1 = 0\}$, and $\Sigma_\alpha = \{ax_0 + x_2 = 0\}$. The indeterminacy locus $\mathcal{I}(f) = \{e_2, e_1, p\}$ consists of the vertices of the triangle $\Sigma_0\Sigma_\alpha\Sigma_\beta$, that is, $e_2 = [0 : 0 : 1], e_1 = [0 : 1 : 0], p = [1, -b, -a]$. Using $f_{a,b}$ and $f_{a,b}^{-1}$ we see that

$$f_{a,b} : \Sigma_0 \mapsto e_1, \Sigma_\beta \mapsto e_2, \Sigma_\alpha \mapsto q = [1 : -a : 0]$$
$$f_{a,b}^{-1} : \{x_2 = 0\} \mapsto e_1, \Sigma_0 \mapsto e_2, \{ax_0 + x_1 = 0\} \mapsto p = [1 : -b : -a].$$

We define the cubic polynomial $j_f := x_0(bx_0 + x_1)(ax_0 + x_2)$, so $\{j_f = 0\}$ is the exceptional locus for f. We define the functions:

$$\varphi_1(t) = \left(\frac{t - t^3 - t^4}{1 + 2t + t^2}, \frac{1 - t^5}{t^2 + t^3}\right), \ \varphi_2(t) = \left(\frac{t + t^2 + t^3}{1 + 2t + t^2}, \frac{-1 + t^3}{t + t^2}\right) \text{ and}$$
$$\varphi_3(t) = \left(1 + t, t - t^{-1}\right)$$

Theorem 2.1. *Let $t \neq 0, \pm 1$ with $t^3 \neq 1$ be given. Then there is a homogeneous cubic polynomial P satisfying (1) if and only if $(a, b) = \varphi_j(t)$ for some $1 \leq j \leq 3$. If this occurs, then (up to a constant multiple) P is given by (2) below.*

Proof. From Lemma 1.2, we know that if P is invariant under $f_{a,b}$ and $f_{a,b}^{-1}$, P must vanish at e_1, e_2, p and $q = f_{a,b}\Sigma_\alpha = [1 : -a : 0]$. Using the conditions $P(e_1) = P(e_2) = P(q) = 0$, we may set

$$P[x_0 : x_1 : x_2] =(-a^2C_1 + aC_2)x_0^3 + C_2x_1x_0^2 + C_3x_2x_0^2 + C_1x_0x_1^2 + C_4x_2x_1^2 + C_5x_0x_2^2 + C_6x_1x_2^2 + C_7x_0x_1x_2$$

for some $C_1, \dots, C_7 \in \mathbb{C}$. Since $e_1, e_2, q \in \{P = 0\}$, we have $P \circ f = j_f \cdot \tilde{P}$ for some cubic $\tilde{P}$. A computation shows that

$$\tilde{P} =(-ab^2C_1 + b^2C_2 + bC_3 + aC_5)x_0^3 + (-2abC_1 + 2bC_2 + C_3)x_1x_0^2 + (bC_1 + C_5 + aC_6 + bC_7)x_2x_0^2 + (-aC_1 + C_2)x_0x_1^2 + C_1x_2x_1^2 + (bC_4 + C_6)x_0x_2^2 + C_4x_1x_2^2 + (2bC_1 + C_7)x_0x_1x_2.$$

Now setting $\tilde{P} = tP$ and comparing coefficients, we get a system of 8 linear equations in $C_1, \dots, C_7$ of the form

$$M \cdot [x_0^3, x_1x_0^2, x_2x_0^2, x_0x_1^2, x_0x_2^2, x_0x_1x_2, x_1x_2^2, x_1^2x_2]^t = 0.$$

We check that there exist cubic polynomials satisfying (1) if and only if the two principal minors of M vanish simultaneously, which means that

$$b(a + abt + abt^4 - b^2t^4 - at^5 + bt^5) = 0$$

$$-1+(1-a-b)t+(a+b)t^2+b^2t^3+b^2t^4+(a-2b)t^5+(1-a+2b)t^6-t^7=0$$

Solving these two equations for a and b, we obtain φ_j, $j=1,2,3$ as the only solutions, and then solving $M=0$ we find that P must have the form:

$$\begin{aligned}P_{t,a,b}(x) =& ax_0^3(-1+t)t^4+x_1x_2(-1+t)t(x_2+x_1t)\\ &+x_0[2bx_1x_2t^3+x_1^2(-1+t)t^3+x_2^2(-1+t)(1+bt)]\\ &+x_0^2(-1+t)t^3[a(x_1+x_2t)+t(x_1+(-2b+t)x_2)].\end{aligned} \tag{2}$$

□

3 Higher Dimension Case

To illustrate the method, let us consider the map known as Lyness process defined by a linear fractional recurrence relation :

$$z_{k+1}=\frac{a+z_2+z_3+\cdots+z_k}{z_1}.$$

For $a\in\mathbb{C}^*$ we define a birational map on $\mathbb{P}^k$:

$$\begin{aligned}f[x_0:\cdots:x_k]&=[x_0x_1:x_2x_1:\cdots:x_kx_1:x_0(ax_0+x_2+\cdots+x_k)]\\ f^{-1}[x_0:\cdots:x_k]&=[x_0x_k:x_0(ax_0+x_1+\cdots+x_{k-1}):x_1x_k:\cdots:x_{k-1}x_k].\end{aligned}$$

The Jacobian of f is a constant multiple of $x_0x_1^{k-1}(ax_0+x_2+\cdots+x_k)$. The Jacobian vanishes on three hypersurfaces $\Sigma_0=\{x_0=0\},\Sigma_1=\{x_1=0\}$, and $\Sigma_a=\{ax_0+x_2+\cdots+x_k=0\}$. These hypersurfaces are exceptional and are mapped to the lower dimensional linear subspaces. Unlike the surface map, some of the images of exceptional hypersurfaces are again in the exceptional locus:

$$\begin{aligned}f\ &:\Sigma_0\mapsto\Sigma_{0k}\mapsto\Sigma_{0\,k-1\,k}\mapsto\cdots\mapsto\Sigma_{03\ldots k}\mapsto e_1=[0:1:0:\cdots:0]\\ &\quad\Sigma_1\mapsto e_k=[0:\cdots:0:1]\\ &\quad\Sigma_a\mapsto\Sigma_k\cap\{ax_0+x_1+\cdots+x_{k-1}=0\}\\ f^{-1}\ &:\Sigma_k\mapsto e_1\\ &\quad\Sigma_0\mapsto\Sigma_{01}\mapsto\Sigma_{012}\mapsto\cdots\mapsto\Sigma_{01\cdots k-2}\mapsto e_k\\ &\quad\{ax_0+x_1+\cdots+x_{k-1}=0\}\mapsto\Sigma_1\cap\{ax_0+x_2+\cdots+x_k=0\}\end{aligned}$$

where $\Sigma_I=\{x_i=0,i\in I\}$. To get an invariant hypersurface, we are looking for homogeneous polynomials P satisfying

$$P\circ f=t\cdot(x_0x_1^{k-1}(ax_0+x_2+\cdots+x_k))\cdot P \tag{3}$$

for some $t\in\mathbb{C}^*$. Since f is a map on $\mathbb{P}^k$, from Lemma 1.1 we look for polynomials P of degree $k+1$. Recall that f maps Σ_β to e_k. Thus by (3) we see that P will vanish to order at least $k-1$ at e_k, since the Jacobian vanishes to order $k-1$ at Σ_1. Similarly, since $f(\Sigma_0)=\Sigma_{0,k}$, we see that P must vanish at $\Sigma_{0,k}$. Now starting with a point $z\in\Sigma_{0,k}$, we have $f(z)\in\Sigma_{0,k-1,k}$, so by (3), P vanishes to order at least 2 on $\Sigma_{0,k-1,k}$. Continuing this way, we see that P vanishes to order at least $k-j$ on $\Sigma_{j+1,j+2,\dots,k}$ for $1 \le j \le k-1$. Finally, since the Jacobian vanishes on Σ_a, and $f(\Sigma_a) = \Sigma_k \cap \{ax_0 + x_1 + \cdots + x_{k-1} = 0\}$, we see that P vanishes on $\Sigma_k \cap \{ax_0 + x_1 + \cdots + x_{k-1} = 0\}$. Iterating this, we see that P must vanish on $f^j(\Sigma_a)$ for $j = 1, \dots, k+1$ since $f^{k+1}\Sigma_a \in \Sigma_1 \cap \{ax_0 + x_2 + \cdots + x_k = 0\}$. This process yields the solutions to (1):

$$\begin{aligned}
P_1 &= x_0 x_1 x_2 \dots x_k \\
P_2 &= (ax_0 + x_1 + x_2 + \cdots + x_k)(x_0 + x_1)(x_0 + x_2) \dots (x_0 + x_k) \\
P_3 &= (x_0(ax_0 + x_1 + x_2 + \cdots + x_k) + x_1 x_k) \times \\
&\quad \times (x_0 + x_1 + x_2)(x_0 + x_2 + x_3) \cdots (x_0 + x_{k-1} + x_k).
\end{aligned}$$

Thus we have rediscovered the invariants which were found earlier in [6], [11], and [14].

References

[1] E. Bedford and KH Kim, *Periodicities in linear fractional recurrences: degree growth of birational surface maps*, Michigan Math. J. **54** (2006), 647–670.

[2] E. Bedford and KH Kim, *Dynamics of rational surface automorphisms: linear fractional recurrences*, J. Geom. Anal. **19** (2009), 553-583.

[3] E. Bedford and KH Kim, *Continuous families of rational surface automorphisms with positive entropy*, Mathematische Annalen **348** (2010), 667–688.

[4] E. Bedford and KH Kim, *Linear fractional recurrences: periodicities and integrability*, Annales de la Faculté des Sciences de Toulouse, to appear

[5] E. Bedford and KH Kim, *Periodicities in 3-step linear fractional recurrences*, preprint arXiv:1101.1614v2.

[6] A. Cima, A. Gasull, V. Mañosa, *Dynamics of some rational discrete dynamical systems via invariants*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **16** (2006), 631–645.

[7] A. Cima, A. Gasull, V. Mañosa, *Dynamics of the third order Lyness' difference equation*, J. Difference Equ. Appl. **13** (2007), 855–884.

[8] A. Cima, A. Gasull, V. Mañosa, *Some properties of the k-dimensional Lyness' map*, J. Phys. A: Math. Theor. **41** (2008), 285205, 18 pp.

[9] J. Diller, *Cremona transformations, surface automorphisms and plane cubics*, preprint arXiv:0811.3038v2

[10] J. Diller and C. Favre, *Dynamics of bimeromorphic maps of surfaces*, Amer. J. Math. **123** (2001), 1135–1169.

[11] M. Gao,Y. Kato and M. Ito, *Some invariants for kth-order Lyness equation.* Appl. Math. Lett. **17** (2004), 1183–1189.

[12] M. Gizatullin, *Rational G-surfaces.* (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **44** (1980), 110–144, 239.

[13] R. Hirota, K. Kimura and H. Yahagi, *How to find the conserved quantities of nonlinear discrete equations*, J. Phys. A: Math. Gen. **34** (2001), 10377–10386.

[14] V.I. Kocic and G. Ladas, *Global Behaviour of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic Publishers, 1993.

[15] V.I. Kocic, G. Ladas, and I.W. Rodrigues, *On rational recursive sequences*, J. Math. Anal. Appl. **173** (1993), 127–157.

[16] C. T. McMullen, *Dynamics on blowups of the projective plane*, Publ. Math. Inst. Hautes Études Sci. **105** (2007), 49–89.