The fundamental theorem of Hopf modules in a non-associative setting

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Outline

Weak Hopf quasigroups

- From now on $\mathcal{C}$ denotes a strict braided monoidal category with tensor product denoted by $\otimes$ and unit object K. With c we will denote the braiding and we also assume that in $\mathcal{C}$ every idempotent morphism $q : Y \to Y$ splits, i.e., there exist an object Z (called the image of q) and morphisms $i : Z \to Y$ and $p : Y \to Z$ such that $q = i \circ p$ and $p \circ i = id_Z$.

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- For simplicity of notation, given three objects V, U, B in $\mathcal{C}$ and a morphism $f : V \to U$, we write

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- (A, η_A, μ_A) is a unital magma, i.e. $\eta_A : K \to A$ (unit) and $\mu_A : A \otimes A \to A$ (product) are morphisms in $\mathcal{C}$ such that

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- $(C, \varepsilon_C, \delta_C)$ is a comonoid with comultiplication δ_C and counit ε_C.
- If $f, g : C \to A$ are morphisms, $f * g$ denotes the convolution product.

$$f * g = \mu_A \circ (f \otimes g) \circ \delta_C.$$

Definition

A weak Hopf quasigroup H in $\mathcal{C}$ is a unital magma (H, η_H, μ_H) and a comonoid $(H, \varepsilon_H, \delta_H)$ such that the following axioms hold:

(a1) $\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H)$.

(a2)
$$\begin{aligned} &\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) \\ &= \varepsilon_H \circ \mu_H \circ (H \otimes \mu_H) \\ &= ((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes \delta_H \otimes H) \\ &= ((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes (c^{-1}_{H,H} \circ \delta_H) \otimes H). \end{aligned}$$

(a3)
$$\begin{aligned} &(\delta_H \otimes H) \circ \delta_H \circ \eta_H \\ &= (H \otimes \mu_H \otimes H) \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H)) \\ &= (H \otimes (\mu_H \circ c^{-1}_{H,H}) \otimes H) \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H)). \end{aligned}$$

(a4) There exists $\lambda_H : H \to H$ in $\mathcal{C}$ (called the antipode of H) such that, if we denote the morphisms $id_H * \lambda_H$ by Π_H^L (target morphism) and $\lambda_H * id_H$ by Π_H^R (source morphism),

(a4-1) $\Pi_H^L = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$.
(a4-2) $\Pi_H^R = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$.
(a4-3) $\lambda_H * \Pi_H^L = \Pi_H^R * \lambda_H = \lambda_H$.
(a4-4) $\mu_H \circ (\lambda_H \otimes \mu_H) \circ (\delta_H \otimes H) = \mu_H \circ (\Pi_H^R \otimes H)$.
(a4-5) $\mu_H \circ (H \otimes \mu_H) \circ (H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes H) = \mu_H \circ (\Pi_H^L \otimes H)$.
(a4-6) $\mu_H \circ (\mu_H \otimes \lambda_H) \circ (H \otimes \delta_H) = \mu_H \circ (H \otimes \Pi_H^L)$.
(a4-7) $\mu_H \circ (\mu_H \otimes H) \circ (H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H) = \mu_H \circ (H \otimes \Pi_H^R)$.

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- (a4-3) $\lambda_H * \Pi_H^L = \Pi_H^R * \lambda_H = \lambda_H$.
- (a4-4) $\mu_H \circ (\lambda_H \otimes \mu_H) \circ (\delta_H \otimes H) = \mu_H \circ (\Pi_H^R \otimes H)$.
- (a4-5) $\mu_H \circ (H \otimes \mu_H) \circ (H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes H) = \mu_H \circ (\Pi_H^L \otimes H)$.
- (a4-6) $\mu_H \circ (\mu_H \otimes \lambda_H) \circ (H \otimes \delta_H) = \mu_H \circ (H \otimes \Pi_H^L)$.
- (a4-7) $\mu_H \circ (\mu_H \otimes H) \circ (H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H) = \mu_H \circ (H \otimes \Pi_H^R)$.

Note that, if in the previous definition the triple (H, η_H, μ_H) is a monoid, we obtain the braided monoidal version (Alonso, Fernández and González (**Indiana U. Math. J. (2008)**)) of the notion of weak Hopf algebra introduced by Böhm, Nill and Szlachányi (**J. Algebra (1999)**). On the other hand, if ε_H and δ_H are morphisms of unital magmas, $\Pi_H^L = \Pi_H^R = \eta_H \otimes \varepsilon_H$. As a consequence, conditions (a2), (a3), (a4-1)-(a4-3) trivialize, and we get the monoidal version of the notion of Hopf quasigroup defined by Klim and Majid (**J. Algebra (2010)**).

Example

Let $\mathcal{B}$ be a bicategory and denote by $x, y, z, \cdots$ the 0 cells, by $f : x \to y$ the 1-cells and by $\alpha : f \Rightarrow g$ the 2-cells. For a 1-cell $f : x \to y$, x is called the source of f, represented by $s(f)$, and y is called the target of f, denoted by $t(f)$.

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A bicategory is normal if the unit isomorphisms

$$l_f : 1_{t(f)} \circ f \Rightarrow f, \quad r_f : f \circ 1_{s(f)} \Rightarrow f,$$

are identities. Every bicategory is biequivalent to a normal one.

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A 1-cell f is called an equivalence if there exists a 1-cell $g : t(f) \to s(f)$ and two isomorphisms $g \circ f \Rightarrow 1_{s(f)}$, $f \circ g \Rightarrow 1_{t(f)}$. In this case we will say that $g \in Inv(f)$ and, equivalently, $f \in Inv(g)$.

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We will say that a bigroupoid $\mathcal{B}$ is finite if the set of 0-cells $\mathcal{B}_0$ is finite and $\mathcal{B}(x, y)$ is small for all x, y.

Let $\mathcal{B}$ be a finite normal bigroupoid and denote by $\mathcal{B}_{\mathbf{1}}$ the set of 1-cells. Let $\mathbb{F}$ be a field and $\mathbb{F}\mathcal{B}$ the direct product

$$\mathbb{F}\mathcal{B} = \bigoplus_{f \in \mathcal{B}_{\mathbf{1}}} \mathbb{F}f.$$

The vector space $\mathbb{F}\mathcal{B}$ is a unital non-associative algebra where the product of two 1-cells is equal to their 1-cell composition if the latter is defined and 0 otherwise, i.e., $g.f = g \circ f$ if $s(g) = t(f)$ and $g.f = 0$ if $s(g) \neq t(f)$. The unit element is

$$1_{\mathbb{F}\mathcal{B}} = \sum_{x \in \mathcal{B}_{\mathbf{0}}} 1_x.$$

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Let $H = \mathbb{F}\mathcal{B}/I(\mathcal{B})$ be the quotient algebra where $I(\mathcal{B})$ is the ideal of $\mathbb{F}\mathcal{B}$ generated by

$$h - g \circ (f \circ h), \;\; p - (p \circ f) \circ g,$$

with $f \in \mathcal{B}_1$, $g \in Inv(f)$, and $h, p \in \mathcal{B}_1$ such that $t(h) = s(f)$, $t(f) = s(p)$. In what follows, for any 1-cell f we denote its class in H by $[f]$. If we define $[f]^{-1}$ by the class of $g \in Inv(f)$, we obtain that $[f]^{-1}$ is well-defined.

Therefore the vector space H with the product

$$\mu_H([g] \otimes [f]) = [g.f]$$

and the unit

$$\eta_H(1_{\mathbb{F}}) = \sum_{x \in \mathcal{B}_{\mathbf{0}}} [1_x]$$

is a unital non-associative algebra.
Also, it is easy to show that H is a coalgebra with coproduct

$$\delta_H([f]) = [f] \otimes [f]$$

and counit

$$\varepsilon_H([f]) = 1_{\mathbb{F}}.$$

Moreover, we have a morphism (the antipode) $\lambda_H : H \to H$ defined by

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Note that, if $\mathcal{B}_0 = \{x\}$ we obtain that H is a Hopf quasigroup. Moreover, if $|\mathcal{B}_0| > 1$ and the product defined in H is associative we have an example of weak Hopf algebra.

Proposition

The antipode of a weak Hopf quasigroup H is unique and leaves the unit and the counit invariant, i.e. $\lambda_H \circ \eta_H = \eta_H$ and $\varepsilon_H \circ \lambda_H = \varepsilon_H$.

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Definition

Let H be a weak Hopf quasigroup. We define the morphisms $\overline{\Pi}_H^L$ and $\overline{\Pi}_H^R$ by

$$\overline{\Pi}_H^L = (H \otimes (\varepsilon_H \circ \mu_H)) \circ ((\delta_H \circ \eta_H) \otimes H),$$

and

$$\overline{\Pi}_H^R = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)).$$

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$$\overline{\Pi}_H^R = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)).$$

Proposition

Let H be a weak Hopf quasigroup. The morphisms Π_H^L, Π_H^R, $\overline{\Pi}_H^L$ and $\overline{\Pi}_H^R$ are idempotent.

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$$\Pi_H^L \circ \overline{\Pi}_H^L = \Pi_H^L, \quad \Pi_H^L \circ \overline{\Pi}_H^R = \overline{\Pi}_H^R, \quad \overline{\Pi}_H^L \circ \Pi_H^L = \overline{\Pi}_H^L, \quad \overline{\Pi}_H^R \circ \Pi_H^L = \Pi_H^L,$$

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$$\Pi_H^L = \overline{\Pi}_H^R \circ \lambda_H = \lambda_H \circ \overline{\Pi}_H^L, \quad \Pi_H^R = \overline{\Pi}_H^L \circ \lambda_H = \lambda_H \circ \overline{\Pi}_H^R.$$

Proposition

Let H be a weak Hopf quasigroup. The antipode of H is antimultiplicative and anticomultiplicative, i.e. the following equalities hold:

$$\lambda_H \circ \mu_H = \mu_H \circ c_{H,H} \circ (\lambda_H \otimes \lambda_H),$$

$$\delta_H \circ \lambda_H = (\lambda_H \otimes \lambda_H) \circ c_{H,H} \circ \delta_H,$$

Proposition

Let H be a weak Hopf quasigroup. Put $H_L = Im(\Pi_H^L)$ and let $p_L : H \to H_L$ and $i_L : H_L \to H$ be the morphisms such that $\Pi_H^L = i_L \circ p_L$ and $p_L \circ i_L = id_{H_L}$. Then,

$$H_L \xrightarrow{\ i_L\ } H \underset{(H \otimes \Pi_H^L) \circ \delta_H}{\overset{\delta_H}{\rightrightarrows}} H \otimes H$$

is an equalizer diagram and

$$H \otimes H \underset{\mu_H \circ (H \otimes \Pi_H^L)}{\overset{\mu_H}{\rightrightarrows}} H \xrightarrow{\ p_L\ } H_L$$

is a coequalizer diagram.

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As a consequence, $(H_L, \eta_{H_L} = p_L \circ \eta_H, \mu_{H_L} = p_L \circ \mu_H \circ (i_L \otimes i_L))$ is a unital magma in $\mathcal{C}$. Also

$$(H_L, \varepsilon_{H_L} = \varepsilon_H \circ i_L, \delta_H = (p_L \otimes p_L) \circ \delta_H \circ i_L)$$

is a comonoid in $\mathcal{C}$.

Proposition

Let H be a weak Hopf quasigroup. The following identities hold:

$$\mu_H \circ ((\mu_H \circ (i_L \otimes H)) \otimes H) = \mu_H \circ (i_L \otimes \mu_H),$$

$$\mu_H \circ (H \otimes (\mu_H \circ (i_L \otimes H))) = \mu_H \circ ((\mu_H \circ (H \otimes i_L)) \otimes H),$$

$$\mu_H \circ (H \otimes (\mu_H \circ (H \otimes i_L))) = \mu_H \circ (\mu_H \otimes i_L).$$

As a consequence, the unital magma H_L is a monoid in $\mathcal{C}$.

Hopf modules for weak Hopf quasigroups

Definition

Let H be a weak Hopf quasigroup and M an object in $\mathcal{C}$. We say that (M, ϕ_M, ρ_M) is a right-right H-Hopf module if the following axioms hold:

(b1) The pair (M, ρ_M) is a right H-comodule, i.e., $\rho_M : M \to M \otimes H$ is a morphism such that $(M \otimes \varepsilon_H) \circ \rho_M = id_M$ and $(\rho_M \otimes H) \circ \rho_M = (M \otimes \delta_H) \circ \rho_M$.

(b2) The morphism $\phi_M : M \otimes H \to M$ satisfies:

- (b2-1) $\phi_M \circ (M \otimes \eta_H) = id_M$.
- (b2-2) $\rho_M \circ \phi_M = (\phi_M \otimes \mu_H) \circ (M \otimes c_{H,H} \otimes H) \circ (\rho_M \otimes \delta_H)$.

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(b4) $\phi_M \circ (\phi_M \otimes H) \circ (M \otimes \lambda_H \otimes H) \circ (M \otimes \delta_H) = \phi_M \circ (M \otimes \Pi_H^R)$.

(b5) $\phi_M \circ (\phi_M \otimes H) \circ (M \otimes \Pi_H^L \otimes H) \circ (M \otimes \delta_H) = \phi_M$.

Obviously,the triple $(H, \phi_H = \mu_H, \rho_H = \delta_H)$ is a right-right H-Hopf module. Moreover, if (M, ϕ_M, ρ_M) is a right-right H-Hopf module, the axiom (b5) is equivalent to

$$\phi_M \circ (\phi_M \otimes \Pi_H^R) \circ (M \otimes \delta_H) = \phi_M.$$

Proposition

Let H be a weak Hopf quasigroup and (M, ϕ_M, ρ_M) a right-right H-Hopf module. The endomorphism $q_M := \phi_M \circ (M \otimes \lambda_H) \circ \rho_M : M \to M$ satisfies

$$\rho_M \circ q_M = (M \otimes \Pi_H^L) \circ \rho_M \circ q_M$$

and, as a consequence, is an idempotent. Moreover, if M^{coH} (object of coinvariants) is the image of q_M and $p_M : M \to M^{coH}$, $i_M : M^{coH} \to M$ the morphisms such that $q_M = i_M \circ p_M$ and $id_{M^{coH}} = p_M \circ i_M$,

$$M^{coH} \xrightarrow{\ i_M\ } M \underset{(M \otimes \Pi_H^L) \circ \rho_M}{\overset{\rho_M}{\rightrightarrows}} M \otimes H$$

is an equalizer diagram.

Proposition

Let H be a weak Hopf quasigroup, (M, ϕ_M, ρ_M) a right-right H-Hopf module. The endomorphism

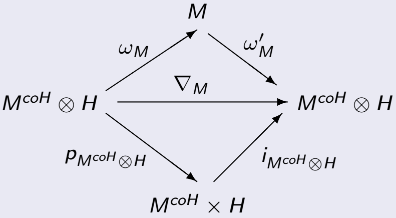
$$\nabla_M := ((p_M \circ \phi_M) \otimes H) \circ (i_M \otimes \delta_H) : M^{coH} \otimes H \to M^{coH} \otimes H$$

is idempotent and the equality

$$(M^{coH} \otimes \delta_H) \circ \nabla_M = (\nabla_M \otimes H) \circ (M^{coH} \otimes \delta_H),$$

hold.

Let be the morphisms $\omega_M : M^{coH} \otimes H \to M$, $\omega'_M : M \to M^{coH} \otimes H$ defined by $\omega_M = \phi_M \circ (i_M \otimes H)$ and $\omega'_M = (p_M \otimes H) \circ \rho_M$. Then, $\omega_M \circ \omega'_M = id_M$ and $\nabla_M = \omega'_M \circ \omega_M$. Also, we have a commutative diagram



where $M^{coH} \times H$ denotes the image of ∇_M and $p_{M^{coH} \otimes H}$, $i_{M^{coH} \otimes H}$ are the morphisms such that $p_{M^{coH} \otimes H} \circ i_{M^{coH} \otimes H} = id_{M^{coH} \times H}$ and $i_{M^{coH} \otimes H} \circ p_{M^{coH} \otimes H} = \nabla_M$. Therefore, the morphism

$$\alpha_M = p_{M^{coH} \otimes H} \circ \omega'_M$$

is an isomorphism of right H-modules (i.e., $\rho_{M^{coH} \times H} \circ \alpha_M = (\alpha_M \otimes H) \circ \rho_M$) with inverse $\alpha_M^{-1} = \omega_M \circ i_{M^{coH} \otimes H}$. The comodule structure of $M^{coH} \times H$ is

$$\rho_{M^{coH} \times H} = (p_{M^{coH} \otimes H} \otimes H) \circ (M^{coH} \otimes \delta_H) \circ i_{M^{coH} \otimes H}.$$

Proposition

Let H be a weak Hopf quasigroup and (M, ϕ_M, ρ_M), (N, ϕ_N, ρ_N) right-right H-Hopf modules. If there exists a right H-comodule isomorphism $\alpha : M \to N$, the triple

$$(M, \phi_M^{\alpha} = \alpha^{-1} \circ \phi_N \circ (\alpha \otimes H), \rho_M)$$

(the α-deformation of (M, ϕ_M, ρ_M)) is a right-right H-Hopf module.

Proposition

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Proposition

Let H be a weak Hopf quasigroup, (M, ϕ_M, ρ_M) a right-right H-Hopf module. The triple $(M^{coH} \times H, \phi_{M^{coH} \times H}, \rho_{M^{coH} \times H})$ where

$$\phi_{M^{coH} \times H} = p_{M^{coH} \otimes H} \circ (M^{coH} \otimes \mu_H) \circ (i_{M^{coH} \otimes H} \otimes H),$$

and

$$\rho_{M^{coH} \times H} = (p_{M^{coH} \otimes H} \otimes H) \circ (M^{coH} \otimes \delta_H) \circ i_{M^{coH} \otimes H},$$

is a right-right H-Hopf module.

Proposition

Let H be a weak Hopf quasigroup, (M, ϕ_M, ρ_M) be a right-right H-Hopf module and $\alpha_M : M \to M^{coH} \times H$ be the isomorphism of right H-comodules defined previously. The triple $(M, \phi_M^{\alpha_M}, \rho_M)$ is a right-right H-Hopf module and the identity

$$\phi_M^{\alpha_M} = \phi_M \circ (q_M \otimes \mu_H) \circ (\rho_M \otimes H)$$

holds and

$$q_M^{\alpha_M} = q_M,$$

where $q_M^{\alpha_M} = \phi_M^{\alpha_M} \circ (M \otimes \lambda_H) \circ \rho_M$ is the idempotent morphism associated to the Hopf module $(M, \phi_M^{\alpha_M}, \rho_M)$. Then, $(M, \phi_M^{\alpha_M}, \rho_M)$ has the same object of coinvariants of (M, ϕ_M, ρ_M). Moreover, for $(M, \phi_M^{\alpha_M}, \rho_M)$ we have that

$$\nabla_M^{\alpha_M} = \nabla_M$$

and then, for $(M, \phi_M^{\alpha_M}, \rho_M)$, the associated isomorphism between M and $M^{coH} \times H$ is α_M. Finally,

$$(\phi_M^{\alpha_M})^{\alpha_M} = \phi_M^{\alpha_M}$$

holds.

Let H be a weak Hopf quasigroup. The triple $(H, \phi_H = \mu_H, \rho_H = \delta_H)$ is a right-right H-Hopf module and $\phi_H^{\alpha_H} = \phi_H$ because

$$\phi_H^{\alpha_H} = \mu_H \circ (\Pi_H^L \otimes \mu_H) \circ (\delta_H \otimes H) = \mu_H.$$

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Proposition

Let H be a weak Hopf quasigroup and let (M, ϕ_M, ρ_M) be an object in $\mathcal{M}_H^H$. Then, for $(M^{coH} \times H, \phi_{M^{coH} \times H}, \rho_{M^{coH} \times H})$ the identity $\phi_{M^{coH} \times H}^{\alpha_{M^{coH} \times H}} = \phi_{M^{coH} \times H}$ holds.

Definition

Let H be a weak Hopf quasigroup and let (M, ϕ_M, ρ_M) and (N, ϕ_N, ρ_N) be right-right H-Hopf modules. A morphism $f : M \to N$ is said to be H-quasilineal if the following identity holds

$$\phi_N^{\alpha_N} \circ (f \otimes H) = f \circ \phi_M^{\alpha_M}.$$

A morphism of right-right H-Hopf modules between M and N is a morphism $f : M \to N$ such that is both a morphism of right H-comodules and H-quasilineal. The collection of all right H-Hopf modules with their morphisms forms a category which will be denoted by

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Theorem: Fundamental Theorem of Hopf modules

Let H be a weak Hopf quasigroup and let (M, ϕ_M, ρ_M) be an object in $\mathcal{M}_H^H$. Then, the right-right H-Hopf modules (M, ϕ_M, ρ_M) and $(M^{coH} \times H, \phi_{M^{coH} \times H}, \rho_{M^{coH} \times H})$ are isomorphic in $\mathcal{M}_H^H$.

Strong Hopf modules for weak Hopf quasigroups

From now on we assume that $\mathcal{C}$ admits coequalizers. Note that this assumption implies that every idempotent morphism splits. Remember that H_L is a monoid and then we can consider $\mathcal{C}_{H_L}$, the category of right H_L-modules.

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Let (N, ψ_N) be an object in $\mathcal{C}_{H_L}$ and consider the coequalizer diagram

$$N \otimes H_L \otimes H \underset{N \otimes \varphi_H}{\overset{\psi_N \otimes H}{\rightrightarrows}} N \otimes H \xrightarrow{n_N} N \otimes_{H_L} H$$

where $\varphi_H = \mu_H \circ (H \otimes i_L)$.

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where $\varphi_H = \mu_H \circ (H \otimes i_L)$.

The morphism $\Gamma_N = (n_N \otimes H) \circ (N \otimes \delta_H)$ is such that $\Gamma_N \circ (\psi_N \otimes H) = \Gamma_N \circ (N \otimes \varphi_H)$. Then, there exists a unique morphism

$$\rho_{N \otimes_{H_L} H} : N \otimes_{H_L} H \to (N \otimes_{H_L} H) \otimes H$$

such that $\rho_{N \otimes_{H_L} H} \circ n_N = \Gamma_N$. The pair $(N \otimes_{H_L} H, \rho_{N \otimes_{H_L} H})$ is a right H-comodule.

On the other hand, if $T_N = n_N \circ (N \otimes \mu_H)$,

$$T_N \circ (\psi_N \otimes H \otimes H) = T_N \circ (N \otimes (\varphi_H \otimes H))$$

and then, if $- \otimes H$ preserves coequalizers, there exists a unique morphism

$$\phi_{N\otimes_{H_L} H} : (N \otimes_{H_L} H) \otimes H \to N \otimes_{H_L} H$$

such that

$$\phi_{N\otimes_{H_L} H} \circ (n_N \otimes H) = T_N.$$

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If $- \otimes H$ preserves coequalizers, the triple $(N \otimes_{H_L} H, \phi_{N \otimes_{H_L} H}, \rho_{N \otimes_{H_L} H})$ is a right-right H-Hopf module.

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and then, if $- \otimes H$ preserves coequalizers, there exists a unique morphism

$$\phi_{N \otimes_{H_L} H} : (N \otimes_{H_L} H) \otimes H \to N \otimes_{H_L} H$$

such that

$$\phi_{N \otimes_{H_L} H} \circ (n_N \otimes H) = T_N.$$

If $- \otimes H$ preserves coequalizers, the triple $(N \otimes_{H_L} H, \phi_{N \otimes_{H_L} H}, \rho_{N \otimes_{H_L} H})$ is a right-right H-Hopf module.

Also

$$\phi_{N \otimes_{H_L} H}^{\alpha_{N \otimes_{H_L} H}} = \phi_{N \otimes_{H_L} H}$$

holds.

Proposition

Let H be a weak Hopf quasigroup such that the functor $-\otimes H$ preserves coequalizers. There exists a functor

$$F : \mathcal{C}_{H_L} \to \mathcal{M}_H^H,$$

called the induction functor, defined on objects by

$$F((N, \psi_N)) = (N \otimes_{H_L} H, \phi_{N\otimes_{H_L} H}, \rho_{N\otimes_{H_L} H})$$

and for morphisms by $F(f) = f \otimes_{H_L} H$.

Definition

Let H be a weak Hopf quasigroup. With $\mathcal{S}M_H^H$ we will denote the full subcategory of $\mathcal{M}_H^H$ whose objects are the right-right H-Hopf modules (M, ϕ_M, ρ_M) such that the following equality holds:

(c1) $\phi_M \circ ((\phi_M \circ (M \otimes i_L)) \otimes H) = \phi_M \circ (M \otimes (\mu_H \circ (i_L \otimes H)))$,

The objects of $\mathcal{S}M_H^H$ will be called right-right strong H-Hopf modules.

Definition

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Note that if H is a Hopf quasigroup, (c1) holds because $i_L = \eta_H$. Then, in this particular setting $\mathcal{S}M_H^H = \mathcal{M}_H^H$.
Also the previous equality holds trivially for any Hopf module associated to a weak Hopf algebra.

Proposition

Let H be a weak Hopf quasigroup such that the functor $-\otimes H$ preserves coequalizers. The induction functor $F : \mathcal{C}_{H_L} \to \mathcal{M}_H^H$ factorizes through the category $\mathcal{S}M_H^H$.

Proposition

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Proposition

Let H be a weak Hopf quasigroup. There exists a functor

$$G : \mathcal{S}M_H^H \to \mathcal{C}_{H_L},$$

called the functor of coinvariants, defined on objects by

$$G((M, \phi_M, \rho_M)) = (M^{coH}, \psi_{M^{coH}} = p_M \circ \phi_M \circ (i_M \otimes i_L))$$

and for morphisms by $G(g) = g^{coH}$.

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and for morphisms by $G(g) = g^{coH}$.

Proposition

Let H be a weak Hopf quasigroup such that the functor $-\otimes H$ preserves coequalizers. For any $(M, \phi_M, \rho_M) \in \mathcal{S}M_H^H$, the objects $M^{coH} \otimes_{H_L} H$ and $M^{coH} \times H$ are isomorphic as right-right H-Hopf modules.

Theorem

For any weak Hopf quasigroup H such the the functor $-\otimes H$ preserves coequalizers, $F \dashv G$ and the category $\mathcal{S}M_H^H$ is equivalent to the category $\mathcal{C}_{H_L}$.

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Thank you