Adjoint actions and quantum commutativity

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Abstract In this paper we explain in detail the main properties of the adjoint action associated to a weak Hopf algebra and the connections of this action with the notion of quantum commutativity. We also clarify some misconceptions that appear in the literature about this action.

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1. Introduction

Weak Hopf algebras (or quantum groupoids in the terminology of Nikshych and Vainerman [6]) were introduced by Böhm, Nill and Szlachányi in [3] as a new generalization of Hopf algebras and groupoid algebras. The main difference with other Hopf algebraic constructions, such as quasi-Hopf algebras and rational Hopf algebras, is that weak Hopf algebras are coassociative but the coproduct is not required to preserve the unit η_H or, equivalently, the counit is not an algebra morphism. Some motivations to study weak Hopf algebras come from the following facts: firstly, as group algebras and their duals are the natural examples of Hopf algebras, groupoid algebras and their duals provide examples of weak Hopf algebras; secondly, these algebraic structures have a remarkable connection with the theory of algebra extensions, important applications in the study of dynamical twists of Hopf algebras and a deep link with quantum field theories and operator algebras (see [6]), as well as they are useful tools in the study of fusion categories in characteristic zero (see [5]).

The notion of Yetter-Drinfeld module was considered to deal with the quantum Yang-Baxter equation, specially in quantum mechanics (see [9] for a detailed exposition of its physical implications). Actually, every Yetter-Drinfeld module gives rise to a solution to the quantum Yang-Baxter equation, i.e. a Yang-Baxter operator, as was proved in [7], and if H is a finite Hopf algebra in a symmetric category $\mathcal{C}$, the category ${}^H_H\mathcal{YD}$ of left-left Yetter-Drinfeld modules is isomorphic to the category of modules over the Drinfeld quantum double, which was originally conceived to find solutions of the Yang-Baxter equation via universal matrices. Continuing with physical applications, any projection of a Hopf algebra provides an example of a Yetter-Drinfeld module and this result is the substrate of the bosonization process introduced by Majid in [8] that gives, for a quasitriangular Hopf algebra, an interpretation of cross products in terms of quantum algebras of observables of dynamical systems, as well as in quantum group gauge theory. Interesting non-trivial examples of Yetter-Drinfeld modules can be obtained in the Hopf algebra setting working with the adjoint action. It is a well-known fact that, if H is a Hopf algebra in an strict braided monoidal category with braid c, the triple (H, φ_H, δ_H) is an object in ${}^H_H\mathcal{YD}$ where $\varphi_H : H \otimes H \to H$ denotes the adjoint action defined by

$$\varphi_H = \mu_H \circ (\mu_H \otimes \lambda_H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H).$$

In the weak setting the notion of Yetter-Drinfeld module was introduced by Böhm in [2] and as in the classical Hopf situation provides examples of an special kind of Yang-Baxter operators defined in [1], called weak Yang-Baxter operators, and closely related with the solutions of the dynamical Yang-Baxter equation. Unfortunately, in the weak setting, the construction of examples of Yetter-Drinfeld modules using the adjoint action does not work as in the classical Hopf algebra case because if H is a weak Hopf algebra in $\mathcal{C}$, the pair (H, φ_H) is not in general a left H-module. The main problem in this setting is the following: The unit condition can fail, i.e. $\varphi_H \circ (\eta_H \otimes H) \neq id_H$ and then we need to find new ways to obtain similar results for the adjoint action associated to a weak Hopf algebra. Then the main motivation of this paper is to show the relevant properties of the adjoint action in the weak setting, in order to obtain examples of Yetter-Drinfed modules for a weak Hopf algebra. We also prove new connections of this action with the notion of quantum commutativity. Finally, we clarify some misconceptions that appear in the literature about this action.

2. Adjoint actions and quantum commutativity

In this paper we denote an strict symmetric monoidal category $\mathcal{C}$ as $(\mathcal{C}, \otimes, K, c)$ where $\mathcal{C}$ is a category and $\otimes$ provides $\mathcal{C}$ with a monoidal structure with unit object K. With c we denote the symmetry natural isomorphism and for each object M in $\mathcal{C}$, $id_M : M \to M$ denotes the identity morphism. For simplicity of notation, given objects M, N, P in $\mathcal{C}$ and a morphism $f : M \to N$, we write $P \otimes f$ for $id_P \otimes f$ and $f \otimes P$ for $f \otimes id_P$.

From now on we assume that $\mathcal{C}$ admits split idempotents, i.e. for every morphism $\nabla_Y : Y \to Y$ such that $\nabla_Y = \nabla_Y \circ \nabla_Y$ there exist an object Z and morphisms $i_Y : Z \to Y$ and $p_Y : Y \to Z$ such that $\nabla_Y = i_Y \circ p_Y$ and $p_Y \circ i_Y = id_Z$.

Definition 2.1. An algebra in $\mathcal{C}$ is a triple $A = (A, \eta_A, \mu_A)$ where A is an object in $\mathcal{C}$ and $\eta_A : K \to A$ (unit), $\mu_A : A \otimes A \to A$ (product) are morphisms in $\mathcal{C}$ such that $\mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A)$, $\mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A)$. Given two algebras $A = (A, \eta_A, \mu_A)$ and $B = (B, \eta_B, \mu_B)$, $f : A \to B$ is an algebra morphism if $\mu_B \circ (f \otimes f) = f \circ \mu_A$, $f \circ \eta_A = \eta_B$. Also, if A, B are algebras in $\mathcal{C}$, the object $A \otimes B$ is an algebra in $\mathcal{C}$ where $\eta_{A\otimes B} = \eta_A \otimes \eta_B$ and $\mu_{A\otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B)$.

A coalgebra in $\mathcal{C}$ is a triple $D = (D, \varepsilon_D, \delta_D)$ where D is an object in $\mathcal{C}$ and $\varepsilon_D : D \to K$ (counit), $\delta_D : D \to D \otimes D$ (coproduct) are morphisms in $\mathcal{C}$ such that $(\varepsilon_D \otimes D) \circ \delta_D = id_D = (D \otimes \varepsilon_D) \circ \delta_D$, $(\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D$. If $D = (D, \varepsilon_D, \delta_D)$ and $E = (E, \varepsilon_E, \delta_E)$ are coalgebras, $f : D \to E$ is a coalgebra morphism if $(f \otimes f) \circ \delta_D = \delta_E \circ f$, $\varepsilon_E \circ f = \varepsilon_D$. When D, E are coalgebras in $\mathcal{C}$, $D \otimes E$ is a coalgebra in $\mathcal{C}$ where $\varepsilon_{D\otimes E} = \varepsilon_D \otimes \varepsilon_E$ and $\delta_{D\otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E)$.

If A is an algebra, B is a coalgebra and $\alpha : B \to A$, $\beta : B \to A$ are morphisms, we define the convolution product by $\alpha \wedge \beta = \mu_A \circ (\alpha \otimes \beta) \circ \delta_B$.

By weak Hopf algebras (or quantum groupoids in the terminology of Nikshych and Vainerman [6]) we understand the objects introduced in [3], as a generalization of ordinary Hopf algebras. In order to clarity, we recall the definition of these objects and some relevant results from [3] without proof.

Definition 2.2. A weak Hopf algebra H is an object in $\mathcal{C}$ with an algebra structure (H, η_H, μ_H) and a coalgebra structure $(H, \varepsilon_H, \delta_H)$ such that the following axioms hold:

- (a1) $\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ \delta_{H\otimes H}$,
- (a2) $\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes \delta_H \otimes H)$
  $= (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes (c_{H,H} \circ \delta_H) \otimes H)$,
- (a3) $(\delta_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \mu_H \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$
  $= (H \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$.
- (a4) There exists a morphism $\lambda_H : H \to H$ in $\mathcal{C}$ (called the antipode of H) verifiying:
  - (a4-1) $id_H \wedge \lambda_H = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$,
  - (a4-2) $\lambda_H \wedge id_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$,
  - (a4-3) $\lambda_H \wedge id_H \wedge \lambda_H = \lambda_H$.

Note that, in this definition, the conditions (a2), (a3) weaken the conditions of multiplicativity of the counit, and comultiplicativity of the unit that we can find in the Hopf algebra definition. On the other hand, axioms (a4-1), (a4-2) and (a4-3) weaken the properties of the antipode in a Hopf algebra. Therefore, a weak Hopf algebra is a Hopf algebra if an only if the morphism δ_H (comultiplication) is unit-preserving and if and only if the counit is a homomorphism of algebras.

2.3. If H is a weak Hopf algebra in $\mathcal{C}$, the antipode λ_H is unique, antimultiplicative, anticomultiplicative and leaves the unit η_H and the counit ε_H invariant:

$$\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}, \quad \delta_H \circ \lambda_H = c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H, \tag{1}$$

$$\lambda_H \circ \eta_H = \eta_H, \quad \varepsilon_H \circ \lambda_H = \varepsilon_H. \tag{2}$$

If we define the morphisms Π_H^L, Π_H^R, $\overline{\Pi}_H^L$ and $\overline{\Pi}_H^R$ by

$\Pi_H^L = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$,
$\Pi_H^R = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$,
$\overline{\Pi}_H^L = (H \otimes (\varepsilon_H \circ \mu_H)) \circ ((\delta_H \circ \eta_H) \otimes H)$,
$\overline{\Pi}_H^R = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$.

it is straightforward to show that they are idempotent and Π_H^L, Π_H^R satisfy the equalities

$$\Pi_H^L = id_H \wedge \lambda_H, \quad \Pi_H^R = \lambda_H \wedge id_H. \tag{3}$$

Moreover, we have that

$$\Pi_H^L \circ \overline{\Pi}_H^L = \Pi_H^L, \quad \Pi_H^L \circ \overline{\Pi}_H^R = \overline{\Pi}_H^R, \quad \Pi_H^R \circ \overline{\Pi}_H^L = \overline{\Pi}_H^L, \quad \Pi_H^R \circ \overline{\Pi}_H^R = \Pi_H^R, \tag{4}$$

$$\overline{\Pi}_H^L \circ \Pi_H^L = \overline{\Pi}_H^L, \quad \overline{\Pi}_H^L \circ \Pi_H^R = \Pi_H^R, \quad \overline{\Pi}_H^R \circ \Pi_H^L = \Pi_H^L, \quad \overline{\Pi}_H^R \circ \Pi_H^R = \overline{\Pi}_H^R. \tag{5}$$

Also it is easy to show the formulas

$$\Pi_H^L = \overline{\Pi}_H^R \circ \lambda_H = \lambda_H \circ \overline{\Pi}_H^L, \quad \Pi_H^R = \overline{\Pi}_H^L \circ \lambda_H = \lambda_H \circ \overline{\Pi}_H^R, \tag{6}$$

$$\Pi_H^L \circ \lambda_H = \Pi_H^L \circ \Pi_H^R = \lambda_H \circ \Pi_H^R, \quad \Pi_H^R \circ \lambda_H = \Pi_H^R \circ \Pi_H^L = \lambda_H \circ \Pi_H^L. \tag{7}$$

If λ_H is an isomorphism (for example, when H is finite), we have the equalities:

$$\overline{\Pi}_H^L = \mu_H \circ (H \otimes \lambda_H^{-1}) \circ c_{H,H} \circ \delta_H, \quad \overline{\Pi}_H^R = \mu_H \circ (\lambda_H^{-1} \otimes H) \circ c_{H,H} \circ \delta_H. \tag{8}$$

Let H be a weak Hopf algebra. We say that (M, φ_M) is a left H-module if M is an object in $\mathcal{C}$ and $\varphi_M : H \otimes M \to M$ is a morphism in $\mathcal{C}$ satisfying $\varphi_M \circ (\eta_H \otimes M) = id_M$, $\varphi_M \circ (H \otimes \varphi_M) = \varphi_M \circ (\mu_H \otimes M)$. Given two left H-modules (M, φ_M) and (N, φ_N), $f : M \to N$ is a morphism of left H-modules if $\varphi_N \circ (H \otimes f) = f \circ \varphi_M$. We denote the category of left H-modules by ${}_H\mathcal{C}$.

We say that (M, ϱ_M) is a left H-comodule if M is an object in $\mathcal{C}$ and $\varrho_M : M \to H \otimes M$ is a morphism in $\mathcal{C}$ satisfying $(\varepsilon_H \otimes M) \circ \varrho_M = id_M$, $(H \otimes \varrho_M) \circ \varrho_M = (\delta_H \otimes M) \circ \varrho_M$. Given two left H-comodules (M, ϱ_M) and (N, ϱ_N), $f : M \to N$ is a morphism of left H-comodules if $\varrho_N \circ f = (H \otimes f) \circ \varrho_M$. We denote the category of left H-comodules by ${}^H\mathcal{C}$.

A well-known result in Hopf algebras says that, if H is a Hopf algebra in $\mathcal{C}$, the triple (H, φ_H) is an object of ${}_H\mathcal{C}$ where $\varphi_H : H \otimes H \to H$ denotes the adjoint action defined by

$$\varphi_H = \mu_H \circ (\mu_H \otimes \lambda_H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H).$$

In a similar way, the triple (H, ϱ_H) is an object of ${}^H\mathcal{C}$ where $\varrho_H : H \to H \otimes H$ denotes the adjoint coaction defined by

$$\varrho_H = (\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes \lambda_H) \circ \delta_H.$$

Unfortunately, in the weak setting, the previous assertions are not true and we can find in the literature some misconceptions about this fact. For example, in [10] the author states erroneously that the pair (H, φ_H) is a left H-module. The true story is the following: If H a weak Hopf algebra in $\mathcal{C}$, the pair (H, φ_H) is not in general a left H-module because the unit condition can fail, i.e.

$$\varphi_H \circ (\eta_H \otimes H) = \mu_H \circ (H \otimes (\lambda_H \circ \Pi_H^L)) \circ \delta_H \neq id_H,$$

and for the adjoint coaction the counit condition may be untrue because

$$(\varepsilon_H \otimes H) \circ \varrho_H = \mu_H \circ (H \otimes (\Pi_H^L \circ \lambda_H)) \circ \delta_H \neq id_H.$$

In this section we shall show that for every weak Hopf algebra H the adjoint action and the adjoint coaction induce idempotent morphisms and as a consequence, using the factorizations of these idempotents, it is possible to construct new examples of objects in the categories ${}_H\mathcal{C}$ and ${}^H\mathcal{C}$. Obviously, if H is a Hopf algebra, the idempotents associated to the adjoint action and coaction are identities and we recover the classical results.

Proposition 2.4. *Let H be a weak Hopf algebra in $\mathcal{C}$. Let $\varphi_H : H\otimes H \to H$ and $\varrho_H : H \to H\otimes H$ be the morphisms defined by*

$$\varphi_H = \mu_H \circ (\mu_H \otimes \lambda_H)\circ (H\otimes c_{H,H})\circ (\delta_H\otimes H)$$

and

$$\varrho_H = (\mu_H \otimes H)\circ (H\otimes c_{H,H})\circ (\delta_H \otimes \lambda_H)\circ \delta_H.$$

Then

1) $\varphi_H \circ (H\otimes \varphi_H) = \varphi_H\circ (\mu_H\otimes H)$.
2) $(H\otimes \varrho_H)\circ \varrho_H = (H\otimes \delta_H)\circ \varrho_H$.

As a consequence,

$$\omega_H^a = \varphi_H\circ (\eta_H\otimes H) : H\to H,$$
$$\omega_H^c = (\varepsilon_H\otimes H)\circ \varrho_H : H\to H$$

are idempotent morphisms in $\mathcal{C}$ and

$$\omega_H^a = \mu_H\circ (H\otimes (\lambda_H\circ \Pi_H^L))\circ \delta_H,$$
$$\omega_H^c = \mu_H\circ (H\otimes (\Pi_H^L\circ \lambda_H))\circ \delta_H.$$

Proof. We prove 1) and the idempotent condition for ω_H^a. The proof for ϱ_H and ω_H^c is analogous and we leave the details to the reader.

$$\begin{aligned}
&\varphi_H\circ (H\otimes \varphi_H)\\
&= \mu_H\circ (\mu_H\otimes (\mu_H\circ (\lambda_H\otimes\lambda_H)\circ c_{H,H}))\circ (\mu_H\otimes c_{H,H}\otimes H)\circ (H\otimes c_{H,H}\otimes c_{H,H})\circ (\delta_H\otimes\delta_H\otimes H)\\
&= \mu_H\circ (\mu_H\otimes \lambda_H)\circ (H\otimes c_{H,H})\circ (((\mu_H\otimes\mu_H)\circ (H\otimes c_{H,H}\otimes H)\circ (\delta_H\otimes\delta_H))\otimes H)\\
&= \varphi_H\circ (\mu_H\otimes H).
\end{aligned}$$

The first equality follows by the naturality of c and by the associativity of μ_D. The second one is a consequence of the naturality of c and (1). Finally, the third one follows by (a1) of Definition 2.2.

Then,

$$\omega_H^a\circ \omega_H^a = \varphi_H\circ ((\mu_H\circ (\eta_H\otimes\eta_H))\otimes H) = \omega_H^a.$$

The equality

$$\omega_D^a = \mu_D\circ (D\otimes (\lambda_D\circ \Pi_D^L))\circ \delta_D,$$

follows from the identity (see [10] for more details)

$$(H\otimes \Pi_H^L)\circ \delta_H = (\mu_H\otimes H)\circ (H\otimes c_{H,H})\circ ((\delta_H\circ\eta_H)\otimes H). \tag{9}$$

□

Examples 2.5. i) As group algebras and their duals are the natural examples of Hopf algebras, groupoid algebras and their duals provide examples of weak Hopf algebras. Recall that a groupoid G is simply a category in which every morphism is an isomorphism. In this example, we consider finite groupoids, i.e. groupoids with a finite number of objects. The set of objects of G will be denoted by G_0 and the set of morphisms by G_1. The identity morphism on $x\in G_0$ will also be denoted by id_x and for a morphism $\sigma : x\to y$ in G_1, we write $s(\sigma)$ and $t(\sigma)$, respectively for the source and the target of σ.

Let G be a groupoid, and R a commutative ring. The groupoid algebra is the direct product

$$RG = \bigoplus_{\sigma\in G_1} R\sigma$$

where the product of two morphisms is equal to their composition if the latter is defined and 0 in otherwise, i.e. $\sigma\tau = \sigma\circ\tau$ if $s(\sigma) = t(\tau)$ and $\sigma\tau = 0$ if $s(\sigma)\neq t(\tau)$. The unit element is $1_{RG} = \sum_{x\in G_0} id_x$. The algebra RG is a cocommutative weak Hopf algebra, with coproduct δ_{RG}, counit ε_{RG} and antipode λ_{RG} given by the formulas:

$$\delta_{RG}(\sigma) = \sigma\otimes\sigma,\quad \varepsilon_{RG}(\sigma) = 1,\quad \lambda_{RG}(\sigma) = \sigma^{-1}.$$

For the weak Hopf algebra RG the morphisms target and source are respectively,

$$\Pi_{RG}^L(\sigma) = id_{t(\sigma)},\quad \Pi_{RG}^R(\sigma) = id_{s(\sigma)}$$

and $\lambda_{RG}\circ\lambda_{RG}=id_{RG}$, i.e. the antipode is an involution.

In this setting the morphisms defined in the previous Proposition are:

$$\omega^a_{RG}(\sigma)=\sigma\circ id_{t(\sigma)}=\begin{cases}\sigma \text{ if } t(\sigma)=s(\sigma)\\ 0 \text{ if } t(\sigma)\neq s(\sigma)\end{cases}$$

$$\omega^c_{RG}(\sigma)=\sigma\circ id_{s(\sigma)}=\sigma.$$

In the particular case of the groupoid algebra on n-objects with one invertible arrow between each ordered pair of objects, we obtain that RG is isomorphic to the $n\times n$ matrix $RG=M_n(R)$. The weak Hopf algebra H has the following structure. If E_{ij} denote the (i,j)- matrix unit, RG has counit given by $\varepsilon_{RG}(E_{ij})=1$, comultiplication by $\delta_{RG}(E_{ij})=E_{ij}\otimes E_{ij}$ and antipode given by $\lambda_{RG}(E_{ij})=E_{ji}$ for each $i,j=1,\cdots,n$. In this case, $\Pi^L_{RG}(E_{ij})=E_{ii}$, $\Pi^R_{RG}(E_{ij})=E_{jj}$ and then $RG_L=RG_R$ is the submodule of the diagonal matrices. Therefore, the image of ω^a_{RG} is RG_L.

In this setting $\varphi_{RG}(\sigma\otimes\tau)=\tau$ if $t(\sigma)=t(\tau)$ and 0 in otherwise. On the other hand, $\rho_{RG}=id_{t(\sigma)}\otimes\sigma$.

ii) In a general setting, if H is a commutative ($\mu_H=\mu_H\circ c_{H,H}$) weak Hopf algebra, we obtain that $\Pi^L_H=\overline{\Pi}^R_H$ and then, we have

$$\begin{aligned}&\omega^a_H\\ &=\mu_H\circ(H\otimes(\mu_H\circ c_{H,H}))\circ(H\otimes\lambda_H\otimes H)\circ((\delta_H\circ\eta_H)\circ H)\\ &=\mu_H\circ(\Pi^L_H\otimes H)\circ(\eta_H\otimes H)\\ &=id_H.\end{aligned}$$

Also,

$$\begin{aligned}&\omega^c_H\\ &=\mu_H\circ(H\otimes(\Pi^L_H\circ\lambda_H))\circ\delta_H\\ &=\mu_H\circ(H\otimes(\overline{\Pi}^R_H\circ\lambda_H))\circ\delta_H\\ &=\mu_H\circ(H\otimes\Pi^L_H)\circ\delta_H.\end{aligned}$$

In a similar way, if H is a cocommutative ($\delta_H=c_{H,H}\circ\delta_H$) weak Hopf algebra, we obtain that $\omega^a_H=\mu_H\circ(H\otimes\Pi^L_H)\circ\delta_H$ and $\omega^c_H=id_H$.

Proposition 2.6. *Let H be a weak Hopf algebra in $\mathcal{C}$. Let $\varphi_H:H\otimes H\to H$ and $\varrho_H:H\to H\otimes H$ be the morphisms defined in Proposition 2.4. Then the following assertions hold:*

1) $$\begin{aligned}&\delta_H\circ\varphi_H\\ &=(\mu_H\otimes H)\circ(H\otimes c_{H,H})\circ(((\mu_H\otimes\varphi_H)\circ(H\otimes c_{H,H}\otimes H)\circ(\delta_H\otimes\delta_H))\otimes\lambda_H)\circ(H\otimes c_{H,H})\circ(\delta_H\otimes H)\end{aligned}$$
2) $$\begin{aligned}&\varrho_H\circ\mu_H\\ &=(\mu_H\otimes H)\circ(H\otimes c_{H,H})\circ(((\mu_H\otimes\mu_H)\circ(H\otimes c_{H,H}\otimes H)\circ(\delta_H\otimes\varrho_H))\otimes\lambda_H)\circ(H\otimes c_{H,H})\circ(\delta_H\otimes H)\end{aligned}$$

Proof. We prove 1). The proof for 2) is analogous and we leave the details to the reader. We have

$$\begin{aligned}&(\mu_H\otimes H)\circ(H\otimes c_{H,H})\circ(((\mu_H\otimes\varphi_H)\circ(H\otimes c_{H,H}\otimes H)\circ(\delta_H\otimes\delta_H))\otimes\lambda_H)\circ(H\otimes c_{H,H})\circ(\delta_H\otimes H)\\ &=(\mu_H\otimes\mu_H)\circ(H\otimes c_{H,H}\otimes H)\circ(((\mu_H\otimes\mu_H)\circ(H\otimes c_{H,H}\otimes H)\circ(\delta_H\otimes\delta_H))\otimes(c_{H,H}\circ(\lambda_H\otimes\lambda_H)\circ\delta_H))\\ &\quad\circ(H\otimes c_{H,H})\circ(\delta_H\otimes H)\\ &=(\mu_H\otimes\mu_H)\circ(H\otimes c_{H,H}\otimes H)\circ((\delta_H\circ\mu_H)\otimes(\delta_H\circ\lambda_H))\circ(H\otimes c_{H,H})\circ(\delta_H\otimes H)\\ &=\delta_H\circ\varphi_H\end{aligned}$$

where the first equality follows by the naturality of c, and the associativity and coassociativity of μ_H and δ_H respectively. The second one follows by (1) and (a1) and the last one by (a1). □

Proposition 2.7. *Let H be a weak Hopf algebra in $\mathcal{C}$. Let ω^a_H, ω^c_H be the idempotent morphisms defined in Proposition 2.4. Then the following assertions hold:*

1) $\varphi_H\circ(H\otimes\omega^a_H)=\varphi_H$.
2) $(H\otimes\omega^a_H)\circ\delta_H\circ\omega^a_H=\delta_H\circ\omega^a_H$.
3) $\varrho_H\circ(H\otimes\omega^c_H)=\varrho_H$.
4) $\omega^c_H\circ\mu_H\circ(H\otimes\omega^c_H)=\omega^c_H\circ\mu_H$.

Proof. As in the previous result we prove 1) and 2) leaving 3) and 4) to the reader. The proof of 1) is a direct consequence of 1) of Proposition 2.4. To check 2), first note that by 1) the equality

$$(H\otimes\omega^a_H)\circ c_{H,H}\circ(\varphi_H\otimes H)=c_{H,H}\circ(\varphi_H\otimes H)\tag{10}$$

holds. Then, composing in 1) of Proposition 2.6 with $\eta_H\otimes H$ and $H\otimes\omega_H^a$ we have

$$\begin{aligned}
&(H\otimes\omega_H^a)\circ\delta_H\circ\omega_H^a\\
=&(H\otimes\omega_H^a)\circ(\mu_H\otimes H)\circ(H\otimes c_{H,H})\circ(((\mu_H\otimes\varphi_H)\circ(H\otimes c_{H,H}\otimes H)\circ(\delta_H\otimes\delta_H))\otimes\lambda_H)\\
&\circ(H\otimes c_{H,H})\circ((\delta_H\circ\eta_H)\otimes H)\\
=&(\mu_H\otimes H)\circ(H\otimes c_{H,H})\circ(((\mu_H\otimes\varphi_H)\circ(H\otimes c_{H,H}\otimes H)\circ(\delta_H\otimes\delta_H))\otimes\lambda_H)\circ(H\otimes c_{H,H})\\
&\circ((\delta_H\circ\eta_H)\otimes H)\\
=&\delta_H\circ\omega_H^a.
\end{aligned}$$

□

Notation 2.8. Let H be a weak Hopf algebra in $\mathcal{C}$. Let ω_H^a, ω_H^c be the idempotent morphisms defined in Proposition 2.4. For $x\in\{a,c\}$, with $\Omega^x(H)$, $p_H^x:H\to\Omega^x(H)$, $i_H^x:\Omega^x(H)\to H$ we denote the object and the morphisms such that $\omega_H^x=i_H^x\circ p_H^x$ and $id_{\Omega^x(H)}=p_H^x\circ {i_H}^x$.

Proposition 2.9. *Let H be a weak Hopf algebra in $\mathcal{C}$. The following assertions hold:*

1) *The object $\Omega^a(H)$ is a left H-module with action*

$$\varphi_{\Omega^a(H)}=p_H^a\circ\varphi_H\circ(H\otimes i_H^a):H\otimes\Omega^a(H)\to\Omega^a(H)$$

and a left H-comodule with coaction

$$\rho_{\Omega^a(H)}=(H\otimes p_H^a)\circ\delta_H\circ i_H^a:\Omega^a(H)\to H\otimes\Omega^a(H).$$

2) *The object $\Omega^c(H)$ is a left H-module with action*

$$\psi_{\Omega^c(H)}={p_H}^c\circ\mu_H\circ(H\otimes i_H^c):H\otimes\Omega^c(H)\to\Omega^c(H)$$

and a left H-comodule with coaction

$$\varrho_{\Omega^c(H)}=(H\otimes {p_H}^c)\circ\varrho_H\circ i_H^c:\Omega^c(H)\to H\otimes\Omega^c(H).$$

Proof. We shall prove 1). The proof for the second assertion is analogous.

Firstly note that

$$\varphi_{\Omega^a(H)}\circ(\eta_H\otimes\Omega^a(H))=p_H^a\circ\omega_H^a\circ i_H^a=id_{\Omega^a(H)}.$$

Secondly, by 1) of Proposition 2.4 and 1) of Proposition 2.7, we have

$$\begin{aligned}
&\varphi_{\Omega^a(H)}\circ(H\otimes\varphi_{\Omega^a(H)})\\
=&p_H^a\circ\varphi_H\circ(H\otimes\omega_H^a)\circ(H\otimes\varphi_H)\circ(H\otimes H\otimes i_H^a)\\
=&p_H^a\circ\varphi_H\circ(H\otimes\varphi_H)\circ(H\otimes H\otimes i_H^a)\\
=&p_H^a\circ\varphi_H\circ(\mu_H\otimes H)\circ(H\otimes H\otimes i_H^a)\\
=&\varphi_{\Omega^a(H)}\circ(\mu_H\otimes\Omega^a(H)).
\end{aligned}$$

On the other hand, trivially $(\varepsilon_H\otimes\Omega^a(H))\circ\rho_{\Omega^a(H)}=id_{\Omega^a(H)}$. Finally, by 2) of Proposition 2.7 we have

$$\begin{aligned}
&(H\otimes\rho_{\Omega^a(H)})\circ\rho_{\Omega^a(H)}\\
=&(H\otimes((H\otimes p_H^a)\circ\delta_H))\circ(H\otimes\omega_H^a)\circ\delta_H\circ i_H^a\\
=&(H\otimes((H\otimes p_H^a)\circ\delta_H))\circ\delta_H\circ i_H^a\\
=&(\delta_H\otimes\Omega^a(H))\circ\rho_{\Omega^a(H)}.
\end{aligned}$$

□

Remark 2.10. Of course, if H is a Hopf algebra we have that $\omega_H^a=\omega_H^c=id_H$ and then $\Omega^x(H)=H$, $x\in\{a,c\}$.

In the following results we connect the adjoint action with a special kind of commutativity (quantum commutativity), related with the notion introduced by M. Cohen and S. Westreich in [4], that we can define for H. For example, if H is quantum commutative the pair (H,φ_H) is a left H-module and then $\Omega^x(H)=H$, $x\in\{a,c\}$.

Definition 2.11. Let H be a weak Hopf algebra in $\mathcal{C}$ and (A,φ_A) an algebra, which is also a left H-module, such that $\varphi_A\circ(H\otimes\mu_A)=\mu_A\circ(\varphi_A\otimes\varphi_A)\circ(H\otimes c_{H,A}\otimes A)\circ(\delta_H\otimes A\otimes A)$. The object (A,φ_A) is called a left H-module algebra if the following equivalent conditions hold:

(b1) $\varphi_A\circ(\mu_H\otimes\eta_A)=(\varphi_A\otimes\varepsilon_H)\circ(H\otimes\eta_A\otimes\mu_H)\circ(\delta_H\otimes H)$.
(b2) $\varphi_A\circ(\mu_H\otimes\eta_A)=(\varepsilon_H\otimes\varphi_A)\circ(\mu_H\otimes H\otimes\eta_A)\circ(H\otimes c_{H,H})\circ(\delta_H\otimes H)$.

(b3) $\varphi_A \circ (\overline{\Pi}_H^L \otimes A) = \mu_A \circ c_{A,A} \circ (\varphi_A \otimes A) \circ (H \otimes \eta_A \otimes A)$.
(b4) $\varphi_A \circ (\Pi_H^L \otimes A) = \mu_A \circ (\varphi_A \otimes A) \circ (H \otimes \eta_A \otimes A)$.
(b5) $\varphi_A \circ (\overline{\Pi}_H^L \otimes A) \circ (H \otimes \eta_A) = \varphi_A \circ (H \otimes \eta_A)$.
(b6) $\varphi_A \circ (\Pi_H^L \otimes A) \circ (H \otimes \eta_A) = \varphi_A \circ (H \otimes \eta_A)$.

Definition 2.12. Let H be a weak Hopf algebra in $\mathcal{C}$. Let (C, ϱ_C) be a coalgebra, which is also a left H-comodule, such that $(H \otimes \delta_C) \circ \varrho_C = (\mu_H \otimes C \otimes C) \circ (H \otimes c_{C,H} \otimes C) \circ (\varrho_C \otimes \varrho_C) \circ \delta_C$. The object (C, ϱ_C) is called a left H-comodule coalgebra if the following equivalent conditions hold:

(c1) $(\delta_H \otimes \varepsilon_C) \circ \varrho_C = (\mu_H \otimes H) \circ (H \otimes \varepsilon_C \otimes \delta_H) \circ (\varrho_C \otimes \eta_H)$.
(c2) $(\delta_H \otimes \varepsilon_C) \circ \varrho_C = (\mu_H \otimes H \otimes \varepsilon_C) \circ (H \otimes c_{H,H} \otimes C) \circ (\delta_H \otimes \varrho_C) \circ (\eta_H \otimes C)$.
(c3) $(\overline{\Pi}_H^R \otimes C) \circ \varrho_C = (H \otimes \varepsilon_C \otimes C) \circ (\varrho_C \otimes C) \circ c_{C,C} \circ \delta_C$.
(c4) $(\Pi_H^L \otimes C) \circ \varrho_C = (H \otimes \varepsilon_C \otimes C) \circ (\varrho_C \otimes C) \circ \delta_C$.
(c5) $(\overline{\Pi}_H^R \otimes \varepsilon_C) \circ \varrho_C = (H \otimes \varepsilon_C) \circ \varrho_C$
(c6) $(\Pi_H^L \otimes \varepsilon_C) \circ \varrho_C = (H \otimes \varepsilon_C) \circ \varrho_C$

Definition 2.13. Let H be a weak Hopf algebra in $\mathcal{C}$. We say that H is quantum commutative if

$$\mu_H \circ (\varphi_H \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H) = \mu_H.$$

If

$$(\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ (\varrho_H \otimes H) \circ \delta_H = \delta_H$$

holds, we say that H is quantum cocommutative.

Example 2.14. If RG is the groupoid algebra defined in i) of Examples 2.5 we have that RG is not quantum commutative but quantum cocommutative.

Theorem 2.15. *Let H be a weak Hopf algebra in $\mathcal{C}$.*

1) *H is quantum commutative if and only if $\varphi_H \circ (\eta_H \otimes H) = id_H$.*
2) *H is quantum cocommutative if and only if $(\varepsilon_H \otimes H) \circ \varrho_H = id_H$.*

Proof. In this case we prove 2). The proof for 1) is similar and we leave the details to the reader.

If H is quantum cocommutative, by the identity,

$$\mu_H \circ (H \otimes \Pi_H^L) = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H) \tag{11}$$

and by the naturality of c and the associativity and coassociativity of μ_H and δ_H respectively, we obtain

$$\begin{aligned} & id_H \\ = & (\varepsilon_H \otimes H) \circ \delta_H \\ = & (\varepsilon_H \otimes H) \circ (\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ (\varrho_H \otimes H) \circ \delta_H \\ = & \mu_H \circ (H \otimes (\Pi_H^L \circ \Pi_H^R)) \circ \delta_H \end{aligned}$$

Therefore,

$$(\varepsilon_H \otimes H) \circ \varrho_H = \mu_H \circ (H \otimes (\Pi_H^L \circ \lambda_H)) \circ \delta_H = \mu_H \circ (H \otimes (\Pi_H^L \circ \Pi_H^R)) \circ \delta_H = id_H.$$

Conversely, if $(\varepsilon_H \otimes H) \circ \varrho_H = id_H$ we have

$$\begin{aligned} & \delta_H \\ = & (H \otimes \varepsilon_H \otimes H) \circ (H \otimes \varrho_H) \circ \delta_H \\ = & (\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ (H \otimes ((H \otimes (\overline{\Pi}_H^L \circ \lambda_H)) \circ \delta_H)) \circ \delta_H \\ = & (\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ (H \otimes ((H \otimes \Pi_H^R) \circ \delta_H)) \circ \delta_H \\ = & (\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ (\varrho_H \otimes H) \circ \delta_H \end{aligned}$$

where the first and the last equalities follow by the naturality of c, and the associativity and coassociativity of μ_H and δ_H respectively. The second one follows by the identity

$$\mu_H \circ (H \otimes \overline{\Pi}_H^L) = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (\delta_H \otimes H) \tag{12}$$

and finally, the third one by (6).

Therefore, H is quantum cocommutative. □

Corollary 2.16. *Let H be a weak Hopf algebra in $\mathcal{C}$.*

1) *H is quantum commutative if and only if (H, φ_H) is a left H-module algebra.*

2) *H is quantum cocommutative if and only if (H, ϱ_H) is a left H-comodule coalgebra.*

Proof. As in the previous Theorem we prove 2). The proof of 1) is similar and we leave the details to the reader. First note that

$$(\Pi_H^L \otimes \varepsilon_H) \circ \varrho_H = \Pi_H^L = (H \otimes \varepsilon_H) \circ \varrho_H.$$

On the other hand,

$$\begin{aligned}
&(\mu_H \otimes H \otimes H) \circ (H \otimes c_{H,H} \otimes H) \circ (\varrho_H \otimes \varrho_H) \circ \delta_H \\
&= (\mu_H \otimes H \otimes H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes (((\mu_H \circ (\Pi_H^R \otimes \lambda_H)) \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H) \circ \delta_H)) \circ \delta_H \\
&= (\mu_H \otimes H \otimes H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes (((H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \lambda_H)) \otimes H) \\
&\quad \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H))) \circ \delta_H \\
&= (\mu_H \otimes ((H \otimes (\varepsilon_H \circ \mu_H)) \circ (\delta_H \otimes H)) \otimes H) \circ (H \otimes c_{H,H} \otimes H \otimes H) \circ (\delta_H \otimes (((\delta_H \circ \lambda_H) \otimes H) \circ c_{H,H} \circ \delta_H)) \circ \delta_H \\
&= (\mu_H \otimes (\mu_H \circ (H \otimes \overline{\Pi}_H^L)) \otimes H) \circ (H \otimes c_{H,H} \otimes H \otimes H) \circ (\delta_H \otimes (((\delta_H \circ \lambda_H) \otimes H) \circ c_{H,H} \circ \delta_H)) \circ \delta_H \\
&= ((\mu_H \circ (H \otimes \lambda_H)) \otimes (\mu_H \circ (H \otimes (\overline{\Pi}_H^L \circ \lambda_H))) \otimes H) \circ (H \otimes c_{H,H} \otimes H \otimes H) \\
&\quad \circ (\delta_H \otimes (((c_{H,H} \circ \delta_H) \otimes H) \circ c_{H,H} \circ \delta_H)) \circ \delta_H \\
&= ((\mu_H \circ (H \otimes \lambda_H)) \otimes (\mu_H \circ (H \otimes \Pi_H^R)) \otimes H) \circ (H \otimes c_{H,H} \otimes H \otimes H) \\
&\quad \circ (\delta_H \otimes (((c_{H,H} \circ \delta_H) \otimes H) \circ c_{H,H} \circ \delta_H)) \circ \delta_H \\
&= ((\mu_H \circ (H \otimes \lambda_H)) \otimes H \otimes H) \circ (H \otimes c_{H,H} \otimes H) \circ (H \otimes H \otimes c_{H,H}) \circ (H \otimes ((((\mu_H \circ (H \otimes \Pi_H^R)) \otimes H) \\
&\quad \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H) \circ \delta_H)) \otimes H) \circ \delta_H \\
&= ((\mu_H \circ (H \otimes \lambda_H)) \otimes H \otimes H) \circ (H \otimes c_{H,H} \otimes H) \circ (H \otimes H \otimes c_{H,H}) \\
&\quad \circ (H \otimes ((\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ (\varrho_H \otimes H) \circ \delta_H) \otimes H) \circ (\delta_H \otimes H) \circ \delta_H
\end{aligned}$$

where the first equality follows by the naturality of c, and the associativity and coassociativity of μ_H and δ_H respectively. The second one follows by

$$\mu_H \circ (\Pi_H^R \otimes H) = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes \delta_H) \tag{13}$$

and the third one by the naturality of c and the coassociativity of δ_H. The fourth one is a consequence of (12) and the fifth identity relies on (1) and the naturality of c. The sixth one follows by (6)) and the seventh one by the naturality of c. Finally in the eight we use the naturality of c and the associativity and coassociativity of μ_H and δ_H respectively.

Using this identities we have the following: If H is quantum cocommutative we have that

$$\begin{aligned}
&(\mu_H \otimes H \otimes H) \circ (H \otimes c_{H,H} \otimes H) \circ (\varrho_H \otimes \varrho_H) \circ \delta_H \\
&= ((\mu_H \circ (H \otimes \lambda_H)) \otimes H \otimes H) \circ (H \otimes c_{H,H} \otimes H) \circ (H \otimes H \otimes c_{H,H}) \\
&\quad \circ (H \otimes ((\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ (\varrho_H \otimes H) \circ \delta_H) \otimes H) \circ (\delta_H \otimes H) \circ \delta_H \\
&= ((\mu_H \circ (H \otimes \lambda_H)) \otimes H \otimes H) \circ (H \otimes c_{H,H} \otimes H) \circ (H \otimes H \otimes c_{H,H}) \circ ((H \otimes \delta_H) \otimes H) \circ (\delta_H \otimes H) \circ \delta_H \\
&= (H \otimes \delta_H) \circ \varrho_H
\end{aligned}$$

and then (H, ϱ_H) is a left H-comodule coalgebra.

Conversely, if (H, ϱ_H) is a left H-comodule coalgebra we get $(\varepsilon_H \otimes H) \circ \varrho_H = id_H$ and then, by the previous Theorem, H is quantum cocommutative. □

3. Yetter-Drinfeld modules induced by the adjoint action

In the last section we show that the adjoint action induces examples of Yetter-Drinfeld modules.

Definition 3.1. Let H be a weak Hopf algebra in $\mathcal{C}$. We say that $(M, \varphi_M, \varrho_M)$ is a left-left Yetter-Drinfeld module over H if (M, φ_M) is a left H-module, (M, ϱ_M) is a left H-comodule and:

(d1) $\varrho_M = (\mu_H \otimes \varphi_M) \circ (H \otimes c_{H,H} \otimes M) \circ (\delta_H \otimes \varrho_M) \circ (\eta_H \otimes M)$

(d2) $(\mu_H \otimes \varphi_M) \circ (H \otimes c_{H,H} \otimes M) \circ (\delta_H \otimes \varrho_M)$
$= (\mu_H \otimes M) \circ (H \otimes c_{M,H}) \circ ((\varrho_M \circ \varphi_M) \otimes H) \circ (H \otimes c_{H,M}) \circ (\delta_H \otimes M)$

The category of left-left Yetter Drinfeld modules over H will be denoted by ${}^H_H\mathcal{YD}$. In this category the morphisms are the obvious, i.e., morphisms of left H-modules and comodules.

It is well-known that (d1) and (d2) are equivalent to

(d3) $\varrho_M \circ \varphi_M$
$= (\mu_H \otimes M) \circ (H \otimes c_{M,H}) \circ (((\mu_H \otimes \varphi_M) \circ (H \otimes c_{H,H} \otimes M) \circ (\delta_H \otimes \varrho_M)) \otimes \lambda_H) \circ (H \otimes c_{H,M}) \circ (\delta_H \otimes M)$

Proposition 3.2. *Let H be a weak Hopf algebra in $\mathcal{C}$. The following assertions hold:*

1) *The object $(\Omega^a(H), \varphi_{\Omega^a(H)}, \rho_{\Omega^a(H)})$ is in ${}^H_H\mathcal{YD}$.*

2) *The object* $(\Omega^c(H), \psi_{\Omega^c(H)}, \varrho_{\Omega^c(H)})$ *is in* ${}^H_H\mathcal{YD}$.

where $\varphi_{\Omega^a(H)}$, $\rho_{\Omega^a(H)}$, $\psi_{\Omega^c(H)}$ *and* $\varrho_{\Omega^c(H)}$ *are the morphisms defined in Proposition 2.9.*

Proof. We prove 1). The proof of 2) are similar and we leave it as an exercise. The triple $(\Omega^a(H), \varphi_{\Omega^a(H)}, \rho_{\Omega^a(H)})$ is a left-left Yetter-Drinfeld module over D because it satisfies (d3). Indeed:

$$\begin{aligned}
&(\mu_H \otimes \Omega^a(H)) \circ (H \otimes c_{\Omega^a(H),H}) \circ (((\mu_H \otimes \varphi_{\Omega^a(H)}) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \rho_{\Omega^a(H)})) \otimes \lambda_H) \\
&\circ (H \otimes c_{H,\Omega^a(H)}) \circ (\delta_H \otimes \Omega^a(H)) \\
&= (\mu_H \otimes p_H^a) \circ (H \otimes c_{H,H}) \circ (((\mu_H \otimes \varphi_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H)) \otimes \lambda_H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes i_H^a) \\
&= (H \otimes p_H^a) \circ \delta_H \circ \varphi_H \circ (H \otimes i_H^a) \\
&= \rho_{\Omega^a(H)} \circ \varphi_{\Omega^a(H)},
\end{aligned}$$

where the first equality follows from 1) of Proposition 2.7 and the naturality of c, the second one by 1) of Proposition 2.6 and the last one by 2) of Proposition 2.7. □

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