The Fundamental Theorem of Hopf Modules

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Outline

In the following $\mathcal{C}$ denotes a strict braided monoidal category with tensor product $\otimes$, unit object K and braiding c. From now on we also assume in $\mathcal{C}$ that every idempotent morphism splits.

Definition.

Let H be a Hopf algebra and M an object in $\mathcal{C}$. We say that (M, ϕ_M, ρ_M) is a right-right H-Hopf module (or a right H-Hopf module for short) if the following axioms hold:

(i) The pair (M, ϕ_M) is a right H-module

(ii) The pair (M, ρ_M) is a right H-comodule.

(iii) $\rho_M \circ \phi_M = (\phi_M \otimes \mu_H) \circ (M \otimes c_{H,H} \otimes H) \circ (\rho_M \otimes \delta_H)$, i.e. ϕ_M is a morphism of right H-comodules with the codiagonal coaction on $M \otimes H$.

Obviously, if H is a Hopf algebra, the triple $(H, \phi_H = \mu_H, \rho_H = \delta_H)$ is a right-right H-Hopf module.

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Obviously, if H is a Hopf algebra, the triple $(H, \phi_H = \mu_H, \rho_H = \delta_H)$ is a right-right H-Hopf module.

A morphism between two right H-Hopf modules is a morphism in $\mathcal{C}$ of right H-modules and right H-comodules. With $\mathcal{M}_H^H$ we will denote the category of right H-Hopf modules and morphisms of right H-Hopf modules.

Proposition.

Let H be a Hopf algebra. If (M, ϕ_M, ρ_M) is a right H-Hopf module, the morphism

$$q_M := \phi_M \circ (M \otimes \lambda_H) \circ \rho_M : M \to M$$

is idempotent. Moreover, if M^{coH} (object of coinvariants) is the image of q_M and $p_M : M \to M^{coH}$, $i_M : M^{coH} \to M$ are the morphisms such that $q_M = i_M \circ p_M$ and $id_{M^{coH}} = p_M \circ i_M$,

$$\begin{tikzcd} M^{coH} \arrow[r, "i_M"] & M \arrow[r, shift left, "\rho_M"] \arrow[r, shift right, "M \otimes \eta_H"'] & M \otimes H \end{tikzcd}$$

is an equalizer diagram. Moreover, the following identities hold:

$$\rho_M \circ q_M = q_M \otimes \eta_H,$$

$$\phi_M \circ (q_M \otimes H) \circ \rho_M = id_M,$$

$$\rho_M \circ \phi_M \circ (i_M \otimes H) = (\phi_M \otimes H) \circ (i_M \otimes \delta_H),$$

$$q_M \circ \phi_M \circ (i_M \otimes H) = i_M \otimes \varepsilon_H.$$

Let If (M, ϕ_M, ρ_M) be a right H-Hopf module. If we define the morphisms

$$\omega_M : M^{coH} \otimes H \to M, \quad \omega'_M : M \to M^{coH} \otimes H,$$

by $\omega_M = \phi_M \circ (i_M \otimes H)$ and $\omega'_M = (p_M \otimes H) \circ \rho_M$. Then, $\omega_M \circ \omega'_M = id_M$ and $\omega'_M \circ \omega_M = id_{M^{coH} \otimes H}$. Therefore, ω_M is an isomorphism and it is also a morphism of right H-Hopf modules if we consider

$$\phi_{M^{coH} \otimes H} = M^{coH} \otimes \mu_H, \;\; \rho_{M^{coH} \otimes H} = M^{coH} \otimes \delta_H.$$

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Theorem. (Fundamental Theorem of Hopf modules)

Let H be a Hopf algebra and assume that (M, ϕ_M, ρ_M) is an object in the category $\mathcal{M}_H^H$. Then, the right-right H-Hopf modules (M, ϕ_M, ρ_M) and $(M^{coH} \otimes H, \phi_{M^{coH} \otimes H}, \rho_{M^{coH} \otimes H})$ are isomorphic in $\mathcal{M}_H^H$.

Let N be an object in $\mathcal{C}$. Then,

$$(N \otimes H, \phi_{N\otimes H} = N \otimes \mu_H, \rho_{N\otimes H} = N \otimes \delta_H)$$

is an object in $\mathcal{M}_H^H$. Also, if $f : N \to P$ is a morphism in $\mathcal{C}$, $f \otimes H$ is a morphism in $\mathcal{M}_H^H$ between $(N \otimes H, \phi_{N\otimes H}, \rho_{N\otimes H})$ and $(N \otimes H, \phi_{N\otimes H}, \rho_{N\otimes H})$.

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Theorem.

Let H be a Hopf algebra. There exists a functor

$$F : \mathcal{C} \to \mathcal{M}_H^H,$$

called the induction functor, defined on objects by $F(N) = (N \otimes H, \phi_{N\otimes H}, \rho_{N\otimes H})$ and for morphisms by $F(f) = f \otimes H$.

Let (M, ϕ_M, ρ_M) be a right-right H-Hopf module and consider the object of coinvariants M^{coH}. Let $g : M \rightarrow T$ be a morphism in $\mathcal{M}_H^H$. Using the comodule morphism condition we obtain that $\rho_T \circ g \circ i_M = (g \circ i_M) \otimes \eta_H$ and this implies that there exists a unique morphism $g^{coH} : M^{coH} \rightarrow T^{coH}$ such that

$$i_T \circ g^{coH} = g \circ i_M.$$

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Theorem.

Let H be a Hopf algebra. There exists a functor

$$G : \mathcal{M}_H^H \rightarrow \mathcal{C},$$

called the functor of coinvariants, defined on objects by $G((M, \phi_M, \rho_M)) = M^{coH}$ and for morphisms by $G(g) = g^{coH}$.

Theorem.

Let H be a Hopf algebra. The induction functor $F : \mathcal{C} \to \mathcal{M}_H^H$ is left adjoint of the functor of coinvariants $G : \mathcal{M}_H^H \to \mathcal{C}$.

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Proof.

The unit and the counit of the adjunction are defined by:

$$u_N = p_{N\otimes H} \circ (N \otimes \eta_H) = id_N : N \to (N \otimes H)^{coH} = N$$

$$v_M = \omega_M : M^{coH} \otimes H \to M$$

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Theorem.

Let H be a Hopf algebra. The induction functor $F : \mathcal{C} \to \mathcal{M}_H^H$ and the functor of coinvariants $G : \mathcal{M}_H^H \to \mathcal{C}$ induce an equivalence of categories.

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Obviously, as in Hopf algebra setting, if H is a weak Hopf algebra, the triple

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Proposition.

Let H be a weak Hopf algebra. If (M, ϕ_M, ρ_M) is a right H-Hopf module, the morphism

$$q_M := \phi_M \circ (M \otimes \lambda_H) \circ \rho_M : M \to M$$

satisfies $\rho_M \circ q_M = (M \otimes \Pi_H^L) \circ \rho_M \circ q_M$ and, as a consequence, q_M is idempotent. Moreover, if M^{coH} (object of coinvariants) is the image of q_M and $p_M : M \to M^{coH}$, $i_M : M^{coH} \to M$ are the morphisms such that $q_M = i_M \circ p_M$ and $id_{M^{coH}} = p_M \circ i_M$,

$$\begin{tikzcd} M^{coH} \arrow[r, "i_M"] & M \arrow[r, shift left, "\rho_M"] \arrow[r, shift right, "(M \otimes \Pi_H^L) \circ \rho_M"'] & M \otimes H \end{tikzcd}$$

is an equalizer diagram. Also,

$$\begin{tikzcd} M^{coH} \arrow[r, "i_M"] & M \arrow[r, shift left, "\rho_M"] \arrow[r, shift right, "(M \otimes \overline{\Pi}_H^R) \circ \rho_M"'] & M \otimes H \end{tikzcd}$$

is an equalizer diagram.

Moreover, the following identities hold:

$$\phi_M \circ (q_M \otimes H) \circ \rho_M = id_M,$$

$$\rho_M \circ \phi_M \circ (i_M \otimes H) = (\phi_M \otimes H) \circ (i_M \otimes \delta_H),$$

$$p_M \circ \phi_M \circ (i_M \otimes H) = p_M \circ \phi_M \circ (i_M \otimes \Pi_H^L).$$

In the conditions of the previous proposition, the morphism

$$\nabla_M := (p_M \otimes H) \circ \rho_M \circ \phi_M \circ (i_M \otimes H) : M^{coH} \otimes H \to M^{coH} \otimes H$$

is idempotent and the equalities

$$\nabla_M = ((p_M \circ \phi_M) \otimes H) \circ (i_M \otimes \delta_H),$$

$$(M^{coH} \otimes \delta_H) \circ \nabla_M = (\nabla_M \otimes H) \circ (M^{coH} \otimes \delta_H).$$

hold. If we define the morphisms

$$\omega_M : M^{coH} \otimes H \to M, \quad \omega'_M : M \to M^{coH} \otimes H,$$

by $\omega_M = \phi_M \circ (i_M \otimes H)$ and $\omega'_M = (p_M \otimes H) \circ \rho_M$. Then, $\omega_M \circ \omega'_M = id_M$ and $\nabla_M = \omega'_M \circ \omega_M$.

Also, we have a commutative diagram

$$
\begin{tikzcd}
 & M \arrow[dr, "\omega'_M"] & \\
M^{coH} \otimes H \arrow[ur, "\omega_M"] \arrow[rr, "\nabla_M"] \arrow[dr, "p_{M^{coH} \otimes H}"'] & & M^{coH} \otimes H \\
 & M^{coH} \times H \arrow[ur, "i_{M^{coH} \otimes H}"'] &
\end{tikzcd}
$$

where $M^{coH} \times H$ denotes the image of ∇_M and $p_{M^{coH} \otimes H}$, $i_{M^{coH} \otimes H}$ are the morphisms such that $p_{M^{coH} \otimes H} \circ i_{M^{coH} \otimes H} = id_{M^{coH} \times H}$ and $i_{M^{coH} \otimes H} \circ p_{M^{coH} \otimes H} = \nabla_M$.

Therefore, the morphism

$$\alpha_M = p_{M^{coH} \otimes H} \circ \omega'_M : M \to M^{coH} \times H$$

is an isomorphism of right H-Hopf modules with inverse $\alpha_M^{-1} = \omega_M \circ i_{M^{coH} \otimes H}$. The comodule structure of $M^{coH} \times H$ is

$$\rho_{M^{coH} \times H} = (p_{M^{coH} \otimes H} \otimes H) \circ (M^{coH} \otimes \delta_H) \circ i_{M^{coH} \otimes H}.$$

and the module structure is defined by

$$\phi_{M^{coH} \times H} = p_{M^{coH} \otimes H} \circ (M^{coH} \otimes \mu_H) \circ (i_{M^{coH} \otimes H} \otimes H).$$

Let H be a weak Hopf algebra in a monoidal braided category. If $H_L = Im(\Pi_H^L)$, $p_L : H \to H_L$, and $i_L : H_L \to H$ are the morphisms such that $\Pi_H^L = i_L \circ p_L$ and $p_L \circ i_L = id_{H_L}$,

$$H_L \xrightarrow{i_L} H \underset{(H \otimes \Pi_H^L) \circ \delta_H}{\overset{\delta_H}{\rightrightarrows}} H \otimes H$$

is an equalizer diagram and

$$H \otimes H \underset{\mu_H \circ (H \otimes \Pi_H^L)}{\overset{\mu_H}{\rightrightarrows}} H \xrightarrow{p_L} H_L$$

is a coequalizer diagram.

As a consequence,

$$(H_L, \eta_{H_L} = p_L \circ \eta_H, \mu_{H_L} = p_L \circ \mu_H \circ (i_L \otimes i_L))$$

is a monoid in $\mathcal{C}$ and

$$(H_L, \varepsilon_{H_L} = \varepsilon_H \circ i_L, \delta_H = (p_L \otimes p_L) \circ \delta_H \circ i_L)$$

is a comonoid in $\mathcal{C}$. Also, we have that

$$\delta_H \circ \mu_H \circ (i_L \otimes H) = (\mu_H \otimes H) \circ (i_L \otimes \delta_H),$$

$$\delta_H \circ \mu_H \circ (H \otimes i_L) = (\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes i_L).$$

From now on we assume that $\mathcal{C}$ admits coequalizers.

With $\mathcal{C}_{H_L}$ we will denote the category of right H_L-modules. Note that the pair $(H, \psi_H = \mu_H \circ (H \otimes i_L))$ is a right H_L-module.

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Let (N, ψ_N) be an object in $\mathcal{C}_{H_L}$ and consider the coequalizer diagram

$$N \otimes H_L \otimes H \underset{N \otimes \varphi_H}{\overset{\psi_N \otimes H}{\rightrightarrows}} N \otimes H \xrightarrow{n_N} N \otimes_{H_L} H$$

where $\varphi_H = \mu_H \circ (i_L \otimes H)$. We have

$$(n_N \otimes H) \circ (\psi_N \otimes \delta_H) = ((n_N \circ (N \otimes \varphi_H)) \otimes H) \circ (N \otimes H_L \otimes \delta_H) = (n_N \otimes H) \circ (N \otimes (\delta_H \circ \varphi_H))$$

and, as a consequence, there exists a unique morphism

$$\rho_{N \otimes_{H_L} H} : N \otimes_{H_L} H \to (N \otimes_{H_L} H) \otimes H$$

such that

$$\rho_{N \otimes_{H_L} H} \circ n_N = (n_N \otimes H) \circ (N \otimes \delta_H).$$

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On the other hand, we have

$$n_N \circ (\psi_N \otimes \mu_H) = n_N \circ (N \otimes (\mu_H \circ (i_L \otimes \mu_H))) = n_N \circ (N \otimes (\mu_H \circ (\varphi_H \otimes H))),$$

and then, if the functor $- \otimes H$ preserves coequalizers, there exists a unique morphism

$$\phi_{N\otimes_{H_L} H} : (N \otimes_{H_L} H) \otimes H \to N \otimes_{H_L} H$$

such that

$$\phi_{N\otimes_{H_L} H} \circ (n_N \otimes H) = n_N \circ (N \otimes \mu_H).$$

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is a right-right H-Hopf module.

On the other hand, if $f : N \to P$ is a morphism in $\mathcal{C}_{H_L}$, we have that

$$n_P \circ (f \otimes H) \circ (\psi_N \otimes H) = n_P \circ (f \otimes H) \circ (N \otimes \varphi_H)$$

and, as a consequence, there exists an unique morphism $f \otimes_{H_L} H : N \otimes_{H_L} H \to P \otimes_{H_L} H$ such that

$$n_P \circ (f \otimes H) = (f \otimes_{H_L} H) \circ n_N.$$

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Theorem.

Let H be a weak Hopf algebra such that the functor $- \otimes H$ preserves coequalizers. There exists a functor

$$F : \mathcal{C}_{H_L} \to \mathcal{M}_H^H,$$

called the induction functor, defined on objects by

$$F((N, \psi_N)) = (N \otimes_{H_L} H, \phi_{N \otimes_{H_L} H}, \rho_{N \otimes_{H_L} H})$$

and for morphisms by $F(f) = f \otimes_{H_L} H$.

Let (M, ϕ_M, ρ_M) be a right-right H-Hopf module. Then, the pair

$$(M^{coH}, \psi_{M^{coH}} = p_M \circ \phi_M \circ (i_M \otimes i_L))$$

is a right H_L-module. Let $g : M \to T$ be a morphism in $\mathcal{M}_H^H$. Using the comodule morphism condition we obtain that

$$\rho_T \circ g \circ i_M = (T \otimes \overline{\Pi}_H^R) \circ \rho_T \circ g \circ i_M$$

and this implies that there exists a unique morphism $g^{coH} : M^{coH} \to T^{coH}$ such that

$$i_T \circ g^{coH} = g \circ i_M.$$

Then,

$$i_T \circ g^{coH} \circ p_M = g \circ q_M = q_T \circ g$$

and, as a consequence,

$$g^{coH} \circ p_M = p_T \circ g.$$

Therefore, we obtain that g^{coH} is a morphism of right H_L-modules.

Theorem.

Let H be a weak Hopf algebra. There exists a functor

$$G : \mathcal{M}_H^H \to \mathcal{C}_{H_L},$$

called the functor of coinvariants, defined on objects by $G((M, \phi_M, \rho_M)) = (M^{coH}, \psi_{M^{coH}})$ and for morphisms by $G(g) = g^{coH}$.

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Proposition.

Let H be a weak Hopf algebra such that the functor $- \otimes H$ preserves coequalizers. For any $(M, \phi_M, \rho_M) \in \mathcal{M}_H^H$, the objects $M^{coH} \otimes_{H_L} H$ and $M^{coH} \times H$ are isomorphic right-right H-Hopf modules.

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Theorem. (Fundamental Theorem of Hopf modules)

Let H be a weak Hopf algebras and assume that (M, ϕ_M, ρ_M) is an object in the category $\mathcal{M}_H^H$. If the functor functor $- \otimes H$ preserves coequalizers, the right-right H-Hopf modules (M, ϕ_M, ρ_M) and $(M^{coH} \otimes_{H_L} H, \phi_{M^{coH} \otimes_{H_L} H}, \rho_{M^{coH} \otimes_{H_L} H})$ are isomorphic in $\mathcal{M}_H^H$.

Theorem.

Let H be a weak Hopf algebra such that the functor $- \otimes H$ preserves coequalizers. The induction functor $F : \mathcal{C}_{H_L} \to \mathcal{M}_H^H$ is left adjoint of the functor of coinvariants $G : \mathcal{M}_H^H \to \mathcal{C}_{H_L}$.

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Proof.

For any right H_L-module (N, ψ_N) define the unit of the adjunction by

$$u_N : N \to GF(N) = (N \otimes_{H_L} H)^{coH}$$

as the unique morphism such that

$$i_{N\otimes_{H_L} H} \circ u_N = n_N \circ (N \otimes \eta_H).$$

For any $(M, \phi_M, \rho_M) \in \mathcal{M}_H^H$ the counit is defined by

$$v_M = \alpha_M^{-1} \circ s_M : M^{coH} \otimes_{H_L} H \to M,$$

where $\alpha_M^{-1} = \omega_M \circ i_{M^{coH} \otimes H}$ and s_M is the isomorphism between $M^{coH} \otimes_{H_L} H$ and $M^{coH} \times H$.

Theorem.

Let H be a Hopf algebra such that the functor $-\otimes H$ preserves coequalizers. The induction functor $F : \mathcal{C}_{H_L} \to \mathcal{M}_H^H$ and the functor of coinvariants $G : \mathcal{M}_H^H \to \mathcal{C}_{H_L}$ induce an equivalence of categories.

Definition.

Let H be a Hopf quasigroup and M an object in $\mathcal{C}$. We say that (M, ϕ_M, ρ_M) is a right-right H-Hopf module (or a right H-Hopf module for short) if the following axioms hold:

(i) The pair (M, ϕ_M) satisfies

(i-1) $\phi_M \circ (M \otimes \eta_H) = id_M$,

(i-2) $\phi_M \circ (\phi_M \otimes H) \circ (M \otimes \lambda_H \otimes H) \circ (M \otimes \delta_H) = M \otimes \varepsilon_H = \phi_M \circ (\phi_M \otimes \lambda_H) \circ (M \otimes \delta_H)$

(ii) The pair (M, ρ_M) is a right H-comodule.

(iii) $\rho_M \circ \phi_M = (\phi_M \otimes \mu_H) \circ (M \otimes c_{H,H} \otimes H) \circ (\rho_M \otimes \delta_H)$, i.e. ϕ_M is a morphism of right H-comodules with the codiagonal coaction on $M \otimes H$.

Obviously, if H is a Hopf quasigroup, the triple $(H, \phi_H = \mu_H, \rho_H = \delta_H)$ is a right-right H-Hopf module.

Definition.

Let H be a Hopf quasigroup. If (M, ϕ_M, ρ_M) is a right H-Hopf module, the morphism

$$q_M := \phi_M \circ (M \otimes \lambda_H) \circ \rho_M : M \to M$$

is idempotent. Moreover, if M^{coH} (object of coinvariants) is the image of q_M and $p_M : M \to M^{coH}$, $i_M : M^{coH} \to M$ are the morphisms such that $q_M = i_M \circ p_M$ and $id_{M^{coH}} = p_M \circ i_M$,

$$M^{coH} \xrightarrow{i_M} M \underset{M \otimes \eta_H}{\overset{\rho_M}{\rightrightarrows}} M \otimes H$$

is an equalizer diagram. Moreover, the following identities hold:

$$\rho_M \circ q_M = q_M \otimes \eta_H,$$

$$\phi_M \circ (q_M \otimes H) \circ \rho_M = id_M,$$

$$\rho_M \circ \phi_M \circ (i_M \otimes H) = (\phi_M \otimes H) \circ (i_M \otimes \delta_H),$$

$$q_M \circ \phi_M \circ (i_M \otimes H) = i_M \otimes \varepsilon_H.$$

Let If (M, ϕ_M, ρ_M) be a right H-Hopf module. If we define the morphisms

$$\omega_M : M^{coH} \otimes H \to M, \quad \omega'_M : M \to M^{coH} \otimes H,$$

by $\omega_M = \phi_M \circ (i_M \otimes H)$ and $\omega'_M = (p_M \otimes H) \circ \rho_M$. Then, $\omega_M \circ \omega'_M = id_M$ and $\omega'_M \circ \omega_M = id_{M^{coH} \otimes H}$. Therefore, ω_M is an isomorphism with inverse ω'_M and it is also a morphism of right H-comodules if we consider

$$\phi_{M^{coH} \otimes H} = M^{coH} \otimes \mu_H, \ \ \rho_{M^{coH} \otimes H} = M^{coH} \otimes \delta_H.$$

If (M, ϕ_M, ρ_M), (N, ϕ_N, ρ_N) are right-right H-Hopf modules and there exists a right H-comodule isomorphism $\alpha : M \to N$, the triple $(M, \phi_M^{\alpha} = \alpha^{-1} \circ \phi_N \circ (\alpha \otimes H), \rho_M)$ is a right-right H-Hopf module. Then, for the isomorphism ω'_M we have that

$$(M, \phi_M^{\omega'_M}, \rho_M)$$

is a right-right H-Hopf module where $\phi_M^{\omega'_M} = \phi_M \circ (q_M \otimes \mu_H) \circ (\rho_M \otimes H)$ holds and

$$q_M^{\omega'_M} = q_M,$$

where $q_M^{\omega'_M} = \phi_M^{\omega'_M} \circ (M \otimes \lambda_H) \circ \rho_M$ is the idempotent morphism associated to the Hopf module $(M, \phi_M^{\omega'_M}, \rho_M)$. Therefore, (M, ϕ_M, ρ_M) and $(M, \phi_M^{\omega'_M}, \rho_M)$ have tha same coinvariants. Finally, $(\phi_M^{\omega'_M})^{\omega'_M} = \phi_M^{\omega'_M}$ holds.

Note that the triple $(H, \phi_H = \mu_H, \rho_H = \delta_H)$ is a right-right H-Hopf module and $\phi_H^{\omega'_H} = \phi_H$.

Definition.

Let H be a Hopf quasigroup and let (M, ϕ_M, ρ_M) and (N, ϕ_N, ρ_N) be right-right H-Hopf modules. A morphism $f : M \to N$ in $\mathcal{C}$ is said to be H-quasilineal if the following identity holds:

$$\phi_N^{\omega'_N} \circ (f \otimes H) = f \circ \phi_M^{\omega'_M}.$$

A morphism of right-right H-Hopf modules between M and N is a morphism $f : M \to N$ in $\mathcal{C}$ such that is both a morphism of right H-comodules and H-quasilineal. The collection of all right H-Hopf modules with their morphisms forms a category which will be denoted by $\mathcal{M}_H^H$.

Definition.

Let H be a Hopf quasigroup and let (M, ϕ_M, ρ_M) and (N, ϕ_N, ρ_N) be right-right H-Hopf modules. A morphism $f : M \to N$ in $\mathcal{C}$ is said to be H-quasilineal if the following identity holds:

$$\phi_N^{\omega_N'} \circ (f \otimes H) = f \circ \phi_M^{\omega_M'}.$$

A morphism of right-right H-Hopf modules between M and N is a morphism $f : M \to N$ in $\mathcal{C}$ such that is both a morphism of right H-comodules and H-quasilineal. The collection of all right H-Hopf modules with their morphisms forms a category which will be denoted by $\mathcal{M}_H^H$.

Theorem. Fundamental Theorem of Hopf modules

Let H be a Hopf quasigroup and assume that (M, ϕ_M, ρ_M) is an object in the category $\mathcal{M}_H^H$. Then, (M, ϕ_M, ρ_M) and $(M^{coH} \otimes H, \phi_{M^{coH} \otimes H}, \rho_{M^{coH} \otimes H})$ are isomorphic in $\mathcal{M}_H^H$.

Let N be an object in $\mathcal{C}$. Then,

$$(N \otimes H, \phi_{N\otimes H} = N \otimes \mu_H, \rho_{N\otimes H} = N \otimes \delta_H)$$

is an object in $\mathcal{M}_H^H$. Also, if $f : N \to P$ is a morphism in $\mathcal{C}$, $f \otimes H$ is a morphism in $\mathcal{M}_H^H$ between $(N \otimes H, \phi_{N\otimes H}, \rho_{N\otimes H})$ and $(N \otimes H, \phi_{N\otimes H}, \rho_{N\otimes H})$.

Let N be an object in $\mathcal{C}$. Then,

$$(N \otimes H, \phi_{N\otimes H} = N \otimes \mu_H, \rho_{N\otimes H} = N \otimes \delta_H)$$

is an object in $\mathcal{M}_H^H$. Also, if $f : N \to P$ is a morphism in $\mathcal{C}$, $f \otimes H$ is a morphism in $\mathcal{M}_H^H$ between $(N \otimes H, \phi_{N\otimes H}, \rho_{N\otimes H})$ and $(N \otimes H, \phi_{N\otimes H}, \rho_{N\otimes H})$.

Theorem.

Let H be a Hopf quasigroup. There exists a functor

$$F : \mathcal{C} \to \mathcal{M}_H^H,$$

called the induction functor, defined on objects by $F(N) = (N \otimes H, \phi_{N\otimes H}, \rho_{N\otimes H})$ and for morphisms by $F(f) = f \otimes H$.

Let (M, ϕ_M, ρ_M) be a right-right H-Hopf module and consider the object of coinvariants M^{coH}. Let $g : M \rightarrow T$ be a morphism in $\mathcal{M}_H^H$. Using the comodule morphism condition we obtain that $\rho_T \circ g \circ i_M = (g \circ i_M) \otimes \eta_H$ and this implies that there exists a unique morphism $g^{coH} : M^{coH} \rightarrow T^{coH}$ such that

$$i_T \circ g^{coH} = g \circ i_M.$$

Let (M, ϕ_M, ρ_M) be a right-right H-Hopf module and consider the object of coinvariants M^{coH}. Let $g : M \to T$ be a morphism in $\mathcal{M}_H^H$. Using the comodule morphism condition we obtain that $\rho_T \circ g \circ i_M = (g \circ i_M) \otimes \eta_H$ and this implies that there exists a unique morphism $g^{coH} : M^{coH} \to T^{coH}$ such that

$$i_T \circ g^{coH} = g \circ i_M.$$

Theorem.

Let H be a Hopf quasigroup. There exists a functor

$$G : \mathcal{M}_H^H \to \mathcal{C},$$

called the functor of coinvariants, defined on objects by $G((M, \phi_M, \rho_M)) = M^{coH}$ and for morphisms by $G(g) = g^{coH}$.

Theorem.

Let H be a Hopf quasigroup. The induction functor $F : \mathcal{C} \to \mathcal{M}_H^H$ is left adjoint of the functor of coinvariants $G : \mathcal{M}_H^H \to \mathcal{C}$.

Theorem.

Let H be a Hopf quasigroup. The induction functor $F : \mathcal{C} \to \mathcal{M}_H^H$ is left adjoint of the functor of coinvariants $G : \mathcal{M}_H^H \to \mathcal{C}$.

Proof.

The unit and the counit of the adjunction are defined by:

$$u_N = p_{N\otimes H} \circ (N \otimes \eta_H) = id_N : N \to (N \otimes H)^{coH} = N$$

$$v_M = \omega_M : M^{coH} \otimes H \to M$$

Theorem.

Let H be a Hopf quasigroup. The induction functor $F : \mathcal{C} \to \mathcal{M}_H^H$ and the functor of coinvariants $G : \mathcal{M}_H^H \to \mathcal{C}$ induce an equivalence of categories.

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