Fundamental theorems of Doi-Hopf modules in a non-associative setting

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MOTIVATION:

1) Let $\mathbb{F}$ be a field and $\mathcal{C} = \mathbb{F} - Vect$. Let H be a Hopf algebra in $\mathcal{C}$ and let B be a right H-comodule algebra with coaction $\rho_B : B \to B \otimes H$, $\rho_B(b) = b_{(0)} \otimes b_{(1)}$. Y. Doi introduced in

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the notion of (H, B)-Hopf module (Doi-Hopf module), as a generalization of the classical notion of Hopf module, defined by Larson and Sweedler, in the following way: Let M be a right B-module and a right H-comodule. If, for all $m \in M$ and $b \in B$, we write $m.b$ for the action and $\rho_M(m) = m_{[0]} \otimes m_{[1]}$ for the coaction, we will say that M is an (H, B)-Hopf module if the equality

$$\rho_M(m.b) = m_{[0]}.b_{(0)} \otimes m_{[1]}b_{(1)}$$

holds, where $m_{[1]}b_{(1)}$ is the product in H of $m_{[1]}$ and $b_{(1)}$.

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holds, where $m_{[1]}b_{(1)}$ is the product in H of $m_{[1]}$ and $b_{(1)}$.

A morphism between two (H, B)-Hopf modules is an $\mathbb{F}$-linear map that is B-linear and H-colinear. Hopf modules and morphisms of Hopf modules constitute the category of (H, B)-Hopf modules denoted by $\mathcal{M}_B^H$.

If there exists a right H-comodule map $h : H \to B$ which is an algebra map (i.e. h is a multiplicative total integral), and

$$M^{coH} = \{m \in M \mid \rho_M(m) = m \otimes 1_H\}, \quad B^{coH} = \{b \in B \mid \rho_B(b) = b \otimes 1_H\}$$

are the subobjects of coinvariants, M^{coH} is a right B^{coH}-module.
Using this property, Doi proved that for all $M \in \mathcal{M}_B^H$

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In this case $H^{coH} = \mathbb{F}$, $M \simeq M^{coH} \otimes H$ in $\mathcal{M}_H^H$, and $\mathcal{M}_H^H \approx \mathbb{F} - Vect$.

2) Let $\mathcal{C} = \mathbb{F} - Vect$. Let H be a weak Hopf algebra in $\mathcal{C}$, let $\Pi_H^L : H \to H$ be the idempotent target morphism

$$\Pi_H^L(h) = \varepsilon_H(1_{(0)}h)1_{(1)}, \quad Im(\Pi_H^L) = H_L$$

and let B be a right H-comodule algebra with coaction $\rho_B : B \to B \otimes H$. We can define the notions of (H, B)-Hopf module and morphism of (H, B)-Hopf modules as in the Hopf algebra setting. Hopf modules and morphisms of Hopf modules constitute the category of (H, B)-Hopf modules denoted by $\mathcal{M}_B^H$.

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Using this fact, Zhang and Zhu proved in

- **L. Zhang, S. Zhu,** Fundamental theorems of weak Doi-Hopf modules and semisimple weak smash product Hopf algebras, Comm. Algebra **32** (2004), 3403-3415.

that, for all $M \in \mathcal{M}_B^H$,

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The previous categorical equivalence for $B = H$ and $h = id_H$, contains as a particular instance the equivalence derived of the Fundamental Theorem of Hopf modules for weak Hopf algebras proved by Böhm, Nill and Szlachányi in

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3) Let $\mathbb{F}$ be a field and $\mathcal{C} = \mathbb{F} - Vect$. Let H be a Hopf quasigroup in $\mathcal{C}$. T. Brzeziński introduced in

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the category of Hopf modules, denoted by $\mathcal{M}_H^H$. In this case the notion of Hopf module reflects the non-associativity of the product defined on H, and the morphisms are H-quasilinear and H-colinear.

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If for $M \in \mathcal{M}_H^H$ we define M^{coH} as in the Hopf algebra setting, T. Brzeziński proved that

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as Hopf modules. Therefore the Fundamental Theorem of Hopf modules also holds for Hopf quasigroups.

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Moreover, there exist two functors

$$F = - \otimes H : \mathcal{C} \to \mathcal{M}_H^H, \quad G = (\ \ )^{coH} : \mathcal{M}_H^H \to \mathcal{C}$$

such that $F \dashv G$, and they induce a categorical equivalence. Thus, as it occurs in the Hopf algebra ambit,

$$\mathcal{M}_H^H \approx \mathbb{F} - Vect$$

4) Let $\mathcal{C}$ be a braided monoidal category where every idempotent morphism splits. Let H be a weak Hopf quasigroup in $\mathcal{C}$. Let $\Pi_H^L : H \to H$ be the idempotent target morphism and $H_L = Im(\Pi_H^L)$.
We can define the notions of H-Hopf module and morphism of H-Hopf modules extending to the weak case the ideas proposed by T. Brzeziński to the Hopf quasigroup setting. In particular we can construct the categories of Hopf modules, denoted by $\mathcal{M}_H^H$, and the category of strong Hopf modules, denoted by $\mathcal{SM}_H^H$.

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- **J.N. Alonso Álvarez, J.M. Fernández Vilaboa, R. González Rodríguez**, Strong Hopf modules for weak Hopf quasigroups, Colloq. Math. Warsaw 148, N. 2, 231-246 (2017) (available in arXiv:1505.04586).

that, if $M \in \mathcal{SM}_H^H$,

$$M \simeq M^{coH} \otimes_{H_L} H$$

as strong H-Hopf modules. Then, the Fundamental Theorem of Hopf modules also holds in this setting.

In the previous conditions there are two functors

$$F = - \otimes_{H_L} H : \mathcal{C}_{H_L} \to \mathcal{SM}_H^H, \quad G = (\ \ )^{coH} : \mathcal{SM}_H^H \to \mathcal{C}_{H_L}$$

such that $F \dashv G$. Moreover, F and G induce a categorical equivalence between $\mathcal{SM}_H^H$ and the category of right H_L-modules:

$$\mathcal{SM}_H^H \approx \mathcal{C}_{H_L}.$$

Hopf algeb.	Weak Hopf algeb.	Hopf quasigroups	Weak Hopf quasigroups
$\mathcal{C} = \mathbb{F} - Vect$ $h : H \to B$ mti $\mathcal{M}_B^H \approx \mathcal{C}_{B^{coH}}$ 1983 Doi	$\mathcal{C} = \mathbb{F} - Vect$ $h : H \to B$ mti $\mathcal{M}_B^H \approx \mathcal{C}_{B^{coH}}$ 2004 Zhang + Zhu	$\mathcal{C} = \mathbb{F} - Vect$	$\mathcal{C} =$ BMC
$B = H,\ h = id_H$ $\mathcal{M}_H^H \approx \mathcal{C}$ 1969 Larson + Sweedler	$B = H,\ h = id_H$ $\mathcal{M}_H^H \approx \mathcal{C}_{H_L}$ 1999 Böhm + Nill + Szlachányi	$\mathcal{M}_H^H \approx \mathcal{C}$ 2010 Brzeziński	$\mathcal{S}\mathcal{M}_H^H \approx \mathcal{C}_{H_L}$ 2016 Alonso + Fernández + González

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Target

Introduce a general theory of Doi-Hopf modules that permits to prove a general categorical equivalence encompassing the previous results.

Outline

Weak Hopf quasigroups

- From now on $\mathcal{C}$ denotes a braided monoidal category with tensor product denoted by $\otimes$ and unit object K. With c we will denote the braiding.
  Without loss of generality, by the coherence theorems, we can assume the monoidal structure of $\mathcal{C}$ strict. Then, in this talk, we omit explicitly the associativity and unit constraints.

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- We also assume that every idempotent morphism $q : Y \rightarrow Y$ in $\mathcal{C}$ splits ($\mathcal{C}$ is Cauchy complete), i.e. there exist an object Z (called the image of q) and morphisms $i : Z \rightarrow Y$ and $p : Y \rightarrow Z$ such that $q = i \circ p$ and $p \circ i = id_Z$.

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- For simplicity of notation, given three objects V, U, B in $\mathcal{C}$ and a morphism $f : V \to U$, we write

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- (A, η_A, μ_A) is a unital magma, i.e. $\eta_A : K \to A$ (unit) and $\mu_A : A \otimes A \to A$ (product) are morphisms in $\mathcal{C}$ such that

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- $(C, \varepsilon_C, \delta_C)$ is a comonoid with comultiplication δ_C and counit ε_C.
- If $f, g : C \to A$ are morphisms, $f * g$ denotes the convolution product.

$$f * g = \mu_A \circ (f \otimes g) \circ \delta_C.$$

- **Alonso Álvarez, J.N., Fernández Vilaboa, J.M. y González Rodríguez, R.**: Weak Hopf quasigroups, Asian Journal of Mathematics 20, N. 4, 665-694 (2016), arXiv:1410.2180.
- **Alonso Álvarez, J.N., Fernández Vilaboa, J.M. y González Rodríguez, R.**: A characterization of weak Hopf (co)quasigroups Mediterranean Journal of Mathematics 13, N. 5, 3747-3764 (2016), arXiv:1506.07664.

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Definition

A weak Hopf quasigroup H in $\mathcal{C}$ is a unital magma (H, η_H, μ_H) and a comonoid $(H, \varepsilon_H, \delta_H)$ such that the following axioms hold:

(a1) $\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H)$.

(a2)
$$\begin{aligned} &\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) \\ &= \varepsilon_H \circ \mu_H \circ (H \otimes \mu_H) \\ &= ((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes \delta_H \otimes H) \\ &= ((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes (c^{-1}_{H,H} \circ \delta_H) \otimes H). \end{aligned}$$

(a3)
$$\begin{aligned} &(\delta_H \otimes H) \circ \delta_H \circ \eta_H \\ &= (H \otimes \mu_H \otimes H) \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H)) \\ &= (H \otimes (\mu_H \circ c^{-1}_{H,H}) \otimes H) \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H)). \end{aligned}$$

(a4) There exists $\lambda_H : H \to H$ in $\mathcal{C}$ (called the antipode of H) such that, if we denote the morphisms $id_H * \lambda_H$ by Π_H^L (target morphism) and $\lambda_H * id_H$ by Π_H^R (source morphism),

- (a4-1) $\Pi_H^L = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$.
- (a4-2) $\Pi_H^R = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$.
- (a4-3) $\lambda_H * \Pi_H^L = \Pi_H^R * \lambda_H = \lambda_H$.
- (a4-4) $\mu_H \circ (\lambda_H \otimes \mu_H) \circ (\delta_H \otimes H) = \mu_H \circ (\Pi_H^R \otimes H)$.
- (a4-5) $\mu_H \circ (H \otimes \mu_H) \circ (H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes H) = \mu_H \circ (\Pi_H^L \otimes H)$.
- (a4-6) $\mu_H \circ (\mu_H \otimes \lambda_H) \circ (H \otimes \delta_H) = \mu_H \circ (H \otimes \Pi_H^L)$.
- (a4-7) $\mu_H \circ (\mu_H \otimes H) \circ (H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H) = \mu_H \circ (H \otimes \Pi_H^R)$.

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- (a4-3) $\lambda_H * \Pi_H^L = \Pi_H^R * \lambda_H = \lambda_H$.
- (a4-4) $\mu_H \circ (\lambda_H \otimes \mu_H) \circ (\delta_H \otimes H) = \mu_H \circ (\Pi_H^R \otimes H)$.
- (a4-5) $\mu_H \circ (H \otimes \mu_H) \circ (H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes H) = \mu_H \circ (\Pi_H^L \otimes H)$.
- (a4-6) $\mu_H \circ (\mu_H \otimes \lambda_H) \circ (H \otimes \delta_H) = \mu_H \circ (H \otimes \Pi_H^L)$.
- (a4-7) $\mu_H \circ (\mu_H \otimes H) \circ (H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H) = \mu_H \circ (H \otimes \Pi_H^R)$.

Note that, if in the previous definition the triple (H, η_H, μ_H) is a monoid, we obtain the braided monoidal version (Alonso, Fernández and González (Indiana U. Math. J. (2008)) of the notion of weak Hopf algebra introduced by Böhm, Nill and Szlachányi (J. Algebra (1999)). On the other hand, if ε_H and δ_H are morphisms of unital magmas, $\Pi_H^L = \Pi_H^R = \eta_H \otimes \varepsilon_H$. As a consequence, conditions (a2), (a3), (a4-1)-(a4-3) trivialize, and we get the monoidal version of the notion of Hopf quasigroup defined by Klim and Majid (J. Algebra (2010)).

Example

Let $\mathcal{B}$ be a bicategory and denote by $x, y, z, \cdots$ the 0 cells, by $f : x \to y$ the 1-cells and by $\alpha : f \Rightarrow g$ the 2-cells. For a 1-cell $f : x \to y$, x is called the source of f, represented by $s(f)$, and y is called the target of f, denoted by $t(f)$.

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A bicategory is normal if the unit isomorphisms

$$l_f : 1_{t(f)} \circ f \Rightarrow f, \quad r_f : f \circ 1_{s(f)} \Rightarrow f,$$

are identities. Every bicategory is biequivalent to a normal one.

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A 1-cell f is called an equivalence if there exists a 1-cell $g : t(f) \to s(f)$ and two isomorphisms $g \circ f \Rightarrow 1_{s(f)}$, $f \circ g \Rightarrow 1_{t(f)}$. In this case we will say that $g \in Inv(f)$ and, equivalently, $f \in Inv(g)$.

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A bigroupoid is a bicategory where every 1-cell is an equivalence and every 2-cell is an isomorphism.

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A bigroupoid is a bicategory where every 1-cell is an equivalence and every 2-cell is an isomorphism.

We will say that a bigroupoid $\mathcal{B}$ is finite if the set of 0-cells $\mathcal{B}_0$ is finite and $\mathcal{B}(x, y)$ is small for all x, y.

Example

Let $\mathcal{B}$ be a finite normal bigroupoid and denote by $\mathcal{B}_{\mathbf{1}}$ the set of 1-cells. Let $\mathbb{F}$ be a field and $\mathbb{F}\mathcal{B}$ the direct product

$$\mathbb{F}\mathcal{B} = \bigoplus_{f \in \mathcal{B}_{\mathbf{1}}} \mathbb{F}f.$$

The vector space $\mathbb{F}\mathcal{B}$ is a unital non-associative algebra where the product of two 1-cells is equal to their 1-cell composition if the latter is defined and 0 otherwise, i.e., $g.f = g \circ f$ if $s(g) = t(f)$ and $g.f = 0$ if $s(g) \neq t(f)$. The unit element is

$$1_{\mathbb{F}\mathcal{B}} = \sum_{x \in \mathcal{B}_{\mathbf{0}}} 1_x.$$

Example

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$$1_{\mathbb{F}\mathcal{B}} = \sum_{x \in \mathcal{B}_0} 1_x.$$

Let $H = \mathbb{F}\mathcal{B}/I(\mathcal{B})$ be the quotient algebra where $I(\mathcal{B})$ is the ideal of $\mathbb{F}\mathcal{B}$ generated by

$$h - g \circ (f \circ h),\ p - (p \circ f) \circ g,$$

with $f \in \mathcal{B}_1$, $g \in Inv(f)$, and $h, p \in \mathcal{B}_1$ such that $t(h) = s(f)$, $t(f) = s(p)$. In what follows, for any 1-cell f we denote its class in H by $[f]$. If we define $[f]^{-1}$ by the class of $g \in Inv(f)$, we obtain that $[f]^{-1}$ is well-defined.

Therefore the vector space H with the product

$$\mu_H([g] \otimes [f]) = [g.f]$$

and the unit

$$\eta_H(1_{\mathbb{F}}) = \sum_{x \in \mathcal{B}_0} [1_x]$$

is a unital non-associative algebra.
Also, it is easy to show that H is a coalgebra with coproduct

$$\delta_H([f]) = [f] \otimes [f]$$

and counit

$$\varepsilon_H([f]) = 1_{\mathbb{F}}.$$

Moreover, we have a morphism (the antipode) $\lambda_H : H \to H$ defined by

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Note that, if $\mathcal{B}_0 = \{x\}$ we obtain that H is a Hopf quasigroup. Moreover, if $|\mathcal{B}_0| > 1$ and the product defined in H is associative we have an example of weak Hopf algebra.

Proposition

The antipode of a weak Hopf quasigroup H is unique and leaves the unit and the counit invariant, i.e. $\lambda_H \circ \eta_H = \eta_H$ and $\varepsilon_H \circ \lambda_H = \varepsilon_H$.

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Definition

Let H be a weak Hopf quasigroup. We define the morphisms $\overline{\Pi}_H^L$ and $\overline{\Pi}_H^R$ by

$$\overline{\Pi}_H^L = (H \otimes (\varepsilon_H \circ \mu_H)) \circ ((\delta_H \circ \eta_H) \otimes H),$$

and

$$\overline{\Pi}_H^R = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)).$$

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Proposition

Let H be a weak Hopf quasigroup. The morphisms Π_H^L, Π_H^R, $\overline{\Pi}_H^L$ and $\overline{\Pi}_H^R$ are idempotent.

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$$\Pi_H^L \circ \overline{\Pi}_H^L = \Pi_H^L, \quad \Pi_H^L \circ \overline{\Pi}_H^R = \overline{\Pi}_H^R, \quad \overline{\Pi}_H^L \circ \Pi_H^L = \overline{\Pi}_H^L, \quad \overline{\Pi}_H^R \circ \Pi_H^L = \Pi_H^L,$$

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Let H be a weak Hopf quasigroup. The following identities hold:

$$\Pi_H^L \circ \lambda_H = \Pi_H^L \circ \Pi_H^R = \lambda_H \circ \Pi_H^R, \quad \Pi_H^R \circ \lambda_H = \Pi_H^R \circ \Pi_H^L = \lambda_H \circ \Pi_H^L,$$

$$\Pi_H^L = \overline{\Pi}_H^R \circ \lambda_H = \lambda_H \circ \overline{\Pi}_H^L, \quad \Pi_H^R = \overline{\Pi}_H^L \circ \lambda_H = \lambda_H \circ \overline{\Pi}_H^R.$$

Proposition

Let H be a weak Hopf quasigroup. The antipode of H is antimultiplicative and anticomultiplicative, i.e. the following equalities hold:

$$\lambda_H \circ \mu_H = \mu_H \circ c_{H,H} \circ (\lambda_H \otimes \lambda_H),$$

$$\delta_H \circ \lambda_H = (\lambda_H \otimes \lambda_H) \circ c_{H,H} \circ \delta_H,$$

Proposition

Let H be a weak Hopf quasigroup. Put $H_L = Im(\Pi_H^L)$ and let $p_L : H \to H_L$ and $i_L : H_L \to H$ be the morphisms such that $\Pi_H^L = i_L \circ p_L$ and $p_L \circ i_L = id_{H_L}$. Then,

$$\begin{tikzcd} H_L \arrow[r, "i_L"] & H \arrow[r, shift left, "\delta_H"] \arrow[r, shift right, "(H \otimes \Pi_H^L) \circ \delta_H"'] & H \otimes H \end{tikzcd}$$

is an equalizer diagram and

$$\begin{tikzcd} H \otimes H \arrow[r, shift left, "\mu_H"] \arrow[r, shift right, "\mu_H \circ (H \otimes \Pi_H^L)"'] & H \arrow[r, "p_L"] & H_L \end{tikzcd}$$

is a coequalizer diagram.

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$$H_L \xrightarrow{i_L} H \underset{(H\otimes \Pi_H^L)\circ \delta_H}{\overset{\delta_H}{\rightrightarrows}} H\otimes H$$

is an equalizer diagram and

$$H\otimes H \underset{\mu_H\circ (H\otimes \Pi_H^L)}{\overset{\mu_H}{\rightrightarrows}} H \xrightarrow{p_L} H_L$$

is a coequalizer diagram.

As a consequence, $(H_L, \eta_{H_L} = p_L \circ \eta_H, \mu_{H_L} = p_L \circ \mu_H \circ (i_L \otimes i_L))$ is a unital magma in $\mathcal{C}$. Also

$$(H_L, \varepsilon_{H_L} = \varepsilon_H \circ i_L, \delta_H = (p_L \otimes p_L)\circ \delta_H \circ i_L)$$

is a comonoid in $\mathcal{C}$.

Proposition

Let H be a weak Hopf quasigroup. The following identities hold:

$$\mu_H \circ ((\mu_H \circ (i_L \otimes H)) \otimes H) = \mu_H \circ (i_L \otimes \mu_H),$$

$$\mu_H \circ (H \otimes (\mu_H \circ (i_L \otimes H))) = \mu_H \circ ((\mu_H \circ (H \otimes i_L)) \otimes H),$$

$$\mu_H \circ (H \otimes (\mu_H \circ (H \otimes i_L))) = \mu_H \circ (\mu_H \otimes i_L).$$

As a consequence, the unital magma H_L is a monoid in $\mathcal{C}$.

Doi-Hopf modules for weak Hopf quasigroups

- **J.N. Alonso Álvarez, J.M. Fernández Vilaboa, & R. González Rodríguez**: Cleft and Galois extensions associated to a weak Hopf quasigroup J. Pure Applied Alg. 220, N. 3, 1002-1034, (2016), arXiv:1412.1622.
- **J.N. Alonso Álvarez, J.M. Fernández Vilaboa, & R. González Rodríguez**: Fundamental theorems of Doi-Hopf modules in a non-associative setting, (2017) arXiv:1703.03229.

Definition

Let H be a weak Hopf quasigroup and let B be a unital magma, which is also a right H-comodule with coaction $\rho_B : B \to B \otimes H$. We will say that (B, ρ_B) is a right H-comodule magma if

$$\mu_{B\otimes H} \circ (\rho_B \otimes \rho_B) = \rho_B \circ \mu_B. \tag{1}$$

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holds.

If (B, ρ_B) is a right H-comodule magma the following equivalent conditions hold:

(b1) $(\rho_B \otimes H) \circ \rho_B \circ \eta_B = (B \otimes (\mu_H \circ c^{-1}_{H,H}) \otimes H) \circ ((\rho_B \circ \eta_B) \otimes (\delta_H \circ \eta_H))$.

(b2) $(\rho_B \otimes H) \circ \rho_B \circ \eta_B = (B \otimes \mu_H \otimes H) \circ ((\rho_B \circ \eta_B) \otimes (\delta_H \circ \eta_H))$.

(b3) $(B \otimes \overline{\Pi}^R_H) \circ \rho_B = (\mu_B \otimes H) \circ (B \otimes (\rho_B \circ \eta_B))$,

(b4) $(B \otimes \Pi^L_H) \circ \rho_B = ((\mu_B \circ c^{-1}_{B,B}) \otimes H) \circ (B \otimes (\rho_B \circ \eta_B))$.

(b5) $(B \otimes \overline{\Pi}^R_H) \circ \rho_B \circ \eta_B = \rho_B \circ \eta_B$.

(b6) $(B \otimes \Pi^L_H) \circ \rho_B \circ \eta_B = \rho_B \circ \eta_B$.

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Note that, if H is a Hopf quasigroup and B is a unital magma which is also a right H-comodule with coaction $\rho_B : B \to B \otimes H$, we will say that (B, ρ_B) is a right H-comodule magma if it satisfies (1). Then, $\rho_B \circ \eta_B = \eta_H \otimes \eta_B$. In this case (b1)-(b6) trivialize.

Example

If H is a (weak) Hopf quasigroup, (H, δ_H) is a right H-comodule magma. Also, if H is cocommutative and $\mathcal{C}$ is symmetric, $(H^{op}, \rho_{H^{op}} = (H \otimes \lambda_H) \circ \delta_H)$ is a right H-comodule magma.

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Definition

Let H be a weak Hopf quasigroup and let (B, ρ_B) be a right H-comodule magma. We will say that $h : H \to B$ is an integral if it is a morphism of right H-comodules. The integral will be called total if $h \circ \eta_H = \eta_B$.

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Proposition

Let H be a weak Hopf quasigroup and let (B, ρ_B) be a right H-comodule magma. Let $h : H \to B$ be a total integral. The endomorphism

$$q_B := \mu_B \circ (B \otimes (h \circ \lambda_H)) \circ \rho_B : B \to B$$

satisfies

$$\rho_B \circ q_B = (B \otimes \Pi_H^L) \circ \rho_B \circ q_B,$$

and, as a consequence, q_B is an idempotent morphism.

Moreover, if B^{coH} (object of coinvariants) is the image of q_B and $p_B : B \to B^{coH}$, $i_B : B^{coH} \to B$ are the morphisms such that $q_B = i_B \circ p_B$ and $id_{B^{\mathbf{coH}}} = p_B \circ i_B$,

$$B^{coH} \xrightarrow{i_B} B \underset{(B \otimes \Pi_H^L) \circ \rho_B}{\overset{\rho_B}{\rightrightarrows}} B \otimes H,$$

is an equalizer diagram.

Moreover, the triple $(B^{coH}, \eta_{B^{coH}}, \mu_{B^{coH}})$ is a unital magma, where

$$\eta_{B^{coH}} : K \to B^{coH}, \quad \mu_{B^{coH}} : B^{coH} \otimes B^{coH} \to B^{coH}$$

are the factorizations through i_B of the morphisms η_B and $\mu_B \circ (i_B \otimes i_B)$, respectively.

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are the factorizations through i_B of the morphisms η_B and $\mu_B \circ (i_B \otimes i_B)$, respectively.

In what follows, the object of coinvariants B^{coH} will be called the submagma of coinvariants of B. Note that, if $B = H$, $\rho_B = \delta_H$ and $h = id_H$, the submagma of coinvariants is $H^{coH} = H_L$ and then, in this case, it is a monoid.

Moreover, the triple $(B^{coH}, \eta_{B^{coH}}, \mu_{B^{coH}})$ is a unital magma, where

$$\eta_{B^{coH}} : K \to B^{coH}, \quad \mu_{B^{coH}} : B^{coH} \otimes B^{coH} \to B^{coH}$$

are the factorizations through i_B of the morphisms η_B and $\mu_B \circ (i_B \otimes i_B)$, respectively.

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If the following equality

$$\mu_B \circ ((\mu_B \circ (B \otimes i_B)) \otimes B) = \mu_B \circ (B \otimes (\mu_B \circ (i_B \otimes B))) \tag{2}$$

holds, the submagma of coinvariants $(B^{coH}, \eta_{B^{coH}}, \mu_{B^{coH}})$ is a monoid.

Definition

Let H be a weak Hopf quasigroup and let (B, ρ_B) be a right H-comodule magma. We will say that $h : H \rightarrow B$ is an anchor morphism if it is a multiplicative total integral (i.e., a right H-comodule morphism such that it is a morphism of unital magmas) and the following equalities hold:

(c1) $\mu_B \circ ((\mu_B \circ (B \otimes h)) \otimes (h \circ \lambda_H)) \circ (B \otimes \delta_H) = \mu_B \circ (B \otimes (h \circ \Pi_H^L))$.

(c2) $\mu_B \circ ((\mu_B \circ (B \otimes (h \circ \lambda_H))) \otimes h) \circ (B \otimes \delta_H) = \mu_B \circ (B \otimes (h \circ \Pi_H^R))$.

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Note that, if the product on B is associative, every multiplicative total integral h satisfies (c1)-(c2) and therefore is an anchor morphism.

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Note that, if the product on B is associative, every multiplicative total integral h satisfies (c1)-(c2) and therefore is an anchor morphism.

Example

The identity morphism id_H is an anchor morphism for the right H-comodule magma (H, δ_H). Also, if H is cocommutative and $\mathcal{C}$ is symmetric, λ_H is an anchor morphism for $(H^{op}, \rho_{H^{op}} = (H \otimes \lambda_H) \circ \delta_H)$.

Definition

Let H be a weak Hopf quasigroup and let (B, ρ_B) be a right H-comodule magma. Let $h : H \to B$ be an anchor morphism and let M be an object in $\mathcal{C}$. We say that (M, ϕ_M, ρ_M) is a strong (H, B, h)-Hopf module if the following axioms hold:

(d1) The pair (M, ρ_M) is a right H-comodule.

(d2) The morphism $\phi_M : M \otimes B \to M$ satisfies:

(d2-1) $\phi_M \circ (M \otimes \eta_B) = id_M$.
(d2-2) $\phi_M \circ ((\phi_M \circ (M \otimes i_B)) \otimes B) = \phi_M \circ (M \otimes (\mu_B \circ (i_B \otimes B)))$.
(d2-3) $\rho_M \circ \phi_M = (\phi_M \otimes \mu_H) \circ (M \otimes c_{H,B} \otimes H) \circ (\rho_M \otimes \rho_B)$.
(d2-4) $\phi_M \circ ((\phi_M \circ (M \otimes h)) \otimes (h \circ \lambda_H)) \circ (M \otimes \delta_H) = \phi_M \circ (M \otimes (h \circ \Pi_H^L))$.
(d2-5) $\phi_M \circ ((\phi_M \circ (M \otimes (h \circ \lambda_H))) \otimes h) \circ (M \otimes \delta_H) = \phi_M \circ (M \otimes (h \circ \Pi_H^R))$.

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- (d2-5) $\phi_M \circ ((\phi_M \circ (M \otimes (h \circ \lambda_H))) \otimes h) \circ (M \otimes \delta_H) = \phi_M \circ (M \otimes (h \circ \Pi_H^R))$.

Example

For example, the triple (H, μ_H, δ_H) is a strong (H, H, id_H)-Hopf module. Also, if the equality (2) holds, the triple (B, μ_B, ρ_B) is a strong (H, B, h)-Hopf module.

Proposition

Let H be a weak Hopf quasigroup and let (B, ρ_B) be a right H-comodule magma. Let $h : H \to B$ be an anchor morphism and let (M, ϕ_M, ρ_M) be a strong (H, B, h)-Hopf module. The endomorphism $q_M := \phi_M \circ (M \otimes (h \circ \lambda_H)) \circ \rho_M : M \to M$ satisfies

$$\rho_M \circ q_M = (M \otimes \Pi_H^L) \circ \rho_M \circ q_M,$$

and, as a consequence, q_M is an idempotent. Moreover, if M^{coH} (object of coinvariants) is the image of q_M and $p_M : M \to M^{coH}$, $i_M : M^{coH} \to M$ are the morphisms such that $q_M = i_M \circ p_M$ and $id_{M^{coH}} = p_M \circ i_M$,

$$\begin{tikzcd} M^{coH} \arrow[r, "i_M"] & M \arrow[r, shift left, "\rho_M"] \arrow[r, shift right, "(M \otimes \Pi_H^L) \circ \rho_M"'] & M \otimes H, \end{tikzcd}$$

is an equalizer diagram.

Proposition

Let H be a weak Hopf quasigroup and let (B, ρ_B) be a right H-comodule magma. Let $h : H \to B$ be an anchor morphism. If (2) holds, for all strong (H, B, h)-Hopf module (M, ϕ_M, ρ_M), the object of coinvariants M^{coH} is a right B^{coH}-module where

$$\phi_{M^{coH}} = p_M \circ \phi_M \circ (i_M \otimes i_B).$$

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Proposition

Let H be a weak Hopf quasigroup and let (B, ρ_B) be a right H-comodule magma. Let $h : H \to B$ be an anchor morphism. Assume that (2) and

$$\mu_B \circ (i_B \otimes \mu_B) = \mu_B \circ ((\mu_B \circ (i_B \otimes B)) \otimes B) \tag{3}$$

hold. Then if the category $\mathcal{C}$ admits coequalizers and the functors $- \otimes B$ and $- \otimes H$ preserve coequalizers, for all strong (H, B, h)-Hopf module (M, ϕ_M, ρ_M), the object $M^{coH} \otimes_{B^{coH}} B$, defined by the coequalizer of

$$T_M^1 = \phi_{M^{coH}} \otimes B, \quad T_M^2 = M^{coH} \otimes (\mu_B \circ (i_B \otimes B)),$$

is a strong (H, B, h)-Hopf module. Moreover, there exists and isomorphism ω_M of right H-comodules between $M^{coH} \otimes_{B^{coH}} B$ and M.

The object $M^{coH} \otimes_{B^{coH}} H$ is defined by the coequalizer diagram

$$M^{coH} \otimes B^{coH} \otimes B \underset{T_M^2}{\overset{T_M^1}{\rightrightarrows}} M^{coH} \otimes B \xrightarrow{n_{M^{coH}}} M^{coH} \otimes_{B^{coH}} B,$$

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- $\phi_{M^{coH} \otimes_{B^{coH}} B} : M^{coH} \otimes_{B^{coH}} B \otimes B \to M^{coH} \otimes_{B^{coH}} B$ is the unique morphism such that

$$\phi_{M^{coH} \otimes_{B^{coH}} B} \circ (n_{M^{coH}} \otimes B) = n_{M^{coH}} \circ (M^{coH} \otimes \mu_B).$$

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- $\phi_{M^{coH} \otimes_{B^{coH}} B} : M^{coH} \otimes_{B^{coH}} B \otimes B \to M^{coH} \otimes_{B^{coH}} B$ is the unique morphism such that

$$\phi_{M^{coH} \otimes_{B^{coH}} B} \circ (n_{M^{coH}} \otimes B) = n_{M^{coH}} \circ (M^{coH} \otimes \mu_B).$$

- $\rho_{M^{coH} \otimes_{B^{coH}} B} : M^{coH} \otimes_{B^{coH}} B \to M^{coH} \otimes_{B^{coH}} B \otimes H$ is the unique morphism such that

$$\rho_{M^{coH} \otimes_{B^{coH}} B} \circ n_{M^{coH}} = (n_{M^{coH}} \otimes H) \circ (M^{coH} \otimes \rho_B).$$

The object $M^{coH} \otimes_{B^{coH}} H$ is defined by the coequalizer diagram

$$M^{coH} \otimes B^{coH} \otimes B \;\overset{T_M^1}{\underset{T_M^2}{\rightrightarrows}}\; M^{coH} \otimes B \xrightarrow{\;n_{M^{coH}}\;} M^{coH} \otimes_{B^{coH}} B,$$

- $\phi_{M^{coH} \otimes_{B^{coH}} B} : M^{coH} \otimes_{B^{coH}} B \otimes B \to M^{coH} \otimes_{B^{coH}} B$ is the unique morphism such that

$$\phi_{M^{coH} \otimes_{B^{coH}} B} \circ (n_{M^{coH}} \otimes B) = n_{M^{coH}} \circ (M^{coH} \otimes \mu_B).$$

- $\rho_{M^{coH} \otimes_{B^{coH}} B} : M^{coH} \otimes_{B^{coH}} B \to M^{coH} \otimes_{B^{coH}} B \otimes H$ is the unique morphism such that

$$\rho_{M^{coH} \otimes_{B^{coH}} B} \circ n_{M^{coH}} = (n_{M^{coH}} \otimes H) \circ (M^{coH} \otimes \rho_B).$$

- $\omega_M : M^{coH} \otimes_{B^{coH}} B \to M$ is the unique morphism such that

$$\omega_M \circ n_{M^{coH}} = \phi_M \circ (i_M \otimes B).$$

ω_M is an isomorphism with inverse $\omega_M^{-1} = n_{M^{coH}} \circ (p_M \otimes h) \circ \rho_M$.

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- $-\otimes B$ and $-\otimes H$ preserve coequalizers.
- There exists an anchor morphism $h : H \to B$.
- The equalities (2), (3)

$$\mu_B \circ ((\mu_B \circ (B \otimes i_B)) \otimes B) = \mu_B \circ (B \otimes (\mu_B \circ (i_B \otimes B))),$$

$$\mu_B \circ (i_B \otimes \mu_B) = \mu_B \circ ((\mu_B \circ (i_B \otimes B)) \otimes B)$$

  hold.

Proposition

Let (P, ϕ_P, ρ_P), (Q, ϕ_Q, ρ_Q) be strong (H, B, h)-Hopf modules. If there exists a right H-comodule isomorphism $\omega : Q \to P$, the triple

$$(P, \phi_P^{\omega} = \omega \circ \phi_Q \circ (\omega^{-1} \otimes B), \rho_P),$$

called the ω-deformation of (P, ϕ_P, ρ_P), is a strong (H, B, h)-Hopf module.

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called the ω-deformation of (P, ϕ_P, ρ_P), is a strong (H, B, h)-Hopf module.

Definition

We define the category of strong (H, B, h)-Hopf modules as the one whose objects are strong (H, B, h)-Hopf modules, and whose morphisms $f : M \to N$ are morphisms of right H-comodules and B-quasilinear, i.e.

$$\phi_N^{\omega_N} \circ (f \otimes B) = f \circ \phi_M^{\omega_M},$$

where $\omega_M : M^{coH} \otimes_{B^{coH}} B \to M$, $\omega_N : N^{coH} \otimes_{B^{coH}} B \to N$ are the isomorphisms of right H-comodules obtained previously. This category will be denoted by $\mathcal{SM}_B^H(h)$.

Proposition

Let (M, ϕ_M, ρ_M) be an object in $\mathcal{SM}_B^H(h)$. Let ω_M be the isomorphism of right H-comodules between $M^{coH} \otimes_{B^{coH}} B$ and M. Then the identity

$$\phi_M^{\omega_M} = \phi_M \circ (q_M \otimes (\mu_B \circ (h \otimes B))) \circ (\rho_M \otimes B)$$

holds and $q_M^{\omega_M} = q_M$, where $q_M^{\omega_M} = \phi_M^{\omega_M} \circ (M \otimes (h \circ \lambda_H)) \circ \rho_M$ is the idempotent morphism associated to the Hopf module $(M, \phi_M^{\omega_M}, \rho_M)$. Then $(M, \phi_M^{\omega_M}, \rho_M)$ has the same object of coinvariants that (M, ϕ_M, ρ_M).
Moreover, for $(M, \phi_M^{\omega_M}, \rho_M)$, the associated isomorphism of right H-comodules between $M^{coH} \otimes_{B^{coH}} B$ and M is ω_M, and the equality

$$(\phi_M^{\omega_M})^{\omega_M} = \phi_M^{\omega_M}$$

holds. Finally, there exists an idempotent functor

$$D : \mathcal{SM}_B^H(h) \to \mathcal{SM}_B^H(h),$$

called the ω-deformation functor, defined on objects by

$$D((M, \phi_M, \rho_M)) = (M, \phi_M^{\omega_M}, \rho_M)$$

and on morphisms by the identity.

Proposition

For any object (M, ϕ_M, ρ_M) in $\mathcal{SM}_B^H(h)$, the strong (H, B, h)-Hopf module

$$(M^{coH} \otimes_{B^{coH}} B, \phi_{M^{coH} \otimes_{B^{coH}} B}, \rho_{M^{coH} \otimes_{B^{coH}} B}),$$

is invariant for the ω-deformation functor, i.e.,

$$D((M^{coH} \otimes_{B^{coH}} B, \phi_{M^{coH} \otimes_{B^{coH}} B}, \rho_{M^{coH} \otimes_{B^{coH}} B}))$$

$$= (M^{coH} \otimes_{B^{coH}} B, \phi_{M^{coH} \otimes_{B^{coH}} B}, \rho_{M^{coH} \otimes_{B^{coH}} B}).$$

Proposition

For any object (M, ϕ_M, ρ_M) in $\mathcal{SM}_B^H(h)$, the strong (H, B, h)-Hopf module

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$$= (M^{coH} \otimes_{B^{coH}} B, \phi_{M^{coH} \otimes_{B^{coH}} B}, \rho_{M^{coH} \otimes_{B^{coH}} B}).$$

Theorem. **Fundamental Theorem of Hopf modules**

Let (M, ϕ_M, ρ_M) be an object in $\mathcal{SM}_B^H(h)$. Then

$$M \simeq M^{coH} \otimes_{B^{coH}} B$$

in $\mathcal{SM}_B^H(h)$.

Categorical equivalences

Let (N, ϕ_N) be an object in $\mathcal{C}_{B^{coH}}$ and consider the coequalizer diagram

$$N \otimes B^{coH} \otimes B \overset{\phi_N \otimes B}{\underset{N \otimes (\mu_B \circ (i_B \otimes B))}{\rightrightarrows}} N \otimes B \xrightarrow{n_N} N \otimes_{B^{coH}} B$$

Let (N, ϕ_N) be an object in $\mathcal{C}_{B^{coH}}$ and consider the coequalizer diagram

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Then, $(n_N \otimes B) \circ (\phi_N \otimes \rho_B) = (n_N \otimes B) \circ (N \otimes (\rho_B \circ (\mu_B \circ (i_B \otimes B))))$ and, as a consequence, there exists a unique morphism

$$\rho_{N \otimes_{B^{coH}} B} : N \otimes_{B^{coH}} B \to N \otimes_{B^{coH}} B \otimes H$$

such that

$$\rho_{N \otimes_{B^{coH}} B} \circ n_N = (n_N \otimes B) \circ (N \otimes \rho_B).$$

On the other hand, we have

$$n_N \circ (\phi_N \otimes \mu_B) = n_N \circ (N \otimes (\mu_B \circ ((\mu_B \circ (i_B \otimes B)) \otimes B))),$$

and then,using that the functor $- \otimes B$ preserves coequalizers, there exists a unique morphism

$$\phi_{N \otimes_{B^{coH}} B} : N \otimes_{B^{coH}} B \otimes B \to N \otimes_{B^{coH}} B$$

such that

$$\phi_{N \otimes_{B^{coH}} B} \circ (n_N \otimes B) = n_N \circ (N \otimes \mu_B).$$

Moreover, we can prove that

$$(N \otimes_{B^{coH}} B, \phi_{N \otimes_{B^{coH}} B}, \rho_{N \otimes_{B^{coH}} B})$$

is a strong (H, B, h)-Hopf module.

Proposition

There exists a functor $F : \mathcal{C}_{B^{coH}} \to \mathcal{SM}_B^H(h)$, called the induction functor, defined on objects by

$$F((N, \phi_N)) = (N \otimes_{B^{coH}} B, \phi_{N\otimes_{B^{coH}}B}, \rho_{N\otimes_{B^{coH}}B})$$

and on morphisms by $F(f) = f \otimes_{B^{coH}} B$

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Proposition

There exists a functor $G : \mathcal{SM}_B^H(h) \rightarrow \mathcal{C}_{B^{coH}}$, called the functor of coinvariants, defined on objects by

$$G((M, \phi_M, \rho_M)) = (M^{coH}, \phi_{M^{coH}})$$

and on morphisms by $G(g) = g^{coH}$.

Proposition

There exists a functor $F : \mathcal{C}_{B^{coH}} \to \mathcal{SM}_B^H(h)$, called the induction functor, defined on objects by

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Proposition

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and on morphisms by $G(g) = g^{coH}$.

Theorem

The functor F is left adjoint of G. Moreover, the categories $\mathcal{S}M_B^H(h)$ and $\mathcal{C}_{B^{coH}}$ are equivalent.

Example

Let H be a Hopf quasigroup and A an unital magma in $\mathcal{C}$. If there exists a morphism $\varphi_A : H \otimes A \to A$ such that

$$\varphi_A \circ (\eta_H \otimes A) = id_A,$$

$$\varphi_A \circ (H \otimes \eta_A) = \varepsilon_H \otimes \eta_A,$$

hold, then the smash product $A\sharp H = (A \otimes H, \eta_{A\sharp H}, \mu_{A\sharp H})$ defined by

$$\eta_{A\sharp H} = \eta_A \otimes \eta_H,$$

$$\mu_{A\sharp H} = (\mu_A \otimes \mu_H) \circ (A \otimes \psi_H^A \otimes H),$$

where

$$\psi_H^A = (\varphi_A \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A),$$

is a right H-comodule magma with comodule structure given by $\varrho_{A\sharp H} = A \otimes \delta_H$.

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where

$$\psi_H^A = (\varphi_A \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A),$$

is a right H-comodule magma with comodule structure given by $\varrho_{A\sharp H} = A \otimes \delta_H$.

Also, $h = \eta_A \otimes H : H \to A\sharp H$ is an anchor morphism. Moreover, $q_{A\sharp H} = A \otimes \eta_H \otimes \varepsilon_H$, $p_{A\sharp H} = A \otimes \varepsilon_H$, $i_{A\sharp H} = A \otimes \eta_H$ and

$$(A\sharp H)^{coH} = A$$

If A is a monoid and the equality

$$\mu_A \circ (\varphi_A \otimes \varphi_A) \circ (H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes A \otimes A) = \varphi_A \circ (H \otimes \mu_A),$$

holds, then (2) and (3)

$$\mu_{A\sharp H} \circ ((\mu_{A\sharp H} \circ (A \otimes H \otimes i_{A\sharp H})) \otimes H) = \mu_{A\sharp H} \circ (A \otimes H \otimes (\mu_{A\sharp H} \circ (i_{A\sharp H} \otimes A \otimes H))),$$

$$\mu_{A\sharp H} \circ (i_{A\sharp H} \otimes \mu_{A\sharp H}) = \mu_{A\sharp H} \circ ((\mu_{A\sharp H} \circ (i_{A\sharp H} \otimes A \otimes H)) \otimes A \otimes H)$$

also hold.

If A is a monoid and the equality

$$\mu_A \circ (\varphi_A \otimes \varphi_A) \circ (H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes A \otimes A) = \varphi_A \circ (H \otimes \mu_A),$$

holds, then (2) and (3)

$$\mu_{A\sharp H} \circ ((\mu_{A\sharp H} \circ (A \otimes H \otimes i_{A\sharp H})) \otimes H) = \mu_{A\sharp H} \circ (A \otimes H \otimes (\mu_{A\sharp H} \circ (i_{A\sharp H} \otimes A \otimes H))),$$

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also hold.

Therefore, if $-\otimes A$ and $-\otimes H$ preserve coequalizers, we have an equivalence of categories

$$\mathcal{S}M^{H}_{A\sharp H}(h) \approx \mathcal{C}_A$$

for $h = \eta_A \otimes H : H \to A\sharp H$.

- **Klim J., Majid S.** Hopf quasigroups and the algebraic 7-sphere. J. Algebra, 2010, 323: 3067-3110.

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Let $\mathbb{K}$ be a field and let $\mathcal{C}$ be the symmetric monoidal category of vector spaces over $\mathbb{K}$. Let G the abelian group $\mathbb{Z}_2^n$ and let $F : G \times G \to \mathbb{K}^*$ be a 2-cochain, i.e. F is a morphism such that $F(\theta, a) = F(a, \theta) = 1$ for all $a \in G$ where θ is the group identity. The group algebra of G, denoted by, $\mathbb{K}G$ is a $\mathbb{K}$-vector space with basis

$$\{e_a \ ; \ a \in G\}$$

and also is a unital magma with the product:

$$e_a e_b = F(a, b) e_{a+b}.$$

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In the following we will denote this magma by

$$\mathbb{K}_F G.$$

Moreover, $\mathbb{K}_F G$ is a composition algebra with respect to the Euclidean norm in basis G if two suitable conditions hold for F (see Klim and Majid). This means that the norm $q(\sum_a u_a e_a) = \sum_a u_a^2$ is multiplicative. Then

$$\mathcal{S}^{2^n-1} = \{\sum_a u_a e_a \ , \ \sum_a u_a^2 = 1_{\mathbb{K}}\}$$

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We know that $\mathcal{S}^{2^n-1}$ is an I.P loop, and then its loop algebra, denoted by

$$\mathbb{K}\mathcal{S}^{2^n-1}$$

is a cocommutative Hopf quasigroup.

Let H be $\mathbb{K}\mathcal{S}^{2^n-1}$ and let A the group algebra of G. Then, A is a monoid (it is a cocommutative Hopf algebra) and we have an action $\varphi_A : H \otimes A \to A$, where $\otimes = \otimes_{\mathbb{K}}$, defined by

$$\varphi_A(e_a \otimes e_b) = (-1)^{a.b} e_b.$$

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It is easy to see that φ_A satisfies the previous conditions and we have a categorical equivalence

$$\mathcal{S}M^{\mathbb{K}\mathcal{S}^{2^n-1}}_{\mathbb{K}\mathbb{Z}_2^n \sharp \mathbb{K}\mathcal{S}^{2^n-1}}(h) \approx \mathcal{C}_{\mathbb{K}\mathbb{Z}_2^n}$$

for $h = \eta_{\mathbb{K}\mathbb{Z}_2^n} \otimes id_{\mathbb{K}\mathcal{S}^{2^n-1}}$.

Example

Let H be a cocommutative weak Hopf quasigroup and assume that $\mathcal{C}$ is symmetric. The pair

$$(H^{op}, \rho_{H^{op}} = (H \otimes \lambda_H) \circ \delta_H)$$

is an example of right H-comodule magma and $h = \lambda_H$ is an anchor morphism. Moreover, we have the equality:

$$q_{H^{op}} = \Pi_H^L.$$

Therefore, $i_{H^{op}} = i_L$, $p_{H^{op}} = p_L$ and $(H^{op})^{coH} = H_L$

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On the other hand,

$$\mu_{H^{op}} \circ ((\mu_{H^{op}} \circ (H \otimes i_L)) \otimes H) = \mu_{H^{op}} \circ (H \otimes (\mu_{H^{op}} \circ (i_L \otimes H))),$$

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$$\mu_{H^{op}} \circ (i_L \otimes \mu_{H^{op}}) = \mu_{H^{op}} \circ ((\mu_{H^{op}} \circ (i_L \otimes H)) \otimes H)$$

also hold.

As a consequence of this facts, if $- \otimes H$ preserve coequalizers, we obtain an equivalence of categories

$$\mathcal{S}M_{H^{op}}^H(\lambda_H) \approx \mathcal{C}_{H_L}.$$

If H is a Hopf quasigroup, the categories $\mathcal{S}M_{H^{op}}^H(\lambda_H)$ and $\mathcal{C}$ are equivalent.

Hopf algeb.	Weak Hopf algeb.	Hopf quasigroups	Weak Hopf quasigroups
$\mathcal{C} = \mathbb{F} - Vect$ $h : H \to B$ mti $\mathcal{M}_B^H \approx \mathcal{C}_{B^{coH}}$ 1983 Doi	$\mathcal{C} = \mathbb{F} - Vect$ $h : H \to B$ mti $\mathcal{M}_B^H \approx \mathcal{C}_{B^{coH}}$ 2004 Zhang + Zhu	$\mathcal{C} = \mathbb{F} - Vect$	$\mathcal{C} = \text{BMC}$
$B = H,\ h = id_H$ $\mathcal{M}_H^H \approx \mathcal{C}$ 1969 Larson + Sweedler	$B = H,\ h = id_H$ $\mathcal{M}_H^H \approx \mathcal{C}_{H_L}$ 1999 Böhm + Nill + Szlachányi	$\mathcal{M}_H^H \approx \mathcal{C}$ 2010 Brzeziński	$\mathcal{SM}_H^H \approx \mathcal{C}_{H_L}$ 2016 Alonso + Fernández + González

Hopf algeb.	Weak Hopf algeb.	Hopf quasigroups	Weak Hopf quasigroups
$\mathcal{C} = \mathbb{F} - Vect$ $h : H \to B$ mti $\mathcal{M}_B^H \approx \mathcal{C}_{B^{coH}}$ 1983 Doi	$\mathcal{C} = \mathbb{F} - Vect$ $h : H \to B$ mti $\mathcal{M}_B^H \approx \mathcal{C}_{B^{coH}}$ 2004 Zhang + Zhu		$\mathcal{C} = \mathrm{BMC}$ $h : H \to B$ anchor $\mathcal{SM}_B^H(h) \approx \mathcal{C}_{B^{coH}}$
$B = H$, $h = id_H$ $\mathcal{M}_H^H \approx \mathcal{C}$ 1969 Larson + Sweedler	$B = H$, $h = id_H$ $\mathcal{M}_H^H \approx \mathcal{C}_{H_L}$ 1999 Böhm + Nill + Szlachányi	$\mathcal{M}_H^H \approx \mathcal{C}$ 2010 Brzeziński	$B = H$, $h = id_H$ $\mathcal{SM}_H^H(id_H) = \mathcal{SM}_H^H \approx \mathcal{C}_{H_L}$ 2016 Alonso + Fernández + González

Hopf algeb.	Weak Hopf algeb.	Hopf quasigroups	Weak Hopf quasigroups
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$B = H,\ h = id_H$ $\mathcal{M}_H^H \approx \mathcal{C}$ 1969 Larson + Sweedler	$B = H,\ h = id_H$ $\mathcal{M}_H^H \approx \mathcal{C}_{H_L}$ 1999 Böhm + Nill + Szlachányi	$B = H,\ h = id_H$ $\mathcal{SM}_H^H(id_H) = \mathcal{M}_H^H \approx \mathcal{C}$ 2010 Brzeziński	$B = H,\ h = id_H$ $\mathcal{SM}_H^H(id_H) = \mathcal{SM}_H^H \approx \mathcal{C}_{H_L}$ 2016 Alonso + Fernández + González

Hopf algeb.	Weak Hopf algeb.	Hopf quasigroups	Weak Hopf quasig.
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$\mathcal{C} = \mathbb{F} - Vect$ $h : H \to B$ mti $\mathcal{M}_B^H \approx \mathcal{C}_{B^{coH}}$ $\approx \mathcal{SM}_B^H(h)$ 1983 Doi	$\mathcal{C} = \mathbb{F} - Vect$ $h : H \to B$ mti $\mathcal{M}_B^H \approx \mathcal{C}_{B^{coH}}$ $\approx \mathcal{SM}_B^H(h)$ 2004 Zhang + Zhu	$\mathcal{C} =$ BMC $h : H \to B$ anchor $\mathcal{SM}_B^H(h) \approx \mathcal{C}_{B^{coH}}$	$\mathcal{C} =$ BMC $h : H \to B$ anchor $\mathcal{SM}_B^H(h) \approx \mathcal{C}_{B^{coH}}$
$B = H,\ h = id_H$ $\mathcal{M}_H^H \approx \mathcal{C}$ $\approx \mathcal{SM}_H^H(id_H)$ 1969 Larson + Sweedler	$B = H,\ h = id_H$ $\mathcal{M}_H^H \approx \mathcal{C}_{H_L}$ $\approx \mathcal{SM}_H^H(id_H)$ 1999 Böhm + Nill + Szlachányi	$B = H,\ h = id_H$ $\mathcal{SM}_H^H(id_H) = \mathcal{M}_H^H \approx \mathcal{C}$ 2010 Brzeziński	$B = H,\ h = id_H$ $\mathcal{SM}_H^H(id_H) = \mathcal{SM}_H^H \approx \mathcal{C}_{H_L}$ 2016 Alonso + Fernández + González

Thank you