Computers and Mathematics with Applications 64 (2012) 2192–2201

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

On stabilization of equilibria using predictive control with and without pulses

Elena Braverman [a], Eduardo Liz [b,*]

[a] Department of Mathematics, University of Calgary, 2500 University Drive N.W., Calgary, AB, Canada T2N 1N4
[b] Departamento de Matemática Aplicada II, Universidade de Vigo, Campus Marcosende, 36310 Vigo, Spain

ARTICLE INFO

Keywords:
Control of chaos
Discrete population model
Global stability
Pulse stabilization
Periodic difference equations
Bifurcations

ABSTRACT

Consider a chaotic difference equation $x_{n+1} = f(x_n)$. We focus on the problem of control of chaos using a prediction-based control (PBC) method. If f has a unique positive equilibrium, it is proved that global stabilization of this equilibrium can be achieved under mild assumptions on the map f; if f has several positive equilibria, we demonstrate that more than one equilibrium can be stabilized simultaneously. We also show that it is still possible to stabilize an unstable equilibrium using a strategy of control with pulses, that is, the control is only applied after a fixed number of iterations. We illustrate our main results with several examples, mainly from population dynamics.

1. Introduction

Stabilization of chaotic systems is an area of research experiencing a fast growth in the past years [1,2]. There seems to be a general agreement in dividing the methods for control of chaos into two main groups. The first one is inspired by the seminal paper of Ott et al. [3], and it relies on the adjustment of an intrinsic parameter of the system to stabilize one of the infinite number of unstable periodic orbits embedded in a chaotic attractor. Two admitted drawbacks of this kind of techniques are that a previous knowledge of the trajectory of the orbit to stabilize is needed, and that it may take a long time to reach such an orbit (resulting in an undesired large number of chaotic transients).

In this paper, we focus our attention on a control technique belonging to a second group of methods based on external perturbations [4]. Among them, we mention the methods more relevant to our discussion, namely, the *constant feedback method* (CF) [5], the *proportional feedback method* (PF) [6,7], and the *prediction-based control* (PBC) [8–10].

An important remark is that all these methods can be considered as parametric control methods, since they are based on the introduction of a new parameter in the dynamical system, in such a way that perturbations of this parameter can stabilize the chaotic attractor.

For the sake of completeness and future reference in this paper, we give a brief review of these methods for the one-dimensional difference equation

$$x_{n+1} = f(x_n), \tag{1.1}$$

where f is a continuous real function.

Constant feedback method (CF). The constant feedback method of control proposed in [5] involves adding a new real parameter c in the form of a constant feedback; thus system (1.1) becomes

$$x_{n+1} = f(x_n) + c.$$

* Corresponding author.
E-mail addresses: maelena@math.ucalgary.ca (E. Braverman), eliz@dma.uvigo.es (E. Liz).

0898-1221/$ – see front matter © 2012 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2012.01.013

If (1.1) is a model for population dynamics, values of c greater than zero mean that a constant migration enters the population at every generation, while negative values of the control parameter c mean migration or harvesting at a constant rate. For recent results and more references concerning this method, see [11–13].

Proportional feedback method (PF). This method was introduced in [6]. It involves multiplying the state variable by a constant factor $\gamma > 0$ for every p iterations, where p is an integer greater than zero. For the case $p = 1$, the map f becomes $f(\gamma x)$ after the control. The biological interpretation is similar to that of the previous method, since a fraction of the population is removed if $\gamma < 1$, whereas a factor $\gamma > 1$ means that a positive migration is added to the population size at each generation. Recently, a number of analytical results for the stabilization of fixed points and periodic orbits using this control technique have been proved [7,14,15].

Prediction-based control (PBC). The prediction-based control (or predictive control) was introduced by Ushio and Yamamoto [10] in order to overcome some limitations of the so-called delayed feedback control [16]. The general form of this method to stabilize a periodic orbit of (1.1) is written as

$$x_{n+1} = f(x_n) - \alpha(f^k(x_n) - x_n),$$

where f^k is the kth iteration of f. When $k = 1$, this method becomes

$$x_{n+1} = f(x_n) - \alpha(f(x_n) - x_n) = \alpha x_n + (1-\alpha)f(x_n) := F_\alpha(x_n). \tag{1.2}$$

This scheme has proved to be very efficient and robust to stabilize an unstable equilibrium of (1.1), see [8,9,17].

In this paper, we will restrict the range of values of the control parameter α to the interval $[0, 1]$. Under this assumption, it is clear that $F_\alpha(x)$ is a convex combination of x and $f(x)$. An important implication, especially when (1.1) is a model of population dynamics, is the following: assume that Eq. (1.1) is permanent, that is, there exists a compact interval $[a, b]$, with $0 < a < b$, such that all solutions $\{x_n\}$ of (1.1) starting at a positive initial condition x_0 satisfy

$$a \le \liminf_{n\to\infty} x_n \le \limsup_{n\to\infty} x_n \le b.$$

Then, the controlled equation (1.2) is permanent too. This is an important difference with other control methods, such as CF, in which the application of a negative control c can induce an Allee effect or even catastrophe bifurcations, driving the population to extinction [12,13].

We also note that it is impossible to stabilize an unstable positive fixed point K of (1.1), if $f'(K) > 1$ using PBC with a value of $\alpha \in [0, 1)$. Indeed, in this case

$$F'_\alpha(K) = (1-\alpha)f'(K) + \alpha > 1 - \alpha + \alpha = 1.$$

A key point which distinguishes PBC scheme from CF and PF is that the former one stabilizes the equilibria of the original (uncontrolled) system (1.1). Actually, if $\alpha \ne 1$, K is an equilibrium of (1.1) if and only if K is an equilibrium of (1.2).

Our main aim in this paper is to improve the existing analytic results on stabilization of equilibria using the PBC scheme (1.2) in two directions. On the one hand, we prove that global stabilization is possible for a wide class of maps with a unique positive equilibrium, avoiding the restrictions in the main result of [8]. On the other hand, we explore the possibility of pulse stabilization, that is, the control is not applied every iteration, but only after a fixed number of m iterations. This aspect is very important in practical situations, because sometimes either it is not feasible or it is very costly to apply the control at each step. That is, we consider the following strategy of control with pulses:

$$x_{n+1} = \begin{cases} f(x_n), & \text{if } n \ne mk, \\ F_\alpha(x_n), & \text{if } n = mk,\ k \in \mathbb{Z}^+, \end{cases} \tag{1.3}$$

where F_α was defined in (1.2). An important consequence of our results is that it is possible to stabilize an unstable equilibrium of (1.1) using (1.3) with a positive number $m > 1$. As we show in Section 3, this goal is impossible to achieve with CF or PF methods. In the recent paper [14], we proved that an application of the control scheme PF every m steps is able to stabilize globally a periodic orbit of (1.1) with minimal period m.

The paper is organized as follows: in Section 2, we prove a result of global stabilization using (1.2) when Eq. (1.1) has a unique positive equilibrium, generalizing in this way Theorem 1 in [8]. When the map f has several positive equilibria, we show that there is a range of values of α for which any positive solution of (1.2) converges to an equilibrium. In Section 3, we address the problem of pulse stabilization, using scheme (1.3). Finally, Section 4 is devoted to the discussion of our results: we highlight the main conclusions and state some directions for further research.

2. Global stabilization

2.1. A unique positive equilibrium

In this subsection, we consider the case when Eq. (1.1) has a unique positive equilibrium. Our main result establishes sufficient conditions for the global stabilization of the positive equilibrium using the PBC scheme (1.2). First, we list and discuss the assumptions for the map f:

(A1) $f : [0, \infty) \to [0, \infty)$ is continuous, $f(0) = 0$, and $f(x) > 0$ for all $x > 0$.

(A2) f has only two nonnegative fixed points $x = 0$ and $x = K > 0$, $f(x) > x$ for $0 < x < K$ and $f(x) < x$ for $x > K$.
(A3) There exists a number $M > 0$ such that

$$\left|\frac{f(x) - K}{x - K}\right| \leq M \tag{2.1}$$

for any $x > 0$.

Assuming that Eq. (1.1) models the growth of a population, the positive equilibrium K is called the carrying capacity, which is the maximum population level naturally sustained by the environment. Conditions (A1)–(A2) have a clear biological meaning. The first hypothesis implies that the stock size at the next stage depends continuously on the size at the present stage. The second assumption means that the population increases if its present size is less than the carrying capacity, and decreases otherwise. The third assumption is not hard to verify. In particular, if f is continuously differentiable and (A1)–(A2) hold, then (2.1) is satisfied, for example, with

$$M = \max\left\{\max_{x\in[0,2K]} |f'(x)|, 1\right\},$$

since $|f(x) - K| = |f'(\zeta)|\,|x - K|$, where ζ is a point between x and K and $|f(x) - K| \leq |x - K|$ for $x > 2K$ due to the inequality $0 < f(x) < x$, $x > K$.

We need the following simple auxiliary result in the proof of the main result of this section.

Lemma 2.1. *Let $g : [0, \infty) \to [0, \infty)$ be a continuous function satisfying* (A1), (A2), *and let $\lambda \in (0, 1)$ be a number such that for any $x > 0$ either*

$$|g(x) - K| \leq \lambda |x - K| \tag{2.2}$$

or $(g(x) - K)(x - K) > 0$.

Then any solution $\{x_n\}$ of the equation

$$x_{n+1} = g(x_n), \tag{2.3}$$

with $x_0 > 0$, converges to K, that is,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} g^n(x_0) = K. \tag{2.4}$$

Proof. For the sequence $x_{n+1} = g(x_n)$ with $x_0 > 0$, there may be two cases: either there exists $n_0 \geq 0$ such that the sign of $x_n - K$ does not change for $n \geq n_0$, or there is an infinite number of points where $(x_n - K)(x_{n+1} - K) < 0$. In the first case, the sequence is strictly increasing if $x_{n_0} < K$ and strictly decreasing if $x_{n_0} > K$, thus it has a limit $d = \lim_{n\to\infty} x_n$. Taking the limit of both sides in (2.3) and using the continuity of g, we deduce that d is a positive fixed point of g, so $d = K$.

Consider the second case when there is an infinite number of points where $x_n - K$ changes its sign. Let us note that, generally (unless some $x_n = K$) the sequence $y_n = |x_n - K|$ is decreasing: if $x_n - K$ and $x_{n+1} - K$ have the same sign, this follows from (A2); if the signs are different, (2.2) implies $y_{n+1} \leq \lambda y_n$. Since there is an infinite number of points n_k, $k \in \mathbb{N}$, where (2.2) holds, we have

$$|x_{n_k} - K| = y_{n_k} \leq \lambda^k |x_0 - K|.$$

Thus $\lim_{n\to\infty}(x_n - K) = 0$, which completes the proof. □

Now we are in a position to prove our main result on global stabilization.

Theorem 2.2. *If f satisfies* (A1)–(A3), *then there exists $A \in (0, 1)$ such that for any $\alpha \in (A, 1)$, all solutions of* (1.2) *with initial condition $x_0 > 0$ converge to K.*

Proof. Without loss of generality, we can assume that $M \geq 1$, where M is defined in (2.1). Otherwise, by Lemma 2.1, the positive equilibrium attracts all positive solutions of Eq. (1.1). Let us define $A = A(M) = (M - 1)/M$. We claim that the conditions of Lemma 2.1 hold for $g = F_\alpha$ and all $\alpha \in (A, 1)$. It is easy to check that F_α meets (A1) and (A2). Note also that if $0 < x < K$ and $0 < f(x) < K$ then, for all $\alpha \in (0, 1)$,

$$0 < F_\alpha(x) = \alpha x + (1 - \alpha)f(x) < \alpha K + (1 - \alpha)K = K.$$

Thus, $(F_\alpha(x) - K)(x - K) > 0$.

The same argument is valid when $x > K$ and $f(x) > K$. Hence, it is enough to consider the cases $0 < x < K$, $f(x) > K$ and $x > K$, $f(x) < K$.

First, let $0 < x < K$ and $f(x) > K$. Note that $0 < 1-\alpha < 1-(M-1)/M = 1/M$ and, by (2.1), $f(x) - K \le M(K - x)$. Thus,

$$\begin{aligned} F_\alpha(x) - K &= \alpha x + (1-\alpha)f(x) - K = (1-\alpha)(f(x) - K) - \alpha(K - x) \\ &\le \frac{1}{M}(f(x) - K) - \alpha(K - x) \le \frac{1}{M}M(K - x) - \alpha(K - x) \\ &= (1-\alpha)(K - x). \end{aligned}$$

This means that either $F_\alpha(x) < K$ or (2.2) holds with $\lambda = 1 - \alpha$.

Next, the case $x > K$ and $f(x) < K$ is handled in a similar way. In this case, inequality (2.1) writes $K - f(x) \le M(x - K)$, and therefore

$$\begin{aligned} K - F_\alpha(x) &= K - \alpha x - (1-\alpha)f(x) = (1-\alpha)(K - f(x)) - \alpha(x - K) \\ &\le \frac{1}{M}(K - f(x)) - \alpha(x - K) \le \frac{1}{M}M(x - K) - \alpha(x - K) \\ &= (1-\alpha)(x - K). \end{aligned}$$

Thus, either $F_\alpha(x) > K$ or (2.2) holds with $\lambda = 1 - \alpha$. The application of Lemma 2.1 completes the proof. □

Remark 1. If we choose $\tilde{A}(M) = (M - \beta)/M$ instead of $A(M) = (M - 1)/M$ in the proof of Theorem 2.2, then the same conclusions are derived with $\lambda = \beta - \alpha$ instead of $\lambda = 1 - \alpha$ in (2.2). The only restrictions on β are $0 < \beta \le M$ and $\beta - \tilde{A}(M) \le 1$. These conditions are equivalent to $0 < \beta \le 2M/(M+1)$, $M \ge 1$. Thus, our approach shows that, under conditions (A1)–(A3), the positive equilibrium of Eq. (1.1) is globally stabilized with the PBC scheme (1.2) if the strength of the parameter control $\alpha \in (0, 1)$ satisfies the inequality

$$\alpha > \alpha_0(M) = \frac{M - (2M/(M+1))}{M} = \frac{M-1}{M+1}.$$

Remark 2. It is easy to prove that F_α is monotone increasing on $(0, K)$ if f is continuously differentiable on $(0, K)$ and $f'(x) > -\alpha/(1-\alpha)$ for all $x \in (0, K)$, that is, if

$$\alpha > \alpha_1 := \max_{x\in[0,K], f'(x)<0} \left\{ \frac{f'(x)}{f'(x) - 1} \right\}. \tag{2.5}$$

In this case, the convergence of any positive solution of (1.2) to K is eventually monotone. This result can be derived from the proof of Theorem 1 in [8].

Example 2.3. Consider the difference equation

$$x_{n+1} = x_n \left(0.55 + \frac{3.45}{1 + x_n^m} \right), \tag{2.6}$$

which models the growth of bobwhite quail populations [18], and is chaotic for $m = 9$. Since the map

$$f(x) = x \left(0.55 + \frac{3.45}{1 + x^9} \right)$$

is bimodal, the results in [8] are not applicable. It is easy to verify that f satisfies (A1)–(A3) with $M = 4.5622$. Thus, the control scheme (1.2) stabilizes (globally) the positive equilibrium $K = 1.2346$ of model (2.6) for all $\alpha \in (\alpha_0, 1)$, where

$$\alpha_0 = \frac{M-1}{M+1} = 0.64043.$$

In view of Remark 2, convergence of any positive solution of the controlled equation to K is eventually monotone if $\alpha > \alpha_1 \approx 0.848101$, where α_1 is defined by (2.5).

2.2. Multistability

An interesting feature of the PBC method is that it allows to stabilize simultaneously several equilibria, in such a way that all positive solutions of the controlled equation (1.2) converge to one of the equilibrium points. We note that this situation was not considered in previous papers on this control technique. In order to generalize the results of Section 2.1 to this situation, we list the assumptions which we will use in this subsection:

(B1) $f : [0, \infty) \to [0, \infty)$ is continuous, $f(0) = 0$, and $f(x) > 0$ for all $x > 0$.
(B2) f has several nonnegative fixed points $0 = K_0 < K_1 < \cdots < K_r$, such that $f(x) > x$ for $0 < x < K_1$, and $f(x) < x$ for $x > K_r$.

(B3) There exists a number $M > 0$ such that

$$\left|\frac{f(x) - K_j}{x - K_j}\right| \le M, \tag{2.7}$$

for any $x > 0$ and every $j = 1, 2, \dots, r$.

Similar to the single equilibrium case, condition (B3) is satisfied for continuously differentiable functions if (B1) and (B2) hold.

To prove the main result of this section, we need the following auxiliary statement, which generalizes Lemma 2.1, and is of independent interest. Let us note that on each interval (K_j, K_{j+1}), either $f(x) > x$ or $f(x) < x$ holds.

Lemma 2.4. *Assume that $f : [0, \infty) \to [0, \infty)$ satisfies* (B1), (B2), *and let $\lambda \in (0, 1)$ be a number such that for any $x > 0$ and $j \in \{1, 2, \dots, r\}$,*

$$\text{either } \left|f(x) - K_j\right| \le \lambda |x - K_j| \text{ or } (f(x) - K_j)(x - K_j) > 0. \tag{2.8}$$

Then any solution $\{x_n\}$ of Eq. (1.1) with $x_0 > 0$ converges to a positive equilibrium, that is, there exists $j \in \{1, 2, \dots, r\}$ such that

$$\lim_{n\to\infty} x_n = K_j. \tag{2.9}$$

Proof. Let us fix an $x^* > 0$. We claim that if $f(x^*) > x^*$, then there exists $\varepsilon > 0$ such that for any $x \in (x^* - \varepsilon, x^* + \varepsilon)$ and $n \ge 1$, we have

$$f^n(x) > x^* + \varepsilon. \tag{2.10}$$

We prove this claim using induction in n. Denoting $\mu = f(x^*) - x^*$, we can find $\varepsilon < \mu/2$ such that $x \in (x^* - \varepsilon, x^* + \varepsilon)$ implies

$$f(x) \in \left(f(x^*) - \frac{\mu}{2}, f(x^*) + \frac{\mu}{2}\right) \subset \left(x^* + \varepsilon, f(x^*) + \frac{\mu}{2}\right),$$

since f is continuous. This means that (2.10) holds for $n = 1$. Let K_j be the least fixed point exceeding x^*. Without loss of generality, we can choose ε which also satisfies $\varepsilon < K_j - x^*$. Then, by (2.8), we can also claim that $|f(x) - K_j| < |x - K_j|$.

For $n = 2$, there may be two cases: $f(f(x)) \ge K_j$ and $f(f(x)) < K_j$. In the former case, (2.10) is obvious. In the latter case, if $x < f(x) < K_j$, then $f(f(x)) > f(x) > x^* + \varepsilon$; if $f(x) > K_j$, then by (2.8), we have

$$K_j - f(f(x)) \le \lambda(f(x) - K_j) \le \lambda(K_j - x^* - \varepsilon),$$

which implies $f(f(x)) \ge (1 - \lambda)K_j + \lambda(x^* + \varepsilon) \ge (1 - \lambda)(x^* + \varepsilon) + \lambda(x^* + \varepsilon) = x^* + \varepsilon$.

For the next induction step, we assume that $f^{n-1}(x) > x^* + \varepsilon$ and $f^n(x) > x^* + \varepsilon$, $n \ge 2$. If $x^* < f^n(x) \le K_j$, then $f^{n+1} \ge f^n(x) > x^* + \varepsilon$. If $f^n(x) > K_j$, then we can choose the maximal i such that $K_i \le f^n(x)$. Similar to the case $n = 2$, we prove that $f^{n+1}(x) < K_i$ implies $|K_i - f^{n-1}(x)| \ge f^n(x) - K_i$ and $K_i - f^{n+1}(x) \le f^n(x) - K_i \le K_i - f^{n-1}(x)$, so $f^{n+1}(x) \ge f^{n-1}(x) > x^* + \varepsilon$. The induction step completes the proof of (2.10).

It can be proved in a similar way that if $f(x^*) < x^*$, then there exists $\varepsilon > 0$ such that for any $x \in (x^* - \varepsilon, x^* + \varepsilon)$ and $n \ge 1$, we have

$$f^n(x) < x^* - \varepsilon.$$

Now, we are in a position to prove (2.9). First, let us note that any solution $\{x_n\}$ is bounded. If $\mathcal{M} = \max_{x\in[0,K_r]} f(x)$, then for any $x > \mathcal{M} + \varepsilon$ we have $x > K_r$, and so $f(x) < x$. Thus $\limsup_{n\to\infty} x_n < \mathcal{M} + \varepsilon$, hence $\{x_n\}$ is bounded.

Any bounded sequence has at least one accumulation point. If the point is unique, this implies (2.9). Otherwise, we can define the minimal accumulation point a and the maximal accumulation point b, that is,

$$a = \liminf_{n\to\infty} x_n; \qquad b = \limsup_{n\to\infty} x_n.$$

If $f(a) < a$, then a is not minimal, since, by (B1), $f(a)$ is also an accumulation point. If $f(a) > a$, then by (2.10) there exists $\varepsilon > 0$ such that for any $x \in (a - \varepsilon, a + \varepsilon)$ we have $f^n(x) > a + \varepsilon$, so a is not an accumulation point. Thus $f(a) = a$. A similar argument leads to the conclusion $f(b) = b$. Thus, we can exclude from consideration the case when some x_n coincides with either a or b.

Next, there may be two cases: there exists $x_i \in (a, b)$ for some i or not. In the former case, if $f(x_i) > x_i$, then for some $\varepsilon > 0$ we have $x_{i+j} > x_i + \varepsilon$, for all $j \ge 1$, so a is not an accumulation point; if $f(x_i) < x_i$, then b is not an accumulation point. In the latter case, for any ε there is n_0 such that $x_n \in (a - \varepsilon, a) \cup (b, b + \varepsilon)$, $n \ge n_0$. Without loss of generality, we assume that $\varepsilon < b - a$. Let $x_k \in (b, b + \varepsilon)$ then $f(x_k) < a$ is impossible since in this case $(f(x_k) - a)(x - a) < 0$ and

$$|f(x_k) - b| = b - f(x_k) > b - a > \varepsilon > x_k - b,$$

so (2.8) fails. Thus $a = b$, which completes the proof. □

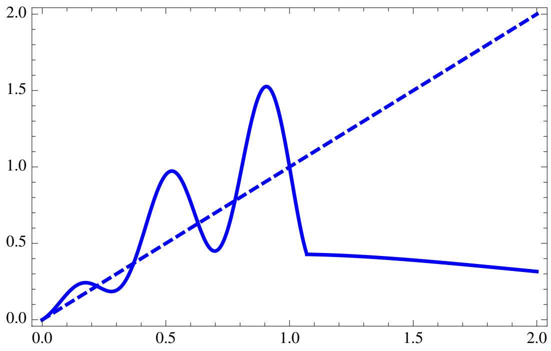

Fig. 1. Representation of the graph $y = f(x)$, where f is defined in (2.11), and the line $y = x$. The five positive equilibria of $x_{n+1} = f(x_n)$ are given by the intersections of these two curves for $x > 0$.

Theorem 2.5. *Assume that f satisfies* (B1)–(B3). *Then there exists $A \in (0, 1)$ such that for all $\alpha \in (A, 1)$, any solution $\{x_n\}$ of* (1.2) *with $x_0 > 0$ converges to one of the positive equilibria.*

Proof. Let M be defined in (2.7). Without loss of generality, we can assume that $M \geq 1$; otherwise, Eq. (1.2) immediately satisfies the conditions of Lemma 2.4. Let us take $\beta = 2M/(M + 1)$ and

$$A = \frac{M - \beta}{M} = \frac{M - 1}{M + 1}.$$

Arguing as in the proof of Theorem 2.2, and having in mind Remark 1, it can be proved that function F_α satisfies the conditions of Lemma 2.4 for all $\alpha \in (A, 1)$. □

Remark 3. Similar to Remark 2, if we assume that conditions (B1), (B2) hold, and f is piecewise monotone and continuously differentiable on $(0, K_r)$, then it is not difficult to prove that the number

$$A_1 := \max_{x\in[0,K_r],f'(x)<0} \left\{ \frac{f'(x)}{f'(x) - 1} \right\}$$

is well defined, and $F'_\alpha(x) \geq 0$ for all $x \in [0, K_r]$ if $\alpha \in [A_1, 1)$. As a consequence, the convergence of any solution of (1.2) with $x_0 > 0$ to one of the positive equilibria of (1.1) is eventually monotone if $\alpha \geq A_1$.

Example 2.6. Define

$$f_1(x) = x\,(1 + 0.3\sin(\pi x) + 0.6\sin(5\pi x))\,, \qquad f_2(x) = \frac{f_1(1.07)}{1.07}\, xe^{1.07-x},$$

and

$$f(x) = \begin{cases} f_1(x), & \text{if } x \in [0, 1.07], \\ f_2(x), & \text{if } x > 1.07. \end{cases} \tag{2.11}$$

A graphic representation of $y = f(x)$ together with the line $y = x$ is plotted in Fig. 1.

The difference equation $x_{n+1} = f(x_n)$ has 6 equilibria

$$0 = K_0 < K_1 < K_2 < K_3 < K_4 < K_5 = 1.$$

The only one stable is K_1, and this attractor coexists with a chaotic attractor. See the bifurcation diagram in Fig. 2.

An application of Theorem 2.5 proves that all solutions of the controlled equation (1.2) converge to an equilibrium if

$$\alpha > A = \frac{M - 1}{M + 1} \approx 0.808348,$$

where M is defined in (2.7). Note that the three attracting equilibria are $K_1 \approx 0.220708$, $K_3 \approx 0.630335$, and $K_5 = 1$. We emphasize that the bound provided by Theorem 2.5 is quite sharp in this case because the value of the parameter α at which $F'_\alpha(1) = -1$, inducing a period-halving bifurcation at $K_5 = 1$, is $\alpha_1 \approx 0.807085$, which is very close to A. For values of α less than α_1, there is an attracting 2-periodic orbit of F_α, so convergence of all positive solutions to one of the equilibria is not possible.

According to Remark 3, we can ensure that convergence of all positive solutions of (1.2) to one of the positive equilibria is eventually monotone if

$$\alpha \geq A_1 := \max_{x\in[0,1],f'(x)<0} \left\{ \frac{f'(x)}{f'(x) - 1} \right\} = \frac{f'(1)}{f'(1) - 1} \approx 0.903542.$$

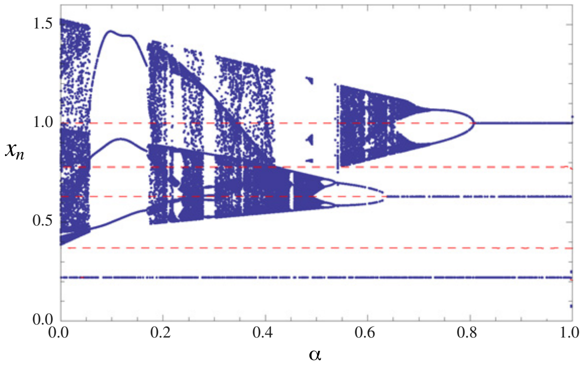

Fig. 2. Bifurcation diagram for the controlled equation $x_{n+1} = \alpha x_n + (1-\alpha)f(x_n)$, where f is defined in (2.11). A random initial condition was chosen for each value of $\alpha \in (0, 1)$, in such a way that the three attracting equilibria can be seen in the bifurcation diagram for $\alpha > 0.807$. The dashed lines represent unstable equilibria. Note that K_2 and K_4 cannot be stabilized using (1.2) because $f'(K_2) > 0, f'(K_4) > 0$.

3. Pulse stabilization

In this section, we analyze the possibility of finding a range of values of the control parameter α that allow to stabilize a positive equilibrium of (1.1) using the scheme with pulses (1.3). The first important remark is that, in general, it is impossible to find a range of values of the parameter control that stabilize a positive equilibrium of (1.1) using CF or PF methods with pulses. Assume, for example, that we want to stabilize a fixed point $K > 0$ using the PF method with pulses

$$x_{n+1} = \begin{cases} f(x_n), & \text{if } n \neq mk, \\ f(\gamma x_n), & \text{if } n = mk, \ k \in \mathbb{Z}^+, \end{cases} \tag{3.1}$$

where $\gamma \in \mathbb{R}$. In particular, K must be a fixed point both of $f(x)$ and $g(x) = f(\gamma x)$, that is, $f(K) = K = f(\gamma K)$. In general, this equality only holds for a finite number of values of γ, unless f is constant over an interval.

If we consider the CF method, then, arguing as before, we arrive at the equality $f(K) = K = f(K) + c$, which is only possible if $c = 0$.

The reason why PBC method allows the possibility of pulse stabilization of equilibria is that the fixed points of the original system are the same as the fixed points of the controlled equation. Before formulating the main result for the PBC method, we recall that it is impossible to stabilize an unstable positive fixed point K of (1.1) if $f'(K) > 1$ using PBC with a value of $\alpha \in [0, 1)$. Thus, we will assume that $f'(K) < -1$.

Theorem 3.1. *Let K be an unstable equilibrium of (1.1). Assume that f is differentiable at K, and $c_1 := f'(K) < -1$. Then K is locally asymptotically stable for the pulse scheme (1.3) if*

$$\alpha \in I_m = \left(\frac{c_1^m - (-1)^m}{c_1^m - c_1^{m-1}}, \frac{c_1^m + (-1)^m}{c_1^m - c_1^{m-1}} \right). \tag{3.2}$$

That is to say, in the following ranges of values of the parameter α:

(a) $\alpha \in \left(\frac{c_1^m+1}{c_1^m - c_1^{m-1}}, \frac{c_1^m-1}{c_1^m - c_1^{m-1}} \right)$, *if m is odd.*

(b) $\alpha \in \left(\frac{c_1^m-1}{c_1^m - c_1^{m-1}}, \frac{c_1^m+1}{c_1^m - c_1^{m-1}} \right)$, *if m is even.*

Proof. We prove the case when m is odd. The arguments for the other case are analogous. The control scheme with pulses (1.3) can be seen as an m-periodic difference equation, so the stability properties of the equilibrium $x = K$ depend on the derivative at K of the period map

$$F_{m,\alpha}(x) = f^{m-1}(F_\alpha(x)) = f^m(x) - \alpha(f^m(x) - f^{m-1}(x)).$$

The fixed point K is asymptotically stable if $|F'_{m,\alpha}(K)| < 1$, and unstable if $|F'_{m,\alpha}(K)| > 1$. See, e.g., [19] and references therein.

Note that

$$F'_{m,\alpha}(K) = c_1^m - \alpha(c_1^m - c_1^{m-1}).$$

Since m is odd and $c_1 < -1$, it follows that $F'_{m,\alpha}(K)$ is an increasing function of α. This means that K is asymptotically stable for $\alpha \in (\alpha_1, \alpha_2)$, where α_1, α_2 solve

$$c_1^m - \alpha_1(c_1^m - c_1^{m-1}) = -1; \qquad c_1^m - \alpha_2(c_1^m - c_1^{m-1}) = 1.$$

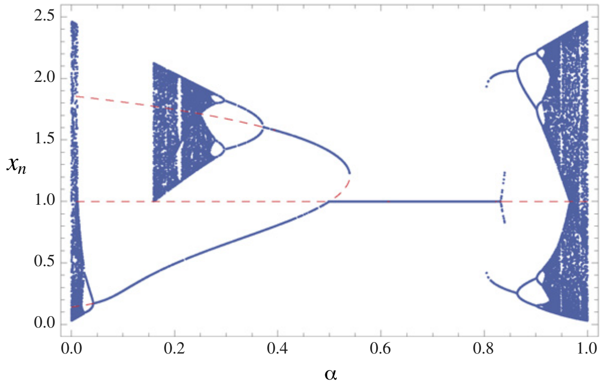

Fig. 3. Bifurcation diagram for the controlled equation by pulses with $m = 2$. The dashed lines correspond to unstable equilibria. There are three positive equilibria of (3.5) for $\alpha < \alpha^* \approx 0.5395$, and only one after α^*. The fixed point $K = 1$ of f becomes stable after a transcritical bifurcation at $\alpha = 0.5$, and becomes unstable again after a period-doubling bifurcation at $\alpha = 0.833$. $K = 1$ seems to be globally stable between α^* and $\alpha^{**} \approx 0.80419$, where an attracting 2-periodic orbit is born in a saddle–node bifurcation.

Hence we get

$$\alpha_1 = \frac{c_1^m + 1}{c_1^m - c_1^{m-1}}; \qquad \alpha_2 = \frac{c_1^m - 1}{c_1^m - c_1^{m-1}}. \quad \square$$

Some remarks are in order. First, it is easy to check that $I_{m+1} \subset I_m$ for all $m \geq 1$. This means that the interval of stabilization becomes smaller as the intervention period increases. Note that the larger interval is attained for $m = 1$, that is, the control without pulses (1.2), and it is

$$I_1 = \left(\frac{c_1 + 1}{c_1 - 1}, 1 \right). \tag{3.3}$$

On the other hand, the length of I_m tends to zero as m tends to infinity, and the only point in the intersection of all intervals I_m is

$$\alpha^{\#} = \frac{c_1}{c_1 - 1}.$$

It is easy to check that $F'_{m,\alpha^{\#}}(K) = 0$ for all $m \geq 1$, which means that the fixed point K is superstable for the control scheme (1.3) regardless of the value of m.

From the proof of Theorem 3.1, it follows that, when m is odd, the fixed point K is stabilized at a period-halving bifurcation, when $f'(K) = -1$, and it is destabilized again after a tangent (transcritical) bifurcation, when $f'(K) = 1$. For even m, the situation is reversed: the tangent bifurcation is stabilizing, and the equilibrium is destabilized in a period-doubling bifurcation (which can be either supercritical or subcritical).

Another important observation is that, while in Section 2, we have proved that global stabilization is possible using the PBC method (1.2) under some mild assumptions on function f, the same conclusion does not hold in general for the pulse scheme (1.3) with $m > 1$. We give an example using the Ricker map usually employed in population dynamics (see, e.g, [20])

$$f(x) = x\, e^{3(1-x)}. \tag{3.4}$$

It is well known that Eq. (1.1) with this function f is chaotic [21]. In [8], it was proved that the positive equilibrium $K = 1$ is locally (and globally) stable for the controlled equation (1.2) if $\alpha \in (1/3, 1)$. Note that $c_1 = f'(1) = -2$, so this interval is exactly interval I_1 defined in (3.3).

An application of Theorem 3.1 with $m = 2$, that is, the control is implemented every two periods, shows that the equilibrium $K = 1$ is stabilized for

$$\alpha \in I_2 = \left(\frac{c_1^2 - 1}{c_1^2 - c_1}, \frac{c_1^2 + 1}{c_1^2 - c_1} \right) = (1/2, 5/6) \approx (0.5, 0.8333).$$

Numerical simulations suggest that the equilibrium is actually globally asymptotically stable for α in a subinterval $J_2 = (0.5395, 0.80419)$ of I_2. See the bifurcation diagram in Fig. 3, where a random initial condition was chosen for the scheme

$$x_{n+1} = \begin{cases} x_n\, e^{3(1-x_n)}, & \text{if } n \neq 2k, \\ \alpha x_n + (1-\alpha) x_n\, e^{3(1-x_n)}, & \text{if } n = 2k,\ k \in \mathbb{Z}^+, \end{cases} \tag{3.5}$$

with $\alpha \in (0, 1)$.

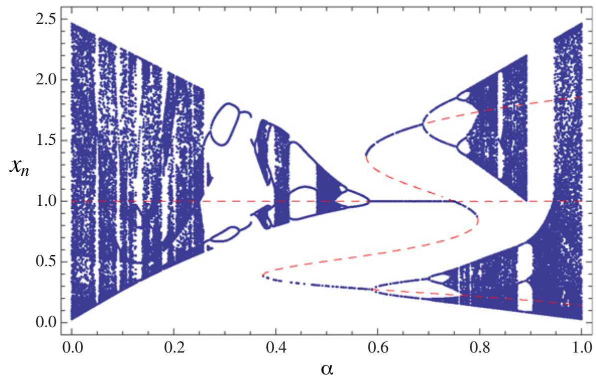

Fig. 4. Bifurcation diagram for the controlled equation by pulses with $m = 3$. The dashed lines correspond to unstable equilibria. The fixed point $K = 1$ of f becomes stable after a period-doubling bifurcation at $\alpha = 0.5833$, and becomes unstable again after a transcritical bifurcation at $\alpha = 0.75$. $K = 1$ is not globally stable for any value of α.

The strategy of pulse control using the PBC method (1.3) with $m = 3$ stabilizes the positive equilibrium $K = 1$ of the Ricker map (3.4) for

$$\alpha \in I_3 = \left(\frac{c_1^3 + 1}{c_1^3 - c_1^2}, \frac{c_1^3 - 1}{c_1^3 - c_1^2} \right) = (7/12, 9/12) \approx (0.5833, 0.75).$$

The bifurcation diagram shown in Fig. 4 suggests that in this case $K = 1$ does not become globally stable for any value of α.

4. Discussion

The method of predictive control (1.2), first introduced by de Sousa Vieira and Lichtenberg [9], is able to stabilize unstable equilibria of a chaotic system using a control which depends on the difference between the current state value x_n of a difference Eq. (1.1) and the value $f(x_n)$, which can be considered as a prediction of the next step x_{n+1}. Although it was already emphasized in [9] that this method is robust against noise, due to the large basis of attraction of the stabilized equilibrium, the first analytical result on global stabilization of fixed points using PBC was recently proved in [8]. In the present paper, we generalized the main result in [8] to a wide class of difference equations of the form (1.1), even without requiring differentiability of the map f governing the associated discrete dynamical system. When there is a unique positive equilibrium of (1.1), the PBC method stabilizes the system about the original equilibrium of the system, which in models of population dynamics usually means the so-called carrying capacity. This fact makes an important difference with other control methods such as constant feedback (CF) and proportional feedback (PF). In addition, persistence of the population is kept after control, avoiding such undesired consequences as the Allee effect and catastrophe bifurcations [5,12,13]. In some systems, there may be several positive equilibria. In this case, we proved that a PBC scheme is able to induce a simple dynamics into a chaotic system, in such a way that all positive solutions converge to one of the equilibria of the system.

Next, in many situations it is very difficult and expensive to apply control at every step. Thus, a strategy of pulse (or periodic) control is necessary; for example, a seasonal intervention in population dynamics. With this motivation in mind, we introduced a strategy of pulse PBC defined by the scheme (1.3), and we have demonstrated that this method is able to stabilize an unstable equilibrium of the original system (1.1). This feature is in contrast with CF and PF methods, in which it is only possible to stabilize orbits of period greater than one if a pulse strategy is used. Our results show that the range of parameters α for which stabilization is achieved using the method (1.3) decreases as the period between interventions increases, and the property of global stabilization is lost because new equilibria and periodic orbits appear; thus, the method is less robust against noise than the usual PBC method. In any case, even if there is multistability, the risk of extinction is prevented if the values of the control parameter α are restricted to the interval $(0, 1)$.

Natural directions for future research are the generalization of our results to systems of difference equations, and to the stabilization of periodic orbits of period greater than one. On the other hand, we note that some generalizations of the delayed feedback control method introduced by Pyragas [16] have been applied using periodic interventions or pulses (also called *oscillating feedback*) [22,23]. It would be interesting to compare these methods with the pulse PBC scheme introduced in this paper.

Acknowledgments

E. Braverman was partially supported by the NSERC grant. E. Liz was supported by the Spanish Ministry of Science and Innovation and FEDER, grant MTM2010–14837.

The authors are grateful to two anonymous referees for their careful review and encouraging comments.

References

[1] S. Boccaletti, C. Grebogi, Y.-C. Lai, H. Mancini, D. Maza, The control of chaos: theory and applications, Phys. Rep. 329 (2000) 103–197.
[2] E.E.N. Macau, C. Grebogi, Controlling chaos, in: E. Schöll, H.G. Schuster (Eds.), Handbook of Chaos Control, second ed., Wiley-VCH, Weinheim, 2008, pp. 3–28.
[3] E. Ott, C. Grebogi, J.A. Yorke, Controlling chaos, Phys. Rev. Lett. 64 (1990) 1196–1199.
[4] R.V. Solé, J.G.P. Gamarra, M. Ginovart, D. López, Controlling chaos in ecology: from discrete maps to individual-based models, Bull. Math. Biol. 61 (1999) 1187–1207.
[5] S. Parthasarathy, S. Sinha, Controlling chaos in unidimensional maps using constant feedback, Phys. Rev. E 51 (1995) 6239–6242.
[6] J. Güémez, M.A. Matías, Control of chaos in unidimensional maps, Phys. Lett. A 181 (1993) 29–32.
[7] E. Liz, How to control chaotic behaviour and population size with proportional feedback, Phys. Lett. A 374 (2010) 725–728.
[8] E. Liz, D. Franco, Global stabilization of fixed points using predictive control, Chaos 20 (2010) 023124. 9 pp.
[9] M. de Sousa Vieira, A.J. Lichtenberg, Controlling chaos using nonlinear feedback with delay, Phys. Rev. E 54 (1996) 1200–1207.
[10] T. Ushio, S. Yamamoto, Prediction-based control of chaos, Phys. Lett. A 264 (1999) 30–35.
[11] S. Gueron, Controlling one-dimensional population maps by harvesting at a constant rate, Phys. Rev. E 57 (1998) 3645–3648.
[12] E. Liz, Complex dynamics of survival and extinction in simple population models with harvesting, Theoretical Ecology 3 (2010) 209–221.
[13] S.J. Schreiber, Chaos and population disappearances in simple ecological models, J. Math. Biol. 42 (2001) 239–260.
[14] E. Braverman, E. Liz, Global stabilization of periodic orbits using a proportional feedback control with pulses, Nonlinear Dynam. (in press) (doi:10.1007/s11071-011-0160-x).
[15] P. Carmona, D. Franco, Control of chaotic behaviour and prevention of extinction using constant proportional feedback, Nonlinear Anal. RWA 12 (2011) 3719–3726.
[16] K. Pyragas, Continuous control of chaos by self-controlling feedback, Phys. Lett. A 170 (1992) 421–428.
[17] B.T. Polyak, Chaos stabilization by predictive control, Autom. Remote Control 66 (2005) 1791–1804.
[18] J.G. Milton, J. Bélair, Chaos, noise, and extincion in models of population growth, Theor. Popul. Biol. 37 (1990) 273–290.
[19] S.N. Elaydi, R. Luís, H. Oliveira, Towards a theory of periodic difference equations and its applications to population dynamics, in: Dynamics, Games and Science I, in: Springer Proceedings in Mathematics, vol. 1, Springer, Heidelberg, 2011 pp. 287–321. (Chapter 19).
[20] H.R. Thieme, Mathematics in Population Biology, Princeton University Press, 2003.
[21] R.M. May, Simple mathematical models with very complicated dynamics, Nature 261 (1976) 459–467.
[22] O. Morgül, On the stabilization of periodic orbits for discrete time chaotic systems, Phys. Lett. A 335 (2005) 127–138.
[23] H.G. Schuster, M.B. Stemmler, Control of chaos by oscillating feedback, Phys. Rev. E 56 (1997) 6410–6417.