Multiplication alteration by two-cocycles. The non associative version

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- **MOTIVATION:** Y. Doi introduced in

  Doi Y., Braided bialgebras and quadratic bialgebras, *Comm. Algebra* **21** (1993), 1731-1749.

  a new contruction to modify the algebra structure of a bialgebra A over a field $\mathbb{F}$ using an invertible 2-cocycle σ in A. In this case if $\sigma : A \otimes A \to \mathbb{F}$ is the 2-cocycle, the new product on A is defined by

$$a * b = \sum \sigma(a_1 \otimes b_1) a_2 b_2 \sigma^{-1}(a_3 \otimes b_3)$$

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  for $a, b \in A$.

- With the new algebra structure and the original coalgebra structure, A is a new bialgebra denoted by A^{σ}, and if A is a Hopf algebra with antipode λ_A, so is A^{σ} whit antipode given by

$$\lambda_{A^{\sigma}}(a) = \sum \sigma(a_1 \otimes \lambda_A(a_2)) \lambda_A(a_3) \sigma^{-1}(\lambda_A(a_4) \otimes a_5).$$

- One of the main remarkable examples of this construction is the Drinfeld double of a Hopf algebra H. If H^* is the dual of H and $A = H^{*cop} \otimes H$, the Drinfeld double $D(H)$ can be obtained as A^σ where σ is defined by

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- As was pointed by Doi and Takeuchi in

  Doi, Y., Takeuchi M., Multiplication alteration by two-cocycles. The quantum version, *Comm. Algebra* **22** (1994), 5175-5732.

  "this will be the shortest description of the multiplication of $D(H)$"

- A particular case of alterations of products by 2-cocycles are provided by invertible skew pairings on bialgebras. If A and H are bialgebras and $\tau : A \otimes H \to \mathbb{F}$ is an invertible skew pairing, Doi and Takeuchi defined, in the paper cited previously, a new bialgebra $A \bowtie_{\tau} H$ in the following way: The morphism $\omega : A \otimes H \otimes A \otimes H \to \mathbb{F}$ defined by

$$\omega((a \otimes g) \otimes (b \otimes h)) = \varepsilon_A(a)\varepsilon_H(h)\tau(b \otimes g),$$

for $a, b \in A$ and $g, h \in H$, is a 2-cocycle in $A \otimes H$ and

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- The construction of $A \bowtie_{\tau} H$ also generalizes the Drinfeld double because H^{*cop} and H are skew-paired.

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1. **Alonso Álvarez J.N., Fernández Vilaboa J.M., González Rodríguez R., Soneira Calvo, C.,** Projections and Yetter-Drinfel'd modules over Hopf (co)quasigroups, *J. Algebra* **443** (2015), 153-199.
2. **Alonso Álvarez J.N., Fernández Vilaboa J.M., González Rodríguez R. and Soneira Calvo, C.**, Cleft comodules over Hopf quasigroups, *Commun. Contemp. Math.* **17** (2015), 1550007.
3. **Alonso Álvarez, J.N., Fernández Vilaboa, J.M. y González Rodríguez, R.,** Multiplication alteration by two-cocycles. The nonassociative version arXiv:1703.01829 (2017).

Outline

Nonassociative bimonoids

- From now on $\mathcal{C}$ denotes a strict symmetric monoidal category with tensor product denoted by $\otimes$ and unit object K. With c we will denote the braiding and we also assume that in $\mathcal{C}$ every idempotent morphism $q : Y \to Y$ splits, i.e., there exist an object Z (called the image of q) and morphisms $i : Z \to Y$ and $p : Y \to Z$ such that $q = i \circ p$ and $p \circ i = id_Z$.

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- For simplicity of notation, given three objects V, U, B in $\mathcal{C}$ and a morphism $f : V \to U$, we write

$$B \otimes f \text{ for } id_B \otimes f \text{ and } f \otimes B \text{ for } f \otimes id_B.$$

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- (A, η_A, μ_A) is a unital magma, i.e. $\eta_A : K \to A$ (unit) and $\mu_A : A \otimes A \to A$ (product) are morphisms in $\mathcal{C}$ such that

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- $(C, \varepsilon_C, \delta_C)$ is a comonoid with comultiplication δ_C and counit ε_C.
- If $f, g : C \to A$ are morphisms, $f * g$ denotes the convolution product.

$$f * g = \mu_A \circ (f \otimes g) \circ \delta_C.$$

Definition

A nonassociative bimonoid in the category $\mathcal{C}$ is a unital magma (H, η_H, μ_H) and a comonoid $(H, \varepsilon_H, \delta_H)$ such that ε_H and δ_H are morphisms of unital magmas (equivalently, η_H and μ_H are morphisms of counital comagmas). Then the following identities hold:

$$\varepsilon_H \circ \eta_H = id_K, \quad \varepsilon_H \circ \mu_H = \varepsilon_H \otimes \varepsilon_H,$$

$$\delta_H \circ \eta_H = \eta_H \otimes \eta_H, \quad \delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ \delta_{H \otimes H}.$$

We say that H has a left division if moreover there exists a morphism $l_H : H \otimes H \to H$ in $\mathcal{C}$ (called the left division of H) such that

$$l_H \circ (H \otimes \mu_H) \circ (\delta_H \otimes H) = \varepsilon_H \otimes H = \mu_H \circ (H \otimes l_H) \circ (\delta_H \otimes H).$$

A morphism $f : H \to B$ between nonassociative bimonoids H and B is a morphism of unital magmas and comonoids.

We have the corresponding notion of nonassociative bimonoid with right division, replacing the left division l_H by a right division $r_H : H \otimes H \to H$ that, satisfies:

$$r_H \circ (\mu_H \otimes H) \circ (H \otimes \delta_H) = H \otimes \varepsilon_H = \mu_H \circ (r_H \otimes H) \circ (H \otimes \delta_H).$$

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Note that, if $\mathcal{C}$ is the category of vector spaces over a field $\mathbb{F}$, the notion of nonassociative bimonoid with left and a right division is the one introduced by Pérez-Izquierdo in

Pérez-Izquierdo, J.M., Algebras, hyperalgebras, nonassociative bialgebras and loops, *Adv. Math.* **208** (2007), 834-876.

with the name of unital H-bialgebra.

Proposition

Let H be a nonassociative bimonoid. There exists a left division l_H if and only if the morphism $h : H \otimes H \to H \otimes H$ defined as $h = (H \otimes \mu_H) \circ (\delta_H \otimes H)$ is an isomorphism. As a consequence, a left division l_H is uniquely determined.
Similarly, there exists a right division r_H if and only if the morphism $d : H \otimes H \to H \otimes H$ defined as $d = (\mu_H \otimes H) \circ (H \otimes \delta_H)$ is an isomorphism. As a consequence, a right division r_H is uniquely determined.

In the conditions of the previous proposition, if h (d) is an isomorphism, we obtain

$$h^{-1} \circ \delta_H = H \otimes \eta_H \; (d^{-1} \circ \delta_H = \eta_H \otimes H), \;\; \mu_H \circ h^{-1} = \varepsilon_H \otimes H \; (\mu_H \circ d^{-1} = H \otimes \varepsilon_H)$$

If $\lambda_H = l_H \circ (H \otimes \eta_H)$ ($\varrho_H = r_H \circ (\eta_H \otimes H)$)

$$id_H * \lambda_H = \varepsilon_H \otimes \eta_H \;\; (\varrho_H * id_H = \varepsilon_H \otimes \eta_H)$$

$$\lambda_H \circ \eta_H = \eta_H \;\; (\rho_H \circ \eta_H = \eta_H), \;\; \varepsilon_H \circ \lambda_H = \varepsilon_H \;\; (\varrho_H \circ \eta_H = \eta_H).$$

Proposition

Let H be a nonassociative bimonoid with left division l_H. It holds that

$$\delta_H \circ l_H = (l_H \otimes l_H) \circ (H \otimes c_{H,H} \otimes H) \circ ((c_{H,H} \circ \delta_H) \otimes \delta_H).$$

As a consequence, if $\lambda_H = l_H \circ (H \otimes \eta_H)$ we have that λ_H is anticomultiplicative, i.e.,

$$\delta_H \circ \lambda_H = (\lambda_H \otimes \lambda_H) \circ c_{H,H} \circ \delta_H.$$

If r_H is a right division for H,

$$\delta_H \circ r_H = (r_H \otimes r_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes (c_{H,H} \circ \delta_H))$$

holds. Then, if $\varrho_H = r_H \circ (\eta_H \otimes H)$, we have that

$$\delta_H \circ \varrho_H = (\varrho_H \otimes \varrho_H) \circ c_{H,H} \circ \delta_H.$$

Example

An essential example of a nonassociative bimonoid arises from Sabinin algebras. Given any Sabinin algebra V, it was explicitly constructed in

> Pérez-Izquierdo, J.M.; Algebras, hyperalgebras, nonassociative bialgebras and loops, *Adv. Math.* **208** (2007), 834-876,

its universal enveloping algebra $U(V)$, and moreover, it was also proved that it can be provided with a cocommutative nonassociative bimonoid structure with left and right division.

This is an interesting example because the definition of Sabinin algebra includes an infinite set of independent operations, but when we take a finite set we recover many other common structures. For example, it includes, as particular instances Lie, Malcev and Bol algebras.

Definition

A left Hopf quasigroup H in $\mathcal{C}$ is a nonassociative bimonoid such that there exists a morphism $\lambda_H : H \to H$ in $\mathcal{C}$ (called the left antipode of H) satisfying:

$$\mu_H \circ (\lambda_H \otimes \mu_H) \circ (\delta_H \otimes H) = \varepsilon_H \otimes H = \mu_H \circ (H \otimes \mu_H) \circ (H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes H).$$

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Note that composing with $H \otimes \eta_H$ in the previous equality we obtain

$$\lambda_H * id_H = \varepsilon_H \otimes \eta_H.$$

Obviously, there is a similar definition for the right side, i.e., H is a right Hopf quasigroup if there is a morphism $\varrho_H : H \to H$ in $\mathcal{C}$ (called the right antipode of H) such that:

$$\mu_H \circ (\mu_H \otimes H) \circ (H \otimes \varrho_H \otimes H) \circ (H \otimes \delta_H) = H \otimes \varepsilon_H = \mu_H \circ (\mu_H \otimes \varrho_H) \circ (H \otimes \delta_H).$$

Then, composing with $\eta_H \otimes H$ in we obtain

$$id_H * \varrho_H = \varepsilon_H \otimes \eta_H.$$

The above definition is the monoidal version of the notion of Hopf quasigroup (also called nonassociative Hopf algebra with the inverse property, or nonassociative IP Hopf algebra) introduced in

> **Klim, J., Majid, S.**, Hopf quasigroups and the algebraic 7-sphere, *J. Algebra* **323** (2010), 3067-3110.

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- Note that if H is both left and right Hopf quasigroup, the left and right antipodes are the same.

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- Note that if H is both left and right Hopf quasigroup, the left and right antipodes are the same.
- If H is a Hopf quasigroup in $\mathcal{C}$, the antipode λ_H is unique and antimultiplicative, i.e.,

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- Note that if H is both left and right Hopf quasigroup, the left and right antipodes are the same.
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$$\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}.$$
- Note that a Hopf quasigroup is associative if an only if is a Hopf algebra.

Proposition

The following assertions are equivalent:

(i) H is a nonassociative bimonoid with left division l_H such that $l_H = \mu_H \circ (\lambda_H \otimes H)$, where $\lambda_H = l_H \circ (H \otimes \eta_H)$.

(ii) H is a nonassociative bimonoid with left division l_H such that

$$\mu_H \circ (\lambda_H \otimes \mu_H) \circ (\delta_H \otimes H) = \varepsilon_H \otimes H, \text{ where } \lambda_H = l_H \circ (H \otimes \eta_H).$$

(iii) H is a left Hopf quasigroup.

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Proposition

The following assertions are equivalent:

(i) H is a nonassociative bimonoid with right division r_H such that $r_H = \mu_H \circ (H \otimes \varrho_H)$, where $\varrho_H = r_H \circ (\eta_H \otimes H)$.

(ii) H is a nonassociative bimonoid with right division r_H such that

$$\mu_H \circ (\mu_H \otimes \varrho_H) \circ (H \otimes \delta_H) = H \otimes \varepsilon_H, \text{ where } \varrho_H = r_H \circ (\eta_H \otimes H).$$

(iii) H is a right Hopf quasigroup.

Example

A loop $(L, \cdot, /, \backslash)$ is a set L equipped with three binary operations of multiplication $\cdot$, right division $/$ and left division $\backslash$, satisfying the identities:

$$v\backslash(v \cdot u) = u, \;\; u = (u \cdot v)/v, \;\; v \cdot (v\backslash u) = u, \;\; u = (u/v) \cdot v,$$

and such that in addition it contains an identity element e_L satisfying that the equations $e_L \cdot x = x = x \cdot e_L$ hold for all x in L.
Let $\mathbb{F}$ be a field and L a loop. Then, the loop algebra

$$\mathbb{F}L = \bigoplus_{u \in L} \mathbb{F}u$$

is a cocommutative nonassociative bimonoid with product, left and right division defined by linear extensions of those defined in L and

$$\delta_{\mathbb{F}L}(u) = u \otimes u, \;\; \varepsilon_{\mathbb{F}L}(u) = 1_{\mathbb{F}}$$

A loop L is said to be a loop with the inverse property (for brevity an IP-loop) if for every element $u \in L$, there corresponds an element $u^{-1} \in L$ such that the equations

$$u^{-1}.(u.v) = v = (v.u).u^{-1},$$

hold for every $v \in L$.
Then, the nonassociative bimonoid $\mathbb{F}L$ is a cocommutative Hopf quasigroup with $\lambda_{\mathbb{F}L}(u) = u^{-1}$.

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Then, the nonassociative bimonoid $\mathbb{F}L$ is a cocommutative Hopf quasigroup with $\lambda_{\mathbb{F}L}(u) = u^{-1}$.

Example

The enveloping algebra $U(L)$ of a Malcev algebra L, introduced in

Pérez-Izquierdo, J.M., Shestakov, I.P., An envelope for Malcev algebras, *J. Algebra* **272** (2004), 379-393,

when the groundfield has characteristic not 2, 3 is an example of cocommutative Hopf quasigroup.

Multiplication alteration by two cocycles for nonassociative bimonoids

Definition

Let H be a nonassociative bimonoid, and let $\sigma : H \otimes H \to K$ be a convolution invertible morphism. We say that σ is a 2-cocycle if the equality

$$\partial^1(\sigma) * \partial^3(\sigma) = \partial^4(\sigma) * \partial^2(\sigma)$$

holds, where $\partial^1(\sigma) = \varepsilon_H \otimes \sigma$, $\partial^2(\sigma) = \sigma \circ (\mu_H \otimes H)$, $\partial^3(\sigma) = \sigma \circ (H \otimes \mu_H)$ and $\partial^4(\sigma) = \sigma \otimes \varepsilon_H$.

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Let H be a nonassociative bimonoid, and let $\sigma : H \otimes H \to K$ be a convolution invertible morphism. We say that σ is a 2-cocycle if the equality

$$\partial^1(\sigma) * \partial^3(\sigma) = \partial^4(\sigma) * \partial^2(\sigma)$$

holds, where $\partial^1(\sigma) = \varepsilon_H \otimes \sigma$, $\partial^2(\sigma) = \sigma \circ (\mu_H \otimes H)$, $\partial^3(\sigma) = \sigma \circ (H \otimes \mu_H)$ and $\partial^4(\sigma) = \sigma \otimes \varepsilon_H$.

Definition

A 2-cocycle σ is called normal if further

$$\sigma \circ (\eta_H \otimes H) = \varepsilon_H = \sigma \circ (H \otimes \varepsilon_H),$$

and it is easy to see that if σ is normal so is σ^{-1}.

Proposition

Let H be a nonassociative bimonoid. Let σ be a normal 2-cocycle. Define the product $\mu_{H^{\sigma}}$ as

$$\mu_{H^{\sigma}} = (\sigma \otimes \mu_H \otimes \sigma^{-1}) \circ (H \otimes H \otimes \delta_{H\otimes H}) \circ \delta_{H\otimes H}.$$

Then $H^{\sigma} = (H, \eta_{H^{\sigma}} = \eta_H, \mu_{H^{\sigma}}, \varepsilon_{H^{\sigma}} = \varepsilon_H, \delta_{H^{\sigma}} = \delta_H)$ is a nonassociative bimonoid.

Proposition

Let H be a nonassociative bimonoid. Let σ be a normal 2-cocycle. Define the product μ_{H^σ} as

$$\mu_{H^\sigma} = (\sigma \otimes \mu_H \otimes \sigma^{-1}) \circ (H \otimes H \otimes \delta_{H \otimes H}) \circ \delta_{H \otimes H}.$$

Then $H^\sigma = (H, \eta_{H^\sigma} = \eta_H, \mu_{H^\sigma}, \varepsilon_{H^\sigma} = \varepsilon_H, \delta_{H^\sigma} = \delta_H)$ is a nonassociative bimonoid.

Proposition

Let H be a left Hopf quasigroup with left antipode λ_H. Let σ be a normal 2-cocycle. Then the nonassociative bimonoid H^σ, is a left Hopf quasigroup with left antipode

$$\lambda_{H^\sigma} = l_{H^\sigma} \circ (H \otimes \eta_H)$$

where

$$l_{H^\sigma} = \mu_{H^\sigma} \circ (f \otimes \lambda_H \otimes f^{-1} \otimes H) \circ (H \otimes \delta_H \otimes H) \circ (\delta_H \otimes H),$$

and

$$f = \sigma \circ (H \otimes \lambda_H) \circ \delta_H, \quad f^{-1} = \sigma^{-1} \circ (\lambda_H \otimes H) \circ \delta_H.$$

Proposition

Let H be a right Hopf quasigroup with right antipode ϱ_H. Let σ be a normal 2-cocycle. Then the nonassociative bimonoid H^{σ}, is a right Hopf quasigroup with right antipode

$$\varrho_{H^{\sigma}} = r_{H^{\sigma}} \circ (\eta_H \otimes H)$$

where

$$r_{H^{\sigma}} = \mu_{H^{\sigma}} \circ (H \otimes g^{-1} \otimes \varrho_H \otimes g) \circ (H \otimes \delta_H \otimes H) \circ (H \otimes \delta_H)$$

and

$$g = \sigma^{-1} \circ (\varrho_H \otimes H) \circ \delta_H, \;\; g^{-1} = \sigma \circ (H \otimes \varrho_H) \circ \delta_H.$$

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and

$$g = \sigma^{-1} \circ (\varrho_H \otimes H) \circ \delta_H, \quad g^{-1} = \sigma \circ (H \otimes \varrho_H) \circ \delta_H.$$

Corollary

Let H be a Hopf quasigroup with antipode λ_H. Let σ be a normal 2-cocycle. Then the nonassociative bimonoid H^{σ}, is a Hopf quasigroup with antipode

$$\lambda_{H^{\sigma}} = l_{H^{\sigma}} \circ (H \otimes \eta_H)$$

where $l_{H^{\sigma}} = \mu_{H^{\sigma}} \circ (f \otimes \lambda_H \otimes f^{-1} \otimes H) \circ (H \otimes \delta_H \otimes H) \circ (\delta_H \otimes H)$, and

$$f = \sigma \circ (H \otimes \lambda_H) \circ \delta_H, \quad f^{-1} = \sigma^{-1} \circ (\lambda_H \otimes H) \circ \delta_H.$$

Two-cocycles and skew pairings

Definition

Let A and H be nonassociative bimonoids in $\mathcal{C}$. A skew pairing between A and H over K is a morphism $\tau : A \otimes H \to K$ such that the equalities

(a1) $\tau \circ (\mu_A \otimes H) = (\tau \otimes \tau) \circ (A \otimes c_{A,H} \otimes H) \circ (A \otimes A \otimes \delta_H)$,

(a2) $\tau \circ (A \otimes \mu_H) = (\tau \otimes \tau) \circ (A \otimes c_{A,H} \otimes H) \circ ((c_{A,A} \circ \delta_A) \otimes H \otimes H)$,

(a3) $\tau \circ (A \otimes \eta_H) = \varepsilon_A$,

(a4) $\tau \circ (\eta_A \otimes H) = \varepsilon_H$,

hold.

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(a2) $\tau\circ(A\otimes\mu_H) = (\tau\otimes\tau)\circ(A\otimes c_{A,H}\otimes H)\circ((c_{A,A}\circ\delta_A)\otimes H\otimes H)$,

(a3) $\tau\circ(A\otimes\eta_H) = \varepsilon_A$,

(a4) $\tau\circ(\eta_A\otimes H) = \varepsilon_H$,

hold.

Proposition

Let A, H be nonassociative bimonoids with left (right) division l_A (r_A) and l_H (r_H), respectively. Let $\tau : A\otimes H\to K$ be a skew pairing. Then τ is convolution invertible with inverse $\tau^{-1} = \tau\circ(\lambda_A\otimes H)$ ($\tau^{-1} = \tau\circ(\varrho_A\otimes H)$) . Moreover, the equalities

$$\tau^{-1}\circ(\eta_A\otimes H) = \varepsilon_H,\ \ \tau^{-1}\circ(A\otimes\eta_H) = \varepsilon_A$$

and

$$\tau^{-1}\circ(A\otimes\mu_H) = (\tau^{-1}\otimes\tau^{-1})\circ(A\otimes c_{A,H}\otimes H)\circ(\delta_A\otimes H\otimes H)$$

hold.

Proposition

Let A, H be nonassociative bimonoids with left (right) division l_A (r_A) and l_H (r_H), respectively. Then

$$A \otimes H = (A \otimes H, \eta_{A\otimes H}, \mu_{A\otimes H}, \varepsilon_{A\otimes H}, \delta_{A\otimes H})$$

is a nonassociative bimonoid with left division $l_{A\otimes H} = (l_A \otimes l_H) \circ (A \otimes c_{H,A} \otimes H)$ ($r_{A\otimes H} = (r_A \otimes r_H) \circ (A \otimes c_{H,A} \otimes H)$). Also, if A, H are left (right) Hopf quasigroups with left (right) antipodes λ_A (ϱ_A), λ_H (ϱ_H) respectively, $A \otimes H$ is a left (right) Hopf quasigroup with left (right) antipode $\lambda_{A\otimes H} = \lambda_A \otimes \lambda_H$ ($\varrho_{A\otimes H} = \varrho_A \otimes \varrho_H$).
Moreover, let $\tau : A \otimes H \to K$ be a skew pairing. The morphism $\omega = \varepsilon_A \otimes (\tau \circ c_{H,A}) \otimes \varepsilon_H$ is a normal 2-cocycle with convolution inverse $\omega^{-1} = \varepsilon_A \otimes (\tau^{-1} \circ c_{H,A}) \otimes \varepsilon_H$.

Proposition

Let A, H be nonassociative bimonoids with left (right) division l_A (r_A) and l_H (r_H), respectively. Then

$$A \otimes H = (A \otimes H, \eta_{A\otimes H}, \mu_{A\otimes H}, \varepsilon_{A\otimes H}, \delta_{A\otimes H})$$

is a nonassociative bimonoid with left division $l_{A\otimes H} = (l_A \otimes l_H) \circ (A \otimes c_{H,A} \otimes H)$ ($r_{A\otimes H} = (r_A \otimes r_H) \circ (A \otimes c_{H,A} \otimes H)$). Also, if A, H are left (right) Hopf quasigroups with left (right) antipodes λ_A (ϱ_A), λ_H (ϱ_H) respectively, $A \otimes H$ is a left (right) Hopf quasigroup with left (right) antipode $\lambda_{A\otimes H} = \lambda_A \otimes \lambda_H$ ($\varrho_{A\otimes H} = \varrho_A \otimes \varrho_H$).
Moreover, let $\tau : A\otimes H \to K$ be a skew pairing. The morphism $\omega = \varepsilon_A \otimes (\tau \circ c_{H,A}) \otimes \varepsilon_H$ is a normal 2-cocycle with convolution inverse $\omega^{-1} = \varepsilon_A \otimes (\tau^{-1} \circ c_{H,A}) \otimes \varepsilon_H$.

Corollary

Let A, H be Hopf quasigroups with antipodes λ_A, λ_H respectively. Then $A \otimes H$ is a Hopf quasigroup with antipode $\lambda_{A\otimes H} = \lambda_A \otimes \lambda_H$.
Moreover, let $\tau : A\otimes H \to K$ be a skew pairing. The morphism $\omega = \varepsilon_A \otimes (\tau \circ c_{H,A}) \otimes \varepsilon_H$ is a normal 2-cocycle with convolution inverse $\omega^{-1} = \varepsilon_A \otimes (\tau^{-1} \circ c_{H,A}) \otimes \varepsilon_H$.

Corollary

Let A, H be left (right) Hopf quasigroups with left (right) antipodes λ_A (ϱ_A), λ_H (ϱ_H) respectively. Let $\tau : A \otimes H \to K$ be a skew pairing. Then

$$A \bowtie_\tau H = (A \otimes H)^\omega$$

has a structure of left (right) Hopf quasigroup.

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Let A, H be Hopf quasigroups with antipodes λ_A, λ_H respectively. Let $\tau : A \otimes H \to K$ be a skew pairing. Then

$$A \bowtie_\tau H = (A \otimes H)^\omega$$

has a structure of Hopf quasigroup.

$$A \bowtie_{\tau} H = (A \otimes H, \eta_{A\bowtie_{\tau} H}, \mu_{A\bowtie_{\tau} H}, \varepsilon_{A\bowtie_{\tau} H}, \delta_{A\bowtie_{\tau} H}, \lambda_{A\bowtie_{\tau} H})$$

$$\eta_{A\bowtie_{\tau} H} = \eta_{A\otimes H},$$

$$\mu_{A\bowtie_{\tau} H} = (\mu_A \otimes \mu_H) \circ (A \otimes \tau \otimes A \otimes H \otimes \tau^{-1} \otimes H)$$

$$\circ (A \otimes \delta_{A\otimes H} \otimes A \otimes H \otimes H) \circ (A \otimes \delta_{A\otimes H} \otimes H) \circ (A \otimes c_{H,A} \otimes H),$$

$$\varepsilon_{A\bowtie_{\tau} H} = \varepsilon_{A\otimes H}, \quad \delta_{A\bowtie_{\tau} H} = \delta_{A\otimes H}$$

and

$$\lambda_{A\bowtie_{\tau} H} = (\tau^{-1} \otimes \lambda_A \otimes \lambda_H \otimes \tau) \circ (A \otimes H \otimes \delta_{A\otimes H}) \circ \delta_{A\otimes H}.$$

Example

Let $\mathbb{F}$ be a field such that Char($\mathbb{F}$)$\neq$ 2 and denote the tensor product over $\mathbb{F}$ as $\otimes$. Consider the nonabelian group $S_3 = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$ where σ_0 is the identity, $o(\sigma_1) = o(\sigma_2) = o(\sigma_3) = 2$ and $o(\sigma_4) = o(\sigma_5) = 3$. Let u be an additional element such that $u^2 = 1$. By

> **Chein O.**, Moufang loops of small order I, *Trans. Amer. Math. Soc.* **188** (1974), 31-51.

the set

$$L = M(S_3, 2) = \{\sigma_i u^{\alpha}\ ;\ \alpha = 0, 1\}$$

is an IP-loop where the product is defined by

$$\sigma_i u^{\alpha} . \sigma_j u^{\beta} = (\sigma_i^{\nu} \sigma_j^{\mu})^{\nu} u^{\alpha+\beta},\ \ \nu = (-1)^{\beta},\ \mu = (-1)^{\alpha+\beta}.$$

Then, $A = \mathbb{F}L$ is a cocommutative Hopf quasigroup.

Let H_4 be the 4-dimensional Taft Hopf algebra. This Hopf algebra is the smallest non commutative, non cocommutative Hopf algebra. The basis of H_4 is $\{1, x, y, w = xy\}$ and the multiplication table is defined by

	x	y	w
x	1	w	y
y	$-w$	0	0
w	$-y$	0	0

The costructure of H_4 is given by

$$\delta_{H_4}(x) = x \otimes x,\ \delta_{H_4}(y) = y \otimes x + 1 \otimes y,\ \delta_{H_4}(w) = w \otimes 1 + x \otimes w,$$

$$\varepsilon_{H_4}(x) = 1_{\mathbb{F}},\ \varepsilon_{H_4}(y) = \varepsilon_{H_4}(w) = 0,$$

and the antipode λ_{H_4} is described by

$$\lambda_{H_4}(x) = x,\ \lambda_{H_4}(y) = w,\ \lambda_{H_4}(w) = -y.$$

Then, $A \otimes H_4$ is a non commutative, non cocommutative Hopf quasigroup and the morphism $\tau : A \otimes H_4 \to \mathbb{F}$ defined by

$$\tau(\sigma_i u^{\alpha} \otimes z) = \begin{cases} 1 & \text{if} \quad z = 1 \\ (-1)^{\alpha} & \text{if} \quad z = x \\ 0 & \text{if} \quad z = y, w \end{cases}$$

is a skew pairing. Then,

$$\omega = \varepsilon_A \otimes (\tau \circ c_{H_4, A}) \otimes \varepsilon_{H_4}$$

is an invertible normal 2-cocycle. Finally,

$$A \bowtie_{\tau} H_4$$

is Hopf quasigroup defined by $(A \otimes H_4)^{\omega}$.

Quasitriangular Hopf quasigroups, skew pairings and projections

Definition

Let H be a Hopf quasigroup. We say that $M = (M, \varphi_M, \rho_M)$ is a left-left Yetter-Drinfeld module over H if (M, φ_M) is a left H-module and (M, ρ_M) is a left H-comodule which satisfies the following equalities:

(b1) $(\mu_H \otimes M) \circ (H \otimes c_{M,H}) \circ ((\rho_M \circ \varphi_M) \otimes H) \circ (H \otimes c_{H,M}) \circ (\delta_H \otimes M)$
$= (\mu_H \otimes \varphi_M) \circ (H \otimes c_{H,H} \otimes M) \circ (\delta_H \otimes \rho_M).$

(b2) $(\mu_H \otimes M) \circ (H \otimes c_{M,H}) \circ (\rho_M \otimes \mu_H)$
$= (\mu_H \otimes M) \circ (\mu_H \otimes c_{M,H}) \circ (H \otimes c_{M,H} \otimes H) \circ (\rho_M \otimes H \otimes H).$

(b3) $(\mu_H \otimes M) \circ (H \otimes \mu_H \otimes M) \circ (H \otimes H \otimes c_{M,H}) \circ (H \otimes \rho_M \otimes H)$
$= (\mu_H \otimes M) \circ (\mu_H \otimes c_{M,H}) \circ (H \otimes \rho_M \otimes H).$

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(b1) $(\mu_H \otimes M) \circ (H \otimes c_{M,H}) \circ ((\rho_M \circ \varphi_M) \otimes H) \circ (H \otimes c_{H,M}) \circ (\delta_H \otimes M)$
$= (\mu_H \otimes \varphi_M) \circ (H \otimes c_{H,H} \otimes M) \circ (\delta_H \otimes \rho_M)$.

(b2) $(\mu_H \otimes M) \circ (H \otimes c_{M,H}) \circ (\rho_M \otimes \mu_H)$
$= (\mu_H \otimes M) \circ (\mu_H \otimes c_{M,H}) \circ (H \otimes c_{M,H} \otimes H) \circ (\rho_M \otimes H \otimes H)$.

(b3) $(\mu_H \otimes M) \circ (H \otimes \mu_H \otimes M) \circ (H \otimes H \otimes c_{M,H}) \circ (H \otimes \rho_M \otimes H)$
$= (\mu_H \otimes M) \circ (\mu_H \otimes c_{M,H}) \circ (H \otimes \rho_M \otimes H)$.

Let M and N be two left-left Yetter-Drinfeld modules over H. We say that $f : M \to N$ is a morphism of left-left Yetter-Drinfeld modules if f is a morphism of H-modules and H-comodules.

We shall denote by ${}_H^H\mathcal{YD}$ the category of left-left Yetter-Drinfeld modules over H. Note that if H is a Hopf algebra, conditions (b2) and (b3) trivialize. In this case, ${}_H^H\mathcal{YD}$ is the classical category of left-left Yetter-Drinfeld modules over H.

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Let (M, φ_M, ρ_M) and (N, φ_N, ρ_N) two objects in ${}_{H}^{H}\mathcal{YD}$. Then $M \otimes N$, with the diagonal structure $\varphi_{M\otimes N}$ and the codiagonal costructure $\rho_{M\otimes N}$, is an object in ${}_{H}^{H}\mathcal{YD}$. Then $({}_{H}^{H}\mathcal{YD}, \otimes, K)$ is a strict monoidal category. If moreover λ_H is an isomorphism, $({}_{H}^{H}\mathcal{YD}, \otimes, K)$ is a strict braided monoidal category where the braiding t and its inverse are defined by

$$t_{M,N} = (\varphi_N \otimes M) \circ (H \otimes c_{M,N}) \circ (\rho_M \otimes N)$$

and

$$t^{-1}_{M,N} = c_{N,M} \circ ((\varphi_N \circ c_{N,H}) \otimes M) \circ (N \otimes \lambda_H^{-1} \otimes M) \circ (N \otimes \rho_M),$$

respectively

Definition

Let H be a Hopf quasigroup such that its antipode is an isomorphism. Let (D, u_D, m_D) be a unital magma in $\mathcal{C}$ such that (D, e_D, Δ_D) is a comonoid in $\mathcal{C}$, and let $s_D : D \to D$ be a morphism in $\mathcal{C}$. We say that the triple $(D, \varphi_D, \varrho_D)$ is a Hopf quasigroup in ${}^H_H\mathcal{YD}$ if:

(c1) The triple (D, φ_D, ρ_D) is a left-left Yetter-Drinfeld H-module.

(c2) The triple (D, u_D, m_D) is a unital magma in ${}^H_H\mathcal{YD}$.

(c3) The triple (D, e_D, Δ_D) is a a comonoid in ${}^H_H\mathcal{YD}$.

(c4) The following identities hold:

- (c4-1) $e_D \circ u_D = id_K$,
- (c4-2) $e_D \circ m_D = e_D \otimes e_D$,
- (c4-3) $\Delta_D \circ e_D = e_D \otimes e_D$,
- (c4-4) $\Delta_D \circ m_D = (m_D \otimes m_D) \circ (D \otimes t_{D,D} \otimes D) \circ (\Delta_D \otimes \Delta_D)$,

where $t_{D,D}$ is the braiding of ${}^H_H\mathcal{YD}$.

(c5) The following identities hold:

- (c5-1) $m_D \circ (s_D \otimes m_D) \circ (\Delta_D \otimes D) = e_D \otimes D = m_D \circ (D \otimes m_D) \circ (D \otimes s_D \otimes D) \circ (\Delta_D \otimes D)$.
- (c5-2) $m_D \circ (m_D \otimes D) \circ (D \otimes s_D \otimes D) \circ (D \otimes \Delta_D) = D \otimes e_D = \mu_D \circ (m_D \otimes s_D) \circ (D \otimes \Delta_D)$.

Note that under these conditions, s_D is a morphism in ${}_H^H\mathcal{YD}$.

Note that under these conditions, s_D is a morphism in ${}^{H}_{H}\mathcal{YD}$.

We know that if $(D, \varphi_D, \varrho_D)$ is a Hopf quasigroup in ${}^{H}_{H}\mathcal{YD}$, then

$$D \rtimes H = (D \otimes H, \eta_{D\rtimes H}, \mu_{D\rtimes H}, \varepsilon_{D\rtimes H}, \delta_{D\rtimes H}, \lambda_{D\rtimes H})$$

is a Hopf quasigroup in $\mathcal{C}$ (the bosonization of D), with the biproduct structure induced by the smash product coproduct, i.e.,

$$\eta_{D\rtimes H} = \eta_D \otimes \eta_H, \quad \mu_{D\rtimes H} = (\mu_D \otimes \mu_H) \circ (D \otimes \Psi_D^H \otimes H),$$

$$\varepsilon_{D\rtimes H} = \varepsilon_D \otimes \varepsilon_H, \quad \delta_{D\rtimes H} = (D \otimes \Gamma_D^H \otimes H) \circ (\delta_D \otimes \delta_H),$$

$$\lambda_{D\rtimes H} = \Psi_D^H \circ (\lambda_H \otimes \lambda_D) \circ \Gamma_D^H,$$

where the morphisms $\Gamma_D^H : D \otimes H \to H \otimes D$, $\Psi_D^H : H \otimes D \to D \otimes H$ are defined by

$$\Gamma_D^H = (\mu_H \otimes D) \circ (H \otimes c_{D,H}) \circ (\varrho_D \otimes H), \quad \Psi_D^H = (\varphi_D \otimes H) \circ (H \otimes c_{H,D}) \circ (\delta_H \otimes D).$$

Let H and B be Hopf quasigroups and let $f : H \to B$ and $g : B \to H$ be morphisms of Hopf quasigroups such that $g \circ f = id_H$. Then

$$q_H^B = id_B * (f \circ \lambda_H \circ g) : B \to B$$

is an idempotent morphism. Moreover, if B_H is the image of q_H^B and $p_H^B : B \to B_H$, $i_H^B : B_H \to B$ a factorization of q_H^B,

$$B_H \xrightarrow{\ i_H^B\ } B \underset{B \otimes \eta_H}{\overset{(B \otimes g) \circ \delta_B}{\rightrightarrows}} B \otimes H$$

is an equalizer diagram. As a consequence, the triple (B_H, u_{B_H}, m_{B_H}) is a unital magma where u_{B_H} and m_{B_H} are the factorizations, through the equalizer i_H^B, of the morphisms η_B and $\mu_B \circ (i_H^B \otimes i_H^B)$, respectively.

Definition

Let H be a Hopf quasigroup. A Hopf quasigroup projection over H is a triple (B, f, g) where B is a Hopf quasigroup, $f : H \to B$ and $g : B \to H$ are morphisms of Hopf quasigroups such that $g \circ f = id_H$, and the equality

$$q_H^B \circ \mu_B \otimes (B \otimes q_H^B) = q_H^B \circ \mu_B$$

holds.

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$$q_H^B \circ \mu_B \otimes (B \otimes q_H^B) = q_H^B \circ \mu_B$$

holds.

A morphism between two Hopf quasigroup projections (B, f, g) and (B', f', g') over H is a Hopf quasigroup morphism $h : B \to B'$ such that $h \circ f = f'$, $g' \circ h = g$. Hopf quasigroup projections over H and morphisms of Hopf quasigroup projections with the obvious composition form a category, denoted by $\mathcal{P}roj(H)$.

If (B, f, g) is a Hopf quasigroup projection over H,

$$B \otimes H \underset{B \otimes \varepsilon_H}{\overset{\mu_B \circ (B \otimes f)}{\rightrightarrows}} B \xrightarrow{p_H^B} B_H$$

is a coequalizer diagram. Moreover, the triple $(B_H, e_{B_H}, \Delta_{B_H})$ is a comonoid, where e_{B_H} and Δ_{B_H} are the factorizations, through the coequalizer p_H^B, of the morphisms ε_B and $(p_H^B \otimes p_H^B) \circ \delta_B$, respectively.

Definition

Let H be a Hopf quasigroup. We say that a Hopf quasigroup projection (B, f, g) over H is strong if it satisfies

$$p_H^B \circ \mu_B \circ (B \otimes \mu_B) \circ (i_H^B \otimes f \otimes i_H^B) = p_H^B \circ \mu_B \circ (\mu_B \otimes B) \circ (i_H^B \otimes f \otimes i_H^B),$$

$$p_H^B \circ \mu_B \circ (B \otimes \mu_B) \circ (f \otimes i_H^B \otimes i_H^B) = p_H^B \circ \mu_B \circ (\mu_B \otimes B) \circ (f \otimes i_H^B \otimes i_H^B),$$

$$p_H^B \circ \mu_B \circ (B \otimes \mu_B) \circ (f \otimes f \otimes i_H^B) = p_H^B \circ \mu_B \circ (\mu_B \otimes B) \circ (f \otimes f \otimes i_H^B).$$

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$$p_H^B \circ \mu_B \circ (B \otimes \mu_B) \circ (i_H^B \otimes f \otimes i_H^B) = p_H^B \circ \mu_B \circ (\mu_B \otimes B) \circ (i_H^B \otimes f \otimes i_H^B),$$

$$p_H^B \circ \mu_B \circ (B \otimes \mu_B) \circ (f \otimes i_H^B \otimes i_H^B) = p_H^B \circ \mu_B \circ (\mu_B \otimes B) \circ (f \otimes i_H^B \otimes i_H^B),$$

$$p_H^B \circ \mu_B \circ (B \otimes \mu_B) \circ (f \otimes f \otimes i_H^B) = p_H^B \circ \mu_B \circ (\mu_B \otimes B) \circ (f \otimes f \otimes i_H^B).$$

We will denote by $\mathcal{SP}roj(H)$ the category of strong Hopf quasigroup projections over H. The morphisms of $\mathcal{SP}roj(H)$ are the morphisms of $\mathcal{P}roj(H)$.

Let H be a Hopf quasigroup with invertible antipode. If D is a Hopf quasigroup in ${}^{H}_{H}\mathcal{YD}$, the triple $(D \rtimes H, f = \eta_D \otimes H, g = \varepsilon_D \otimes H)$ is a strong Hopf quasigroup projection over H. In this case $q_H^{D \rtimes H} = D \otimes \eta_H \otimes \varepsilon_H$. As a consequence,

$$p_H^{D \rtimes H} = D \otimes \varepsilon_H, \quad i_H^{D \rtimes H} = D \otimes \eta_H$$

and then

$$(D \rtimes H)_H = D.$$

Let H be a Hopf quasigroup with invertible antipode. If D is a Hopf quasigroup in ${}^{H}_{H}\mathcal{YD}$, the triple $(D \rtimes H, f = \eta_D \otimes H, g = \varepsilon_D \otimes H)$ is a strong Hopf quasigroup projection over H. In this case $q_H^{D\rtimes H} = D \otimes \eta_H \otimes \varepsilon_H$. As a consequence,

$$p_H^{D\rtimes H} = D \otimes \varepsilon_H, \quad i_H^{D\rtimes H} = D \otimes \eta_H$$

and then

$$(D \rtimes H)_H = D.$$

If (B, f, g) is a strong Hopf quasigroup projection over H, the triple $(B_H, \varphi_{B_H}, \varrho_{B_H})$ is a Hopf quasigroup in ${}^{H}_{H}\mathcal{YD}$, where

$$\varphi_{B_H} = p_H^B \circ \mu_B \circ (f \otimes i_H^B), \quad \rho_{B_H} = (g \otimes p_H^B) \circ \delta_B \circ i_H^B,$$

and $s_{B_H} = p_H^B \circ ((f \circ g) * \lambda_B) \circ i_H^B$.
Moreover,

$$w = \mu_B \circ (i_H^B \otimes f) : B_H \rtimes H \to B$$

is an isomorphism of Hopf quasigroups in $\mathcal{C}$ with inverse

$$w^{-1} = (p_H^B \otimes g) \circ \delta_B.$$

Theorem

Let H be a Hopf quasigroup in $\mathcal{C}$ with invertible antipode. The categories $\mathcal{SP}roj(H)$ and the category of Hopf quasigroups in ${}_{H}^{H}\mathcal{YD}$ are equivalent.

Definition

Let H be a Hopf quasigroup. We will say that H is quasitriangular if there exists a morphism $R : K \to H \otimes H$ such that:

(d1) $(\delta_H \otimes H) \circ R = (H \otimes H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (R \otimes R)$,

(d2) $(H \otimes \delta_H) \circ R = (\mu_H \otimes c_{H,H}) \circ (H \otimes c_{H,H} \otimes H) \circ (R \otimes R)$,

(d3) $\mu_{H \otimes H} \circ ((c_{H,H} \circ \delta_H) \otimes R) = \mu_{H \otimes H} \circ (R \otimes \delta_H)$,

(d4) $(\varepsilon_H \otimes H) \circ R = (H \otimes \varepsilon_H) \circ R = \eta_H$.

In the Hopf algebra setting, the morphism R is convolution invertible with inverse $R^{-1} = (\lambda_H \otimes H) \circ R$. In our nonassociative context we have that if $S = (\lambda_H \otimes H) \circ R$ and $T = (\lambda_H \otimes \lambda_H) \circ R$, the following identities hold:

$$R * S = S * R = \eta_H \otimes \eta_H, \quad S * T = T * S = \eta_H \otimes \eta_H.$$

Theorem

Let A, H be Hopf quasigroups and let $\tau : A \otimes H \to K$ be a skew pairing. Assume that H is quasitriangular with morphism R. Let $A \bowtie_{\tau} H$ be the Hopf quasigroup associated to τ and let $g : A \bowtie_{\tau} H \to H$ be the morphism defined by

$$g = (\tau \otimes \mu_H) \circ (A \otimes R \otimes H).$$

If the following equalities hold

$$\mu_H \circ (g \otimes H) = g \circ (A \otimes \mu_H), \tag{1}$$

$$\mu_H \circ (H \otimes g) = \mu_H \circ (\mu_H \circ H) \circ (H \otimes ((\tau \otimes H) \circ (A \otimes R)) \otimes H), \tag{2}$$

the triple $(A \bowtie_{\tau} H, f, g)$, where $f = \eta_A \otimes H$ is a strong Hopf quasigroup projection over H.

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Note that, if H is a Hopf algebra (1) and (2) always hold.

Corollary

In the conditions of the previous theorem, $A = (A \bowtie_{\tau} H)_H$ and, as a consequence, there exist an action φ_A and a coaction ρ_A such that (A, φ_A, ρ_A) is a Hopf quasigroup in ${}^{H}_{H}\mathcal{YD}$. Moreover, $A \rtimes H$ and $A \bowtie_{\tau} H$ are isomorphic Hopf quasigroups in $\mathcal{C}$.

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Example

Consider the Hopf quasigroup $A \bowtie_{\tau} H_4$ constructed in the previously. The Hopf algebra H_4 is quasitriangular.
Therefore, A admits a structure of Hopf quasigroup in the category ${}^{H_4}_{H_4}\mathcal{YD}$. Moreover, $A \bowtie_{\tau} H_4 \backsimeq A \rtimes H_4$.

Corollary

In the conditions of the previous theorem, $A = (A \bowtie_\tau H)_H$ and, as a consequence, there exist an action φ_A and a coaction ρ_A such that (A, φ_A, ρ_A) is a Hopf quasigroup in ${}^{H}_{H}\mathcal{YD}$. Moreover, $A \rtimes H$ and $A \bowtie_\tau H$ are isomorphic Hopf quasigroups in $\mathcal{C}$.

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Therefore, A admits a structure of Hopf quasigroup in the category ${}^{H_4}_{H_4}\mathcal{YD}$. Moreover, $A \bowtie_\tau H_4 \simeq A \rtimes H_4$.

Thank you