The Next Step: Weak Hopf Quasigroups

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Outline

Definition.

A bicategory $\mathcal{B}$ consists of :

(i) A class $\mathcal{B}_0$, whose elements x are called 0-cells.

(ii) For each x, $y \in \mathcal{B}_0$, a category $\mathcal{B}(x,y)$ whose objects $f : x \to y$ are called 1-cells and whose morphisms $\alpha : f \Rightarrow g$ are called 2-cells. The composition of 2-cells is called the vertical composition of 2-cells and if f is a 1-cell in $\mathcal{B}(x,y)$, x is called the source of f, represented by $s(f)$, and y is called the target of f, denoted by $t(f)$.

(ii) For each $x \in \mathcal{B}_0$, an object $1_x \in \mathcal{B}(x,x)$, called the identity of x; and for each $x, y, z \in \mathcal{B}_0$, a functor

$$\mathcal{B}(y,z) \times \mathcal{B}(x,y) \to \mathcal{B}(x,z)$$

which in objects is called the 1-cell composition $(g,f) \mapsto g \circ f$, and on arrows is called horizontal composition of 2-cells:

$$f, f' \in \mathcal{B}(x,y),\;\; g, g' \in \mathcal{B}(y,z),\; \alpha : f \Rightarrow f',\; \beta : g \Rightarrow g'$$

$$(\beta, \alpha) \mapsto \beta \bullet \alpha : g \circ f \Rightarrow g' \circ f'.$$

(iv) For each $f \in \mathcal{B}(x, y)$, $g \in \mathcal{B}(y, z)$, $h \in \mathcal{B}(z, w)$, an associative isomorphisms

$$\xi_{h,g,f} : (h \circ g) \circ f \Rightarrow h \circ (g \circ f);$$

and for each 1-cell f, unit isomorphisms

$$l_f : 1_{t(f)} \circ f \Rightarrow f, \; r_f : f \circ 1_{s(f)} \Rightarrow f,$$

satisfying the following coherence axioms:

(iv-1) The morphism $\xi_{h,g,f}$ is natural in h, f and g and l_f, r_f are natural in f.
(iv-2) Pentagon axiom: $\xi_{k,h,g \circ f} \circ \xi_{k \circ h,g,f} = (id_k \bullet \xi_{h,g,f}) \circ \xi_{k,h \circ g,f} \circ (\xi_{k,h,g} \bullet id_f)$.
(iv-3) Triangle axiom: $r_g \bullet id_f = (id_g \bullet l_f) \circ \xi_{g,\mathbf{1}_{t(f)},f}$.

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A 1-cell f is called an equivalence if there exists a 1-cell $g : t(f) \rightarrow s(f)$ and two isomorphisms $g \circ f \Rightarrow 1_{s(f)}$, $f \circ g \Rightarrow 1_{t(f)}$. In this case we will say that $g \in Inv(f)$ and, equivalently, $f \in Inv(g)$.

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Examples.

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- There is a 2-category **Cat** whose 0-cells are small categories, whose 1-cells are functors, and whose 2-cells are natural transformations.
- Let $(\mathcal{C}, \otimes, K, a, l, r)$ be a monoidal category. Then we can construct a bicategory $\mathcal{B}$ in the following way:
  - $\mathcal{B}_{\mathbf{0}} = \{1\}$.
  - $\mathcal{B}(1,1) = \mathcal{C}_{\mathbf{0}}$. The composition in $\mathcal{B}(1,1)$ is given by $V \circ U = V \otimes U$.
  - The 2-cells between to objects of $\mathcal{C}$ (1-cells), U, V are the morphisms in $Hom_{\mathcal{C}}(U, V)$. The horizontal composition in 2-cells is the tensor product.

- Another example of bicategory can be found in the theory of bimodules. We chose as 0-cells the rings with identity. A 1-cell from the ring R to the ring S is a (R, S)-bimodule M. Given two (R, S)-bimodules M, N, we chosse as a 2-cells from M to N the (R, S)-linear mappings $f : M \to N$. This yields a bicategory $Bim(R, S)$ of (R, S)-bimodules and their morphisms.
  For a third ring T, the composition
  $$Bim(R, S) \times Bim(S, T) \to Bim(R, T)$$
  is defined by the tensor product. If M is a (R, S)-bimodule and N is a (S, T)-bimodule, we have that $M \otimes_S N$ is a (R, T)-bimodule.

Definition.

A bigroupoid is a bicategory where every 1-cell is an equivalence and every 2-cell is an isomorphism. We will say that a bigroupoid $\mathcal{B}$ is finite if $\mathcal{B}_0$ is finite and $\mathcal{B}(x, y)$ is small for all x, y.

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Note that if $\mathcal{B}$ is a bigroupoid where $\mathcal{B}(x, y)$ is small for all x, y and we pick a finite number of 0-cells, considering the full sub-bicategory generated by these 0-cells, we have an example of finite bigroupoid.

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Definition.

A weak Hopf quasigroup H in $\mathcal{C}$ is a unital magma (H, η_H, μ_H) and a comonoid $(H, \varepsilon_H, \delta_H)$ such that the following axioms hold:

(i) $\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ \delta_{H \otimes H}$.

(ii) $\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = \varepsilon_H \circ \mu_H \circ (H \otimes \mu_H)$
$= ((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes \delta_H \otimes H)$
$= ((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes (c^{-1}_{H,H} \circ \delta_H) \otimes H)$.

(iii) $(\delta_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \mu_H \otimes H) \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H))$
$= (H \otimes (\mu_H \circ c^{-1}_{H,H}) \otimes H) \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H))$.

(iv) There exists $\lambda_H : H \to H$ in $\mathcal{C}$ (called the antipode of H) such that, if we denote the morphisms $id_H * \lambda_H$ by Π_H^L (target morphism) and $\lambda_H * id_H$ by Π_H^R (source morphism),

- (iv-1) $\Pi_H^L = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$.
- (iv-2) $\Pi_H^R = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$.
- (iv-3) $\lambda_H * \Pi_H^L = \Pi_H^R * \lambda_H = \lambda_H$.
- (iv-4) $\mu_H \circ (\lambda_H \otimes \mu_H) \circ (\delta_H \otimes H) = \mu_H \circ (\Pi_H^R \otimes H)$.
- (iv-5) $\mu_H \circ (H \otimes \mu_H) \circ (H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes H) = \mu_H \circ (\Pi_H^L \otimes H)$.
- (iv-6) $\mu_H \circ (\mu_H \otimes \lambda_H) \circ (H \otimes \delta_H) = \mu_H \circ (H \otimes \Pi_H^L)$.
- (iv-7) $\mu_H \circ (\mu_H \otimes H) \circ (H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H) = \mu_H \circ (H \otimes \Pi_H^R)$.

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- (iv-3) $\lambda_H * \Pi_H^L = \Pi_H^R * \lambda_H = \lambda_H$.
- (iv-4) $\mu_H \circ (\lambda_H \otimes \mu_H) \circ (\delta_H \otimes H) = \mu_H \circ (\Pi_H^R \otimes H)$.
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Note that, if in the previous definition the triple (H, η_H, μ_H) is a monoid, we obtain the notion of weak Hopf algebra introduced by Böhm, Nill and Szlachányi. On the other hand, if ε_H and δ_H are morphisms of unital magmas, $\Pi_H^L = \Pi_H^R = \eta_H \otimes \varepsilon_H$. As a consequence, conditions (ii), (iii), (iv-1)-(iv-3) trivialize, and we get the notion of Hopf quasigroup defined by Klim and Majid.

Let $\mathcal{B}$ be a finite normal bigroupoid and denote by $\mathcal{B}_1$ the set of 1-cells. Let $\mathbb{F}$ be a field and $\mathbb{F}\mathcal{B}$ the direct product

$$\mathbb{F}\mathcal{B} = \bigoplus_{f \in \mathcal{B}_{\mathbf{1}}} \mathbb{F}f.$$

The vector space $\mathbb{F}\mathcal{B}$ is a unital non-associative algebra where the product of two 1-cells is equal to their 1-cell composition if the latter is defined and 0 otherwise, i.e., $g.f = g \circ f$ if $s(g) = t(f)$ and $g.f = 0$ if $s(g) \neq t(f)$. The unit element is

$$1_{\mathbb{F}\mathcal{B}} = \sum_{x \in \mathcal{B}_{\mathbf{0}}} 1_x.$$

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Let $H = \mathbb{F}\mathcal{B}/I(\mathcal{B})$ be the quotient algebra where $I(\mathcal{B})$ is the ideal of $\mathbb{F}\mathcal{B}$ generated by

$$h - g \circ (f \circ h),\ p - (p \circ f) \circ g,$$

with $f \in \mathcal{B}_1$, $g \in Inv(f)$, and $h, p \in \mathcal{B}_1$ such that $t(h) = s(f)$, $t(f) = s(p)$. In what follows, for any 1-cell f we denote its class in H by $[f]$. If we assume that $I(\mathcal{B})$ is a proper ideal and for $[f]$ we define $[f]^{-1}$ by the class of $g \in Inv(f)$, we obtain that $[f]^{-1}$ is well-defined.

Therefore the vector space H with the product

$$\mu_H([g] \otimes [f]) = [g.f]$$

and the unit

$$\eta_H(1_{\mathbb{F}}) = \sum_{x \in \mathcal{B}_{\mathbf{0}}} [1_x]$$

is a unital non-associative algebra.
Also, it is easy to show that H is a coalgebra with coproduct and counit

$$\delta_H([f]) = [f] \otimes [f], \ \varepsilon_H([f]) = 1_{\mathbb{F}}.$$

Moreover, we have a morphism (the antipode) $\lambda_H : H \to H$ defined by

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Note that, if $\mathcal{B}_0 = \{x\}$ we obtain that H is a Hopf quasigroup. Moreover, if $|\mathcal{B}_0| > 1$ and the product defined in H is associative we have an example of weak Hopf algebra.

Proposition

The antipode of a weak Hopf quasigroup H is unique and leaves the unit and the counit invariant, i.e. $\lambda_H \circ \eta_H = \eta_H$ and $\varepsilon_H \circ \lambda_H = \varepsilon_H$.

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Definition.

Let H be a weak Hopf quasigroup. We define the morphisms $\overline{\Pi}_H^L$ and $\overline{\Pi}_H^R$ by

$$\overline{\Pi}_H^L = (H \otimes (\varepsilon_H \circ \mu_H)) \circ ((\delta_H \circ \eta_H) \otimes H),$$

and

$$\overline{\Pi}_H^R = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)).$$

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Proposition.

Let H be a weak Hopf quasigroup. The morphisms Π_H^L, Π_H^R, $\overline{\Pi}_H^L$ and $\overline{\Pi}_H^R$ are idempotent.

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Let H be a weak Hopf quasigroup. The following identities hold:

$$\Pi_H^L \circ \overline{\Pi}_H^L = \Pi_H^L, \quad \Pi_H^L \circ \overline{\Pi}_H^R = \overline{\Pi}_H^R, \quad \overline{\Pi}_H^L \circ \Pi_H^L = \overline{\Pi}_H^L, \quad \overline{\Pi}_H^R \circ \Pi_H^L = \Pi_H^L,$$

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$$\Pi_H^L = \overline{\Pi}_H^R \circ \lambda_H = \lambda_H \circ \overline{\Pi}_H^L, \quad \Pi_H^R = \overline{\Pi}_H^L \circ \lambda_H = \lambda_H \circ \overline{\Pi}_H^R.$$

Proposition.

Let H be a weak Hopf quasigroup. The antipode of H is antimultiplicative and anticomultiplicative, i.e. the following equalities hold:

$$\lambda_H \circ \mu_H = \mu_H \circ c_{H,H} \circ (\lambda_H \otimes \lambda_H),$$

$$\delta_H \circ \lambda_H = (\lambda_H \otimes \lambda_H) \circ c_{H,H} \circ \delta_H,$$

Proposition.

Let H be a weak Hopf quasigroup. Put $H_L = Im(\Pi_H^L)$ and let $p_L : H \to H_L$ and $i_L : H_L \to H$ be the morphisms such that $\Pi_H^L = i_L \circ p_L$ and $p_L \circ i_L = id_{H_L}$. Then,

$$\begin{tikzcd} H_L \arrow[r, "i_L"] & H \arrow[r, shift left, "\delta_H"] \arrow[r, shift right, "(H\otimes \Pi_H^L)\circ\delta_H"'] & H\otimes H \end{tikzcd}$$

is an equalizer diagram and

$$\begin{tikzcd} H\otimes H \arrow[r, shift left, "\mu_H"] \arrow[r, shift right, "\mu_H\circ(H\otimes \Pi_H^L)"'] & H \arrow[r, "p_L"] & H_L \end{tikzcd}$$

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is a coequalizer diagram.

As a consequence, $(H_L, \eta_{H_L} = p_L \circ \eta_H, \mu_{H_L} = p_L \circ \mu_H \circ (i_L \otimes i_L))$ is a unital magma in $\mathcal{C}$. Also

$$(H_L, \varepsilon_{H_L} = \varepsilon_H \circ i_L, \delta_H = (p_L \otimes p_L) \circ \delta_H \circ i_L)$$

is a comonoid in $\mathcal{C}$.

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Let H be a weak Hopf quasigroup. The following identities hold:

$$\mu_H \circ ((\mu_H \circ (i_L \otimes H)) \otimes H) = \mu_H \circ (i_L \otimes \mu_H),$$

$$\mu_H \circ (H \otimes (\mu_H \circ (i_L \otimes H))) = \mu_H \circ ((\mu_H \circ (H \otimes i_L)) \otimes H),$$

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As a consequence, the unital magma H_L is a monoid in $\mathcal{C}$.

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As a consequence, the unital magma H_L is a monoid in $\mathcal{C}$.

As in the weak Hopf algebra setting, H_L is a Frobenius separable monoid.

Definition.

Let H be a weak Hopf quasigroup and M an object in $\mathcal{C}$. We say that (M, ϕ_M, ρ_M) is a right-right H-Hopf module if the following axioms hold:

(i) The pair (M, ρ_M) is a right H-comodule, i.e., $\rho_M : M \to M \otimes H$ is a morphism such that $(M \otimes \varepsilon_H) \circ \rho_M = id_M$ and $(\rho_M \otimes H) \circ \rho_M = (M \otimes \delta_H) \circ \rho_M$.

(ii) The morphism $\phi_M : M \otimes H \to M$ satisfies:

(ii-1) $\phi_M \circ (M \otimes \eta_H) = id_M$.
(ii-2) $\rho_M \circ \phi_M = (\phi_M \otimes \mu_H) \circ (M \otimes c_{H,H} \otimes H) \circ (\rho_M \otimes \delta_H)$.

(iii) $\phi_M \circ (\phi_M \otimes \lambda_H) \circ (M \otimes \delta_H) = \phi_M \circ (M \otimes \Pi_H^L)$.

(iv) $\phi_M \circ (\phi_M \otimes H) \circ (M \otimes \lambda_H \otimes H) \circ (M \otimes \delta_H) = \phi_M \circ (M \otimes \Pi_H^R)$.

(v) $\phi_M \circ (\phi_M \otimes H) \circ (M \otimes \Pi_H^L \otimes H) \circ (M \otimes \delta_H) = \phi_M$.

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(v) $\phi_M \circ (\phi_M \otimes H) \circ (M \otimes \Pi_H^L \otimes H) \circ (M \otimes \delta_H) = \phi_M$.

Obviously,the triple $(H, \phi_H = \mu_H, \rho_H = \delta_H)$ is a right-right H-Hopf module. Moreover, if (M, ϕ_M, ρ_M) is a right-right H-Hopf module, the axiom (v) is equivalent to

$$\phi_M \circ (\phi_M \otimes \Pi_H^R) \circ (M \otimes \delta_H) = \phi_M.$$

Proposition.

Let H be a weak Hopf quasigroup and (M, ϕ_M, ρ_M) a right-right H-Hopf module. The endomorphism $q_M := \phi_M \circ (M \otimes \lambda_H) \circ \rho_M : M \to M$ satisfies

$$\rho_M \circ q_M = (M \otimes \Pi_H^L) \circ \rho_M \circ q_M$$

and, as a consequence, is an idempotent. Moreover, if M^{coH} (object of coinvariants) is the image of q_M and $p_M : M \to M^{coH}$, $i_M : M^{coH} \to M$ the morphisms such that $q_M = i_M \circ p_M$ and $id_{M^{coH}} = p_M \circ i_M$,

$$\begin{tikzcd} M^{coH} \arrow[r, "i_M"] & M \arrow[r, shift left, "\rho_M"] \arrow[r, shift right, "(M \otimes \Pi_H^L) \circ \rho_M"'] & M \otimes H \end{tikzcd}$$

is an equalizer diagram.

Proposition.

Let H be a weak Hopf quasigroup, (M, ϕ_M, ρ_M) a right-right H-Hopf module. The endomorphism

$$\nabla_M := (p_M \otimes H) \circ \rho_M \circ \phi_M \circ (i_M \otimes H) : M^{coH} \otimes H \to M^{coH} \otimes H$$

is idempotent and the equalities

$$\nabla_M = ((p_M \circ \phi_M) \otimes H) \circ (i_M \otimes \delta_H),$$

$$(M^{coH} \otimes \delta_H) \circ \nabla_M = (\nabla_M \otimes H) \circ (M^{coH} \otimes \delta_H),$$

hold.

Let be the morphisms $\omega_M : M^{coH} \otimes H \to M$, $\omega'_M : M \to M^{coH} \otimes H$ defined by $\omega_M = \phi_M \circ (i_M \otimes H)$ and $\omega'_M = (p_M \otimes H) \circ \rho_M$. Then, $\omega_M \circ \omega'_M = id_M$ and $\nabla_M = \omega'_M \circ \omega_M$. Also, we have a commutative diagram

$$\begin{tikzcd}
 & M \arrow[dr, "\omega'_M"] & \\
M^{coH} \otimes H \arrow[ur, "\omega_M"] \arrow[rr, "\nabla_M"] \arrow[dr, "p_{M^{coH} \otimes H}"'] & & M^{coH} \otimes H \\
 & M^{coH} \times H \arrow[ur, "i_{M^{coH} \otimes H}"'] &
\end{tikzcd}$$

where $M^{coH} \times H$ denotes the image of ∇_M and $p_{M^{coH} \otimes H}$, $i_{M^{coH} \otimes H}$ are the morphisms such that $p_{M^{coH} \otimes H} \circ i_{M^{coH} \otimes H} = id_{M^{coH} \times H}$ and $i_{M^{coH} \otimes H} \circ p_{M^{coH} \otimes H} = \nabla_M$. Therefore, the morphism

$$\alpha_M = p_{M^{coH} \otimes H} \circ \omega'_M$$

is an isomorphism of right H-modules (i.e., $\rho_{M^{coH} \times H} \circ \alpha_M = (\alpha_M \otimes H) \circ \rho_M$) with inverse $\alpha_M^{-1} = \omega_M \circ i_{M^{coH} \otimes H}$. The comodule structure of $M^{coH} \times H$ is the one induced by the isomorphism α_M and it is equal to

$$\rho_{M^{coH} \times H} = (p_{M^{coH} \otimes H} \otimes H) \circ (M^{coH} \otimes \delta_H) \circ i_{M^{coH} \otimes H}.$$

Proposition.

Let H be a weak Hopf quasigroup and (M, ϕ_M, ρ_M), (N, ϕ_N, ρ_N) right-right H-Hopf modules. If there exists a right H-comodule isomorphism $\alpha : M \to N$, the triple

$$(M, \phi_M^{\alpha} = \alpha^{-1} \circ \phi_N \circ (\alpha \otimes H), \rho_M)$$

(the α-deformation of (M, ϕ_M, ρ_M)) is a right-right H-Hopf module.

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Proposition.

Let H be a weak Hopf quasigroup, (M, ϕ_M, ρ_M) a right-right H-Hopf module. The triple $(M^{coH} \times H, \phi_{M^{coH} \times H}, \rho_{M^{coH} \times H})$ where

$$\phi_{M^{coH} \times H} = p_{M^{coH} \otimes H} \circ (M^{coH} \otimes \mu_H) \circ (i_{M^{coH} \otimes H} \otimes H),$$

and

$$\rho_{M^{coH} \times H} = (p_{M^{coH} \otimes H} \otimes H) \circ (M^{coH} \otimes \delta_H) \circ i_{M^{coH} \otimes H},$$

is a right-right H-Hopf module.

Proposition.

Let H be a weak Hopf quasigroup, (M, ϕ_M, ρ_M) be a right-right H-Hopf module and $\alpha_M : M \to M^{coH} \times H$ be the isomorphism of right H-comodules defined previously. The triple $(M, \phi_M^{\alpha_M}, \rho_M)$ is a right-right H-Hopf module and the identity

$$\phi_M^{\alpha_M} = \phi_M \circ (q_M \otimes \mu_H) \circ (\rho_M \otimes H)$$

holds and

$$q_M^{\alpha_M} = q_M,$$

where $q_M^{\alpha_M} = \phi_M^{\alpha_M} \circ (M \otimes \lambda_H) \circ \rho_M$ is the idempotent morphism associated to the Hopf module $(M, \phi_M^{\alpha_M}, \rho_M)$. Then, $(M, \phi_M^{\alpha_M}, \rho_M)$ has the same object of coinvariants of (M, ϕ_M, ρ_M). Moreover, for $(M, \phi_M^{\alpha_M}, \rho_M)$ we have that

$$\nabla_M^{\alpha_M} = \nabla_M$$

and then, for $(M, \phi_M^{\alpha_M}, \rho_M)$, the associated isomorphism between M and $M^{coH} \times H$ is α_M. Finally,

$$(\phi_M^{\alpha_M})^{\alpha_M} = \phi_M^{\alpha_M}$$

holds.

Let H be a weak Hopf quasigroup. The triple $(H, \phi_H = \mu_H, \rho_H = \delta_H)$ is a right-right H-Hopf module and $\phi_H^{\alpha_H} = \phi_H$ because

$$\phi_H^{\alpha_H} = \mu_H \circ (\Pi_H^L \otimes \mu_H) \circ (\delta_H \otimes H) = \mu_H.$$

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Proposition.

Let H be a weak Hopf quasigroup and let (M, ϕ_M, ρ_M) be an object in $\mathcal{M}_H^H$. Then, for $(M^{coH} \times H, \phi_{M^{coH} \times H}, \rho_{M^{coH} \times H})$ the identity $\phi_{M^{coH} \times H}^{\alpha_{M^{coH} \times H}} = \phi_{M^{coH} \times H}$ holds.

Definition.

Let H be a weak Hopf quasigroup and let (M, ϕ_M, ρ_M) and (N, ϕ_N, ρ_N) be right-right H-Hopf modules. A morphism $f : M \to N$ is said to be H-quasilineal if the following identity holds

$$\phi_N^{\alpha_N} \circ (f \otimes H) = f \circ \phi_M^{\alpha_M}.$$

A morphism of right-right H-Hopf modules between M and N is a morphism $f : M \to N$ such that is both a morphism of right H-comodules and H-quasilineal. The collection of all right H-Hopf modules with their morphisms forms a category which will be denoted by

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Theorem: Fundamental Theorem of Hopf modules

Let H be a weak Hopf quasigroup and let (M, ϕ_M, ρ_M) be an object in $\mathcal{M}_H^H$. Then, the right-right H-Hopf modules (M, ϕ_M, ρ_M) and $(M^{coH} \times H, \phi_{M^{coH} \times H}, \rho_{M^{coH} \times H})$ are isomorphic in $\mathcal{M}_H^H$.

From now on we assume that $\mathcal{C}$ admits coequalizers. Remember that H_L is a monoid and then we can consider $\mathcal{C}_{H_L}$, the category of right H_L-modules.

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Let (N, ψ_N) be an object in $\mathcal{C}_{H_L}$ and consider the coequalizer diagram

$$\begin{tikzcd} N \otimes H_L \otimes H \arrow[r, shift left, "\psi_N \otimes H"] \arrow[r, shift right, "N \otimes \varphi_H"'] & N \otimes H \arrow[r, "n_N"] & N \otimes_{H_L} H \end{tikzcd}$$

where $\varphi_H = \mu_H \circ (H \otimes i_L)$.

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Let (N, ψ_N) be an object in $\mathcal{C}_{H_L}$ and consider the coequalizer diagram

$$N \otimes H_L \otimes H \underset{N \otimes \varphi_H}{\overset{\psi_N \otimes H}{\rightrightarrows}} N \otimes H \xrightarrow{n_N} N \otimes_{H_L} H$$

where $\varphi_H = \mu_H \circ (H \otimes i_L)$.

The morphism $\Gamma_N = (n_N \otimes H) \circ (N \otimes \delta_H)$ is such that $\Gamma_N \circ (\psi_N \otimes H) = \Gamma_N \circ (N \otimes \varphi_H)$. Then, there exists a unique morphism

$$\rho_{N \otimes_{H_L} H} : N \otimes_{H_L} H \to (N \otimes_{H_L} H) \otimes H$$

such that $\rho_{N \otimes_{H_L} H} \circ n_N = \Gamma_N$. The pair $(N \otimes_{H_L} H, \rho_{N \otimes_{H_L} H})$ is a right H-comodule.

On the other hand, if $T_N = n_N \circ (N \otimes \mu_H)$,

$$T_N \circ (\psi_N \otimes H \otimes H) = T_N \circ (N \otimes (\varphi_H \otimes H))$$

and then, if $- \otimes H$ preserves coequalizers, there exists a unique morphism

$$\phi_{N \otimes_{H_L} H} : (N \otimes_{H_L} H) \otimes H \to N \otimes_{H_L} H$$

such that

$$\phi_{N \otimes_{H_L} H} \circ (n_N \otimes H) = T_N.$$

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If $- \otimes H$ preserves coequalizers, the triple $(N \otimes_{H_L} H, \phi_{N \otimes_{H_L} H}, \rho_{N \otimes_{H_L} H})$ is a right-right H-Hopf module.

On the other hand, if $T_N = n_N \circ (N \otimes \mu_H)$,

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$$\phi_{N \otimes_{H_L} H} : (N \otimes_{H_L} H) \otimes H \to N \otimes_{H_L} H$$

such that

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If $- \otimes H$ preserves coequalizers, the triple $(N \otimes_{H_L} H, \phi_{N \otimes_{H_L} H}, \rho_{N \otimes_{H_L} H})$ is a right-right H-Hopf module.

Also

$$\phi_{N \otimes_{H_L} H}^{\alpha_{N \otimes_{H_L} H}} = \phi_{N \otimes_{H_L} H}$$

holds.

Proposition.

Let H be a weak Hopf quasigroup such that the functor $-\otimes H$ preserve coequalizers. There exists a functor

$$F : \mathcal{C}_{H_L} \to \mathcal{M}_H^H,$$

called the induction functor, defined on objects by

$$F((N, \psi_N)) = (N \otimes_{H_L} H, \phi_{N\otimes_{H_L} H}, \rho_{N\otimes_{H_L} H})$$

and for morphisms by $F(f) = f \otimes_{H_L} H$.

Definition.

Let H be a weak Hopf quasigroup. With $\mathcal{S}M_H^H$ we will denote the full subcategory of $\mathcal{M}_H^H$ whose objects are the right-right H-Hopf modules (M, ϕ_M, ρ_M) such that the following equality holds:

(1) $\phi_M \circ ((\phi_M \circ (M \otimes i_L)) \otimes H) = \phi_M \circ (M \otimes (\mu_H \circ (i_L \otimes H))),$

The objects of $\mathcal{S}M_H^H$ will be called right-right strong H-Hopf modules.

Definition.

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Note that if H is a Hopf quasigroup, (1) holds because $i_L = \eta_H$. Then, in this particular setting $\mathcal{S}M_H^H = \mathcal{M}_H^H$.
Also the previous equality holds trivially for any Hopf module associated to a weak Hopf algebra.

Proposition.

Let H be a weak Hopf quasigroup such that the functor $-\otimes H$ preserve coequalizers. The induction functor $F : \mathcal{C}_{H_L} \to \mathcal{M}_H^H$ factorizes through the category $\mathcal{S}M_H^H$.

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Proposition.

Let H be a weak Hopf quasigroup. There exists a functor

$$G : \mathcal{S}M_H^H \to \mathcal{C}_{H_L},$$

called the functor of coinvariants, defined on objects by

$$G((M, \phi_M, \rho_M)) = (M^{coH}, \psi_{M^{coH}} = p_M \circ \phi_M \circ (i_M \otimes i_L))$$

and for morphisms by $G(g) = g^{coH}$.

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and for morphisms by $G(g) = g^{coH}$.

Proposition.

Let H be a weak Hopf quasigroup such that the functor $-\otimes H$ preserve coequalizers. For any $(M, \phi_M, \rho_M) \in \mathcal{S}M_H^H$, the objects $M^{coH} \otimes_{H_L} H$ and $M^{coH} \times H$ are isomorphic as right-right H-Hopf modules.

Theorem.

For any weak Hopf quasigroup H such the the functor $-\otimes H$ preserve coequalizers, $F \dashv G$ and the category $\mathcal{S}M_H^H$ is equivalent to the category $\mathcal{C}_{H_L}$.

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