Cleft extensions, integrals and crossed products in a weak setting

Ramón González Rodríguez

Universidade de Vigo

Departamento de Matemática Aplicada II

Based in a joint work with J.N. Alonso Álvarez and J.M. Fernández Vilaboa

Research supported by Ministerio de Ciencia e Innovación: MTM2010-15634.

Recend Trends in Rings and Algebras
Murcia, 3-7 June 2013

Outline

Some notation and conventions.

- From now on $\mathcal{C}$ denotes a strict symmetric category with tensor product denoted by $\otimes$ and unit object K. With c we will denote the natural isomorphism of symmetry and we also assume that every idempotent morphism $q : Y \to Y$ splits, i.e., there exist an object Z and morphisms $i : Z \to Y$ and $p : Y \to Z$ such that $q = i \circ p$ and $p \circ i = id_Z$.

Some notation and conventions.

- From now on $\mathcal{C}$ denotes a strict symmetric category with tensor product denoted by $\otimes$ and unit object K. With c we will denote the natural isomorphism of symmetry and we also assume that every idempotent morphism $q : Y \to Y$ splits, i.e., there exist an object Z and morphisms $i : Z \to Y$ and $p : Y \to Z$ such that $q = i \circ p$ and $p \circ i = id_Z$.
- (A, η_A, μ_A) is an associative algebra with multiplication μ_A and unit η_A.

Some notation and conventions.

- From now on $\mathcal{C}$ denotes a strict symmetric category with tensor product denoted by $\otimes$ and unit object K. With c we will denote the natural isomorphism of symmetry and we also assume that every idempotent morphism $q : Y \to Y$ splits, i.e., there exist an object Z and morphisms $i : Z \to Y$ and $p : Y \to Z$ such that $q = i \circ p$ and $p \circ i = id_Z$.
- (A, η_A, μ_A) is an associative algebra with multiplication μ_A and unit η_A.
- $(C, \varepsilon_C, \delta_C)$ is a coassociative coalgebra with comultiplication δ_C and counit ε_C.

Some notation and conventions.

- From now on $\mathcal{C}$ denotes a strict symmetric category with tensor product denoted by $\otimes$ and unit object K. With c we will denote the natural isomorphism of symmetry and we also assume that every idempotent morphism $q : Y \to Y$ splits, i.e., there exist an object Z and morphisms $i : Z \to Y$ and $p : Y \to Z$ such that $q = i \circ p$ and $p \circ i = id_Z$.
- (A, η_A, μ_A) is an associative algebra with multiplication μ_A and unit η_A.
- $(C, \varepsilon_C, \delta_C)$ is a coassociative coalgebra with comultiplication δ_C and counit ε_C.
- For simplicity of notation, given three objects V, U, B in $\mathcal{C}$ and a morphism $f : V \to U$, we write

$$B \otimes f \text{ for } id_B \otimes f \text{ and } f \otimes B \text{ for } f \otimes id_B.$$

Some notation and conventions.

- From now on $\mathcal{C}$ denotes a strict symmetric category with tensor product denoted by $\otimes$ and unit object K. With c we will denote the natural isomorphism of symmetry and we also assume that every idempotent morphism $q : Y \to Y$ splits, i.e., there exist an object Z and morphisms $i : Z \to Y$ and $p : Y \to Z$ such that $q = i \circ p$ and $p \circ i = id_Z$.
- (A, η_A, μ_A) is an associative algebra with multiplication μ_A and unit η_A.
- $(C, \varepsilon_C, \delta_C)$ is a coassociative coalgebra with comultiplication δ_C and counit ε_C.
- For simplicity of notation, given three objects V, U, B in $\mathcal{C}$ and a morphism $f : V \to U$, we write

$$B \otimes f \text{ for } id_B \otimes f \text{ and } f \otimes B \text{ for } f \otimes id_B.$$

- For an algebra A we denote by $Z(A)$ the center of A and by $i_{Z(A)}$ the inclusion morphism of $Z(A)$ on A.

Cleft extensions for weak Hopf algebras

Definition.

A weak Hopf algebra in $\mathcal{C}$ is an object in $\mathcal{C}$ with an algebra structure (H, η_H, μ_H) and a coalgebra structure $(H, \varepsilon_H, \delta_H)$ satisfying:

(1) $\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H)$.

(2) $\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = ((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes \delta_H \otimes H)$
$= ((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes (c_{H,H} \circ \delta_H) \otimes H)$.

(3) $(\delta_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \mu_H \otimes H) \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H))$
$= (H \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H))$.

(4) There exists a morphism $\lambda_H : H \to H$ in $\mathcal{C}$ (called the antipode of H) satisfying:

(4-1) $id_H \wedge \lambda_H = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$.

(4-2) $\lambda_H \wedge id_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$.

(4-3) $\lambda_H \wedge id_H \wedge \lambda_H = \lambda_H$.

If H is a weak Hopf algebra in $\mathcal{C}$, the antipode λ_H is unique, antimultiplicative, anticomultiplicative and leaves the unit and the counit invariant:

$$\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}, \quad \delta_H \circ \lambda_H = c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H,$$

$$\lambda_H \circ \eta_H = \eta_H, \quad \varepsilon_H \circ \lambda_H = \varepsilon_H.$$

If H is a weak Hopf algebra in $\mathcal{C}$, the antipode λ_H is unique, antimultiplicative, anticomultiplicative and leaves the unit and the counit invariant:

$$\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}, \quad \delta_H \circ \lambda_H = c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H,$$

$$\lambda_H \circ \eta_H = \eta_H, \quad \varepsilon_H \circ \lambda_H = \varepsilon_H.$$

If we define the morphisms Π_H^L (target), Π_H^R (source), by

$$\Pi_H^L = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H),$$

$$\Pi_H^R = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)),$$

they are idempotent and we denote by H_L, p_L and i_L the object and the morphisms such that $i_L \circ p_L = \Pi_H^L$ and $p_L \circ i_L = id_{H_L}$.

If H is a weak Hopf algebra in $\mathcal{C}$, the antipode λ_H is unique, antimultiplicative, anticomultiplicative and leaves the unit and the counit invariant:

$$\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}, \quad \delta_H \circ \lambda_H = c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H,$$

$$\lambda_H \circ \eta_H = \eta_H, \quad \varepsilon_H \circ \lambda_H = \varepsilon_H.$$

If we define the morphisms Π_H^L (target), Π_H^R (source), by

$$\Pi_H^L = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H),$$

$$\Pi_H^R = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)),$$

they are idempotent and we denote by H_L, p_L and i_L the object and the morphisms such that $i_L \circ p_L = \Pi_H^L$ and $p_L \circ i_L = id_{H_L}$.

$$\overline{\Pi}_H^L = \Pi_{H^{coop}}^L, \quad \overline{\Pi}_H^R = \Pi_{H^{coop}}^R$$

Definition.

A weak Hopf algebra H is cocommutative if $\delta_H = c_{H,H} \circ \delta_H$.

Definition.

A weak Hopf algebra H is cocommutative if $\delta_H = c_{H,H} \circ \delta_H$.

Example.

Let G be a finite groupoid and R a commutative ring. Let G_0 be the set of objects and G_1 the set of morphisms.
The groupoid algebra is the direct product

$$RG = \bigoplus_{\sigma \in G_1} R\sigma$$

with the product of two morphisms being equal to their composition if the latter is defined and 0 in otherwise, i.e. $\sigma\tau = \sigma \circ \tau$ if $s(\sigma) = t(\tau)$ and $\sigma\tau = 0$ if $s(\sigma) \neq t(\tau)$.
The unit element is $1_{RG} = \sum_{x \in G_0} id_x$. RG is a cocommutative weak Hopf algebra, with

$$\delta_{RG}(\sigma) = \sigma \otimes \sigma, \quad \varepsilon_{RG}(\sigma) = 1, \quad \lambda_{RG}(\sigma) = \sigma^{-1}.$$

The morphisms target and source are $\Pi^L_{RG}(\sigma) = id_{t(\sigma)}, \quad \Pi^R_{RG}(\sigma) = id_{s(\sigma)}$.

Definition.

Let H be a weak Hopf algebra. We will say that a right H-comodule (A, ρ_A) is a right H-comodule algebra if it satisfies

$$\rho_A \circ \mu_A = (\mu_A \otimes \mu_H) \circ (A \otimes c_{H,A} \otimes H) \circ (\rho_A \otimes \rho_A)$$

and any of the following equivalent conditions hold:

(1) $(A \otimes \Pi_H^L) \circ \rho_A = (\mu_A \otimes H) \circ (A \otimes c_{H,A}) \circ ((\rho_A \circ \eta_A) \otimes A)$.

(2) $(A \otimes \overline{\Pi}_H^R) \circ \rho_A = (\mu_A \otimes H) \circ (A \otimes (\rho_A \circ \eta_A))$.

(3) $(A \otimes \Pi_H^L) \circ \rho_A \circ \eta_A = \rho_A \circ \eta_A$.

(4) $(A \otimes \overline{\Pi}_H^R) \circ \rho_A \circ \eta_A = \rho_A \circ \eta_A$.

(5) $(\rho_A \otimes H) \circ \rho_A \circ \eta_A = (A \otimes \mu_H \otimes H) \circ (\rho_A \otimes \delta_H) \circ (\eta_A \otimes \eta_H)$.

(6) $(\rho_A \otimes H) \circ \rho_A \circ \eta_A = (A \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ (\rho_A \otimes \delta_H) \circ (\eta_A \otimes \eta_H)$.

If (A, ρ_A) is a right H-comodule algebra, the triple (A, H, Ψ) is a weak entwining structure where the entwining morphism is

$$\Psi = (A \otimes \mu_H) \circ (c_{H,A} \otimes H) \circ (H \otimes \rho_A) : H \otimes A \to A \otimes H$$

Therefore the following identies hold:

(1) $\Psi \circ (H \otimes \mu_A) = (\mu_A \otimes H) \circ (A \otimes \Psi) \circ (\Psi \otimes A)$,

(2) $(A \otimes \delta_H) \circ \Psi = (\Psi \otimes H) \circ (H \otimes \Psi) \circ (\delta_H \otimes A)$,

(3) $\Psi \circ (H \otimes \eta_A) = (e_A \otimes H) \circ \delta_H$,

(4) $(A \otimes \varepsilon_H) \circ \Psi = \mu_A \circ (e_A \otimes A)$,

where

$$e_A = (A \otimes \varepsilon_H) \circ \Psi \circ (H \otimes \eta_A) : H \to A.$$

We denote by $\mathcal{M}_A^H(\Psi)$ the category of weak entwined modules, i.e., the objects M in $\mathcal{C}$ together with two morphisms $\phi_M : M \otimes A \to A$ and $\rho_M : M \to M \otimes H$ such that (M, ϕ_M) is a right A-module, (M, ρ_M) is a right H-comodule and such that the following equality

$$\rho_M \circ \phi_M = (\phi_M \otimes H) \circ (M \otimes \Psi) \circ (\rho_M \otimes A)$$

holds.
Then, if (A, ρ_A) is a right H-comodule algebra, (A, μ_A, ρ_A) is an object of $\mathcal{M}_A^H(\Psi)$.

If (A, ρ_A) is a right H-comodule algebra, we define the subalgebra of coinvariants of A as the equalizer:

$$\begin{tikzcd} A_H \arrow[r, "i_A"] & A \arrow[r, shift left, "\rho_A"] \arrow[r, shift right, "\zeta_A"'] & A \otimes H \end{tikzcd}$$

where

$$\zeta_A = (\mu_A \otimes H) \circ (A \otimes c_{H,A}) \circ ((\rho_A \circ \eta_A) \otimes A).$$

Note that

$$\zeta_A = (A \otimes \Pi_H^L) \circ \rho_A.$$

If (A, ρ_A) is a right H-comodule algebra, we define the subalgebra of coinvariants of A as the equalizer:

$$\begin{tikzcd} A_H \arrow[r, "i_A"] & A \arrow[r, shift left, "\rho_A"] \arrow[r, shift right, "\zeta_A"'] & A \otimes H \end{tikzcd}$$

where

$$\zeta_A = (\mu_A \otimes H) \circ (A \otimes c_{H,A}) \circ ((\rho_A \circ \eta_A) \otimes A).$$

Note that

$$\zeta_A = (A \otimes \Pi_H^L) \circ \rho_A.$$

Also

$$\begin{tikzcd} A_H \arrow[r, "i_A"] & A \arrow[r, shift left, "\rho_A"] \arrow[r, shift right, "(A \otimes \overline{\Pi}_H^R) \circ \rho_A"'] & A \otimes H \end{tikzcd}$$

is an equalizer diagram.

The triple

$$(A_H, \eta_{A_H}, \mu_{A_H})$$

is an algebra, being η_{A_H} and μ_{A_H} the factorizations through the equalizer i_A of the morphisms η_A and $\mu_A \circ (i_A \otimes i_A)$, respectively.

The triple

$$(A_H, \eta_{A_H}, \mu_{A_H})$$

is an algebra, being η_{A_H} and μ_{A_H} the factorizations through the equalizer i_A of the morphisms η_A and $\mu_A \circ (i_A \otimes i_A)$, respectively.

For example, the weak Hopf algebra H is a right H-comodule algebra with right comodule structure giving by $\rho_H = \delta_H$ and subalgebra of coinvariants H_H, the image of the idempotent morphism Π_H^L, which we denoted by H_L.

Definition

Let H be a weak Hopf algebra and let (A, ρ_A) be a right H-comodule algebra. We define an integral as a morphism of right H-comodules $f : H \to A$. If moreover $f \circ \eta_H = \eta_A$ we will say that the integral is total.

An integral $f : H \to A$ is convolution invertible if there exists a morphism $f^{-1} : H \to A$ (called the convolution inverse of f) such that

(1) $f^{-1} \wedge f = e_A$.

(2) $f \wedge f^{-1} = (A \otimes (\varepsilon_H \circ \mu_H)) \circ ((\rho_A \circ \eta_A) \otimes H)$.

(3) $f^{-1} \wedge f \wedge f^{-1} = f^{-1}$.

Definition

Let H be a weak Hopf algebra and let (A, ρ_A) be a right H-comodule algebra. We define an integral as a morphism of right H-comodules $f : H \to A$. If moreover $f \circ \eta_H = \eta_A$ we will say that the integral is total.
An integral $f : H \to A$ is convolution invertible if there exists a morphism $f^{-1} : H \to A$ (called the convolution inverse of f) such that

(1) $f^{-1} \wedge f = e_A$.
(2) $f \wedge f^{-1} = (A \otimes (\varepsilon_H \circ \mu_H)) \circ ((\rho_A \circ \eta_A) \otimes H)$.
(3) $f^{-1} \wedge f \wedge f^{-1} = f^{-1}$.

Trivially, f^{-1} is unique and by (1), if f is an integral convolution invertible, we get that

$$f \wedge f^{-1} \wedge f = f.$$

Finally, when f is a total integral we can rewrite equality (1) as

$$f^{-1} \wedge f = f \circ \Pi_H^R$$

and (2) as

$$f \wedge f^{-1} = f \circ \overline{\Pi}_H^L.$$

Example

Let H be a weak Hopf algebra such that $\Pi_H^L = \overline{\Pi}_H^L$ (equivalently, $\Pi_H^R = \overline{\Pi}_H^R$). Then the identity id_H is a total integral convolution invertible with inverse λ_H. Note that this equality is always true in the Hopf algebra setting. In our case it holds, for example, if H is a cocommutative weak Hopf algebra.

Example

Let H be a weak Hopf algebra such that $\Pi_H^L = \overline{\Pi}_H^L$ (equivalently, $\Pi_H^R = \overline{\Pi}_H^R$). Then the identity id_H is a total integral convolution invertible with inverse λ_H. Note that this equality is always true in the Hopf algebra setting. In our case it holds, for example, if H is a cocommutative weak Hopf algebra.

Definition

Let H be a weak Hopf algebra and (A, ρ_A) a right H-comodule algebra. We say that $A_H \hookrightarrow A$ is a H-cleft extension if there exists an integral $f : H \to A$ convolution invertible and such that the morphism $f \wedge f^{-1}$ factorizes through the equalizer i_A. In what follows, the morphism f will be called a cleaving morphism associated to the H-cleft extension $A_H \hookrightarrow A$.

Definition

Two H-cleft extensions $A_H \hookrightarrow A$ and $B_H \hookrightarrow B$ are equivalent

$$A_H \hookrightarrow A \sim B_H \hookrightarrow B$$

if $A_H = B_H$ and there is a morphism of right H-comodule algebras $T : A \to B$ such that $T \circ i_A = i_B$.

Definition

Two H-cleft extensions $A_H \hookrightarrow A$ and $B_H \hookrightarrow B$ are equivalent

$$A_H \hookrightarrow A \sim B_H \hookrightarrow B$$

if $A_H = B_H$ and there is a morphism of right H-comodule algebras $T : A \to B$ such that $T \circ i_A = i_B$.

Remark

If $A_H \hookrightarrow A$ and $B_H \hookrightarrow B$ are equivalent, the morphism T is an isomorphism.

Proposition

Let H be a weak Hopf algebra and (A, ρ_A) a right H-comodule algebra such that $A_H \hookrightarrow A$ is a H-cleft extension with cleaving morphism f. Then the equality

$$\rho_A \circ f^{-1} = (f^{-1} \otimes \lambda_H) \circ c_{H,H} \circ \delta_H$$

holds.

Proposition

Let H be a weak Hopf algebra and (A, ρ_A) a right H-comodule algebra such that $A_H \hookrightarrow A$ is a H-cleft extension with cleaving morphism f. Then the equality

$$\rho_A \circ f^{-1} = (f^{-1} \otimes \lambda_H) \circ c_{H,H} \circ \delta_H$$

holds.

Proposition

Let H be a cocommutative weak Hopf algebra and let (A, ρ_A) be a right H-comodule algebra. If there exists a convolution invertible integral $f : H \to A$, then $A_H \hookrightarrow A$ is an H-cleft extension.

Let H be a weak Hopf algebra and let (A, ρ_A) be a right H-comodule algebra. In

- Alonso Álvarez J.N., Fernández Vilaboa J.M., González Rodríguez R., Rodríguez Raposo A.B., Weak C-cleft extensions, weak entwining structures and weak Hopf algebras. J. of Algebra, **284**, 2005, 679-704.

we introduce the set $Reg^{WR}(H, A)$ as the one whose elements are the morphisms

$$h : H \to A$$

such that there exists a morphism $h^{-1} : H \to A$, called the left weak inverse of h, such that

$$h^{-1} \wedge h = e_A$$

where e_A is the morphism associated to the right-right weak entwining structure Ψ associated to (A, ρ_A).

Definition

Let H be a weak Hopf algebra and let (A, ρ_A) be a right H-comodule algebra. We say that $A_H \hookrightarrow A$ is a weak H-cleft extension if there exists a morphism $h : H \to A$ in $Reg^{WR}(H, A)$ of right H-comodules such that

$$\Psi \circ (H \otimes h^{-1}) \circ \delta_H = \zeta_A \circ h^{-1}.$$

Definition

Let H be a weak Hopf algebra and let (A, ρ_A) be a right H-comodule algebra. We say that $A_H \hookrightarrow A$ is a weak H-cleft extension if there exists a morphism $h : H \to A$ in $Reg^{WR}(H, A)$ of right H-comodules such that

$$\Psi \circ (H \otimes h^{-1}) \circ \delta_H = \zeta_A \circ h^{-1}.$$

Proposition

Let H be a weak Hopf algebra and let (A, ρ_A) be a right H-comodule algebra. If there exists $h \in Reg^{WR}(H, A)$ of right H-comodules such that $e_A \wedge h^{-1} = h^{-1}$, the following assertions are equivalent:

(i) The morphism $h \wedge h^{-1}$ factorizes through the equalizer i_A and h^{-1} satisfies

$$\rho_A \circ f^{-1} = (f^{-1} \otimes \lambda_H) \circ c_{H,H} \circ \delta_H.$$

(ii) The equality

$$\Psi \circ (H \otimes h^{-1}) \circ \delta_H = \zeta_A \circ h^{-1}$$

holds.

Corollary

Let H be a weak Hopf algebra and let (A, ρ_A) be a right H-comodule algebra. If $A_H \hookrightarrow A$ is an H-cleft extension then it is a weak H-cleft extension.

Proposition

Let H be a weak Hopf algebra with invertible antipode. If $A_H \hookrightarrow A$ is an H-cleft extension with cleaving morphism f, then $h = \mu_A \circ (f \otimes (f^{-1} \circ \eta_H))$ is a total integral. Moreover, if H is cocommutative h is convolution invertible.

Proposition

Let H be a weak Hopf algebra with invertible antipode. If $A_H \hookrightarrow A$ is an H-cleft extension with cleaving morphism f, then $h = \mu_A \circ (f \otimes (f^{-1} \circ \eta_H))$ is a total integral. Moreover, if H is cocommutative h is convolution invertible.

Remark

As a consequence of the previous proposition, in the cocommutative setting we can assume that the cleaving morphism is a total integral.

Definition

Let H be a weak Hopf algebra. We will say that A is a weak left H-module algebra if there exists a morphism $\varphi_A : H \otimes A \to A$ satisfying:

(1) $\varphi_A \circ (\eta_H \otimes A) = id_A$.

(2) $\varphi_A \circ (H \otimes \mu_A) = \mu_A \circ (\varphi_A \otimes \varphi_A) \circ (H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes A \otimes A)$.

(3) $\varphi_A \circ (\mu_H \otimes \eta_A) = \varphi_A \circ (H \otimes (\varphi_A \circ (H \otimes \eta_A)))$.

and any of the following equivalent conditions hold:

(4) $\varphi_A \circ (\Pi_H^L \otimes A) = \mu_A \circ ((\varphi_A \circ (H \otimes \eta_A) \otimes A)$.

(5) $\varphi_A \circ (\overline{\Pi}_H^L \otimes A) = \mu_A \circ c_{A,A} \circ ((\varphi_A \circ (H \otimes \eta_A) \otimes A)$.

(6) $\varphi_A \circ (\Pi_H^L \otimes \eta_A) = \varphi_A \circ (H \otimes \eta_A)$.

(7) $\varphi_A \circ (\overline{\Pi}_H^L \otimes \eta_A) = \varphi_A \circ (H \otimes \eta_A)$.

(8) $\varphi_A \circ (H \otimes (\varphi_A \circ (H \otimes \eta_A))) = ((\varphi_A \circ (H \otimes \eta_A)) \otimes (\varepsilon_H \circ \mu_H)) \circ (\delta_H \otimes H)$.

(9) $\varphi_A \circ (H \otimes (\varphi_A \circ (H \otimes \eta_A))) = ((\varepsilon_H \circ \mu_H) \otimes (\varphi_A \circ (H \otimes \eta_A))) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)$.

If we replace (3) by

(3-1) $\varphi_A \circ (\mu_H \otimes A) = \varphi_A \circ (H \otimes \varphi_A)$

we will say that (A, φ_A) is a left H-module algebra.

Proposition

Let H be a cocommutative weak Hopf algebra. If $A_H \hookrightarrow A$ is an H-cleft extension with cleaving morphism f, the pair (A_H, φ_{A_H}) is a weak left H-module algebra, being φ_{A_H} the factorization of the morphism

$$\varphi_A = \mu_A \circ (A \otimes (\mu_A \circ c_{A,A})) \circ (((f \otimes f^{-1}) \circ \delta_H) \otimes i_A)$$

through the equalizer i_A.

Crossed systems for weak Hopf algebras

Definition

Let H be a cocommutative weak Hopf algebra and (A, φ_A) be a weak left H-module algebra.
By $Reg_{\varphi_A}(H_L, A)$ we denote the set of morphisms $g : H_L \to A$ such that there exists a morphism $g^{-1} : H_L \to A$ satisfying

$$g \wedge g^{-1} = g^{-1} \wedge g = u_0, \;\; g \wedge g^{-1} \wedge g = g, \;\; g^{-1} \wedge g \wedge g^{-1} = g^{-1}$$

where $u_0 = u_1 \circ i_L$ where $u_1 = \varphi_A \circ (H \otimes \eta_A)$.
By $Reg_{\varphi_A}(H, A)$, as the set of morphisms $h : H \to A$ such that there exists a morphism $h^{-1} : H \to A$ satisfying the following equalities:

(1) $h \wedge h^{-1} = h^{-1} \wedge h = u_1$,

(2) $h \wedge h^{-1} \wedge h = h$,

(3) $h^{-1} \wedge h \wedge h^{-1} = h^{-1}$,

Note that

$$u_1 = u_0 \circ p_L.$$

For $n > 1$, we denote by H^n the n-fold tensor power $H \otimes \cdots \otimes H$. $Reg_{\varphi_A}(H^n, A)$ is the set of morphisms $\sigma : H^n \to A$ such that there exists a morphism $\sigma^{-1} : H^n \to A$ satisfying:

(1) $\sigma \wedge \sigma^{-1} = \sigma^{-1} \wedge \sigma = u_n$

(2) $\sigma \wedge \sigma^{-1} \wedge \sigma = \sigma$.

(3) $\sigma^{-1} \wedge \sigma \wedge \sigma^{-1} = \sigma^{-1}$.

where $u_n = \varphi_A \circ (H \otimes u_{n-1})$.

For $n > 1$, we denote by H^n the n-fold tensor power $H \otimes \cdots \otimes H$. $Reg_{\varphi_A}(H^n, A)$ is the set of morphisms $\sigma : H^n \to A$ such that there exists a morphism $\sigma^{-1} : H^n \to A$ satisfying:

(1) $\sigma \wedge \sigma^{-1} = \sigma^{-1} \wedge \sigma = u_n$

(2) $\sigma \wedge \sigma^{-1} \wedge \sigma = \sigma$.

(3) $\sigma^{-1} \wedge \sigma \wedge \sigma^{-1} = \sigma^{-1}$.

where $u_n = \varphi_A \circ (H \otimes u_{n-1})$.

If we denote by H^0 the object H_L, $u_n \in Reg_{\varphi_A}(H^n, A)$ and $Reg_{\varphi_A}(H^n, A)$ is a group with neutral element u_n for all $n \geq 0$. Also, if A is commutative, we have that $Reg_{\varphi_A}(H^n, A)$ is an abelian group for all $n \geq 0$.

Definition

Let H be a cocommutative weak Hopf algebra, (A, φ_A) a weak left H-module algebra and $\sigma \in Reg_{\varphi_A}(H^2, A)$. We say that

$$(\varphi_A, \sigma)$$

is a crossed system for H over A if the following conditions hold:

(1) Twisted condition

$$\mu_A \circ (A \otimes \varphi_A) \circ (\sigma \otimes \mu_H \otimes A) \circ (\delta_{H\otimes H} \otimes A)$$

$$= \mu_A \circ ((\varphi_A \circ (H \otimes \varphi_A)) \otimes A) \circ (H \otimes H \otimes c_{A,A}) \circ (H \otimes H \otimes \sigma \otimes A) \circ (\delta_{H\otimes H} \otimes A).$$

(2) Two cocycle condition

$$(\varphi_A \circ (H \otimes \sigma)) \wedge (\sigma \circ (H \otimes \mu_H)) = (\sigma \circ (H \otimes (\mu_H \circ (H \otimes \Pi_H^L)))) \wedge (\sigma \circ (\mu_H \otimes H)).$$

(3) Normal condition

$$\sigma \circ (H \otimes \eta_H) = \sigma \circ (\eta_H \otimes H) = u_1.$$

Two crossed systems for H over A, (φ_A, σ) and (ϕ_A, τ) are said to be equivalent, denoted by

$$(\varphi_A, \sigma) \approx (\phi_A, \tau),$$

if

$$\varphi_A \circ (H \otimes \eta_A) = \phi_A \circ (H \otimes \eta_A)$$

and there exists h in $Reg_{\varphi_A}(H, A) \cap Reg_{\phi_A}(H, A)$ with $h \circ \eta_H = \eta_A$ and such that

$$\varphi_A = \mu_A \circ (\mu_A \otimes A) \circ (h \otimes \phi_A \otimes h^{-1}) \circ (\delta_H \otimes c_{H,A}) \circ (\delta_H \otimes A),$$

$$\sigma = \mu_A \circ (\mu_A \otimes h^{-1}) \circ (\mu_A \otimes \tau \otimes \mu_H) \circ (h \otimes \phi_A \otimes \delta_{H\otimes H}) \circ (\delta_H \otimes h \otimes H \otimes H) \circ \delta_{H\otimes H}.$$

Two crossed systems for H over A, (φ_A, σ) and (ϕ_A, τ) are said to be equivalent, denoted by

$$(\varphi_A, \sigma) \approx (\phi_A, \tau),$$

if

$$\varphi_A \circ (H \otimes \eta_A) = \phi_A \circ (H \otimes \eta_A)$$

and there exists h in $Reg_{\varphi_A}(H, A) \cap Reg_{\phi_A}(H, A)$ with $h \circ \eta_H = \eta_A$ and such that

$$\varphi_A = \mu_A \circ (\mu_A \otimes A) \circ (h \otimes \phi_A \otimes h^{-1}) \circ (\delta_H \otimes c_{H,A}) \circ (\delta_H \otimes A),$$

$$\sigma = \mu_A \circ (\mu_A \otimes h^{-1}) \circ (\mu_A \otimes \tau \otimes \mu_H) \circ (h \otimes \phi_A \otimes \delta_{H\otimes H}) \circ (\delta_H \otimes h \otimes H \otimes H) \circ \delta_{H\otimes H}.$$

Proposition

Let H be a cocommutative weak Hopf algebra. Then $\approx$ is an equivalence relation.

Proposition

Let H be a cocommutative weak Hopf algebra, (A, φ_A) a weak left H-module algebra and $\sigma \in Reg_{\varphi_A}(H^2, A)$ satisfying the twisted condition. The following assertions are equivalent:

(i) (A, φ_A) is a left H-module algebra.

(ii) The morphism σ factors through the center of A.

Proposition

Let H be a cocommutative weak Hopf algebra, (A, φ_A) a weak left H-module algebra and $\sigma \in Reg_{\varphi_A}(H^2, A)$ satisfying the twisted condition. The following assertions are equivalent:

(i) (A, φ_A) is a left H-module algebra.

(ii) The morphism σ factors through the center of A.

Corollary

Let H be a cocommutative weak Hopf algebra and (A, φ_A) a weak left H-module algebra. The following assertions are equivalent:

(i) (A, φ_A) is a left H-module algebra.

(ii) (φ_A, u_2) is a crossed system for H over A.

Let H be a weak Hopf algebra, (A, φ_A) a weak left H-module algebra and $\sigma : H^2 \to A$ a morphism. We define the morphisms

$$\psi_H^A : H \otimes A \to A \otimes H, \quad \sigma_H^A : H \otimes H \to A \otimes H,$$

by

$$\psi_H^A = (\varphi_A \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A) : H \otimes A \to A \otimes H$$

and

$$\sigma_H^A = (\sigma \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H) : H \otimes H \to A \otimes H$$

The morphism $\nabla_{A\otimes H} : A \otimes H \to A \otimes H$ defined by

$$\nabla_{A\otimes H} = (\mu_A \otimes H) \circ (A \otimes \psi_H^A) \circ (A \otimes H \otimes \eta_A)$$

is idempotent.

With $A \times H$, $i_{A\otimes H} : A \times H \to A \otimes H$ and $p_{A\otimes H} : A \otimes H \to A \times H$ we denote the object, the injection and the projection associated to the factorization of $\nabla_{A\otimes H}$.

If σ satisfies the twisted and the cocycle conditions the object $A \times H$ admits an associative product defined by

$$\mu_{A\times_\sigma H} = p_{A\otimes H} \circ \mu_{A\otimes_\sigma H} \circ (i_{A\otimes H} \otimes i_{A\otimes H})$$

where

$$\mu_{A\otimes_\sigma H} = (\mu_A \otimes H) \circ (\mu_A \otimes \sigma_H^A) \circ (A \otimes \psi_H^A \otimes H).$$

Moreover, if the normal condition holds $A \times H$ is an algebra with unit

$$\eta_{A\times_\sigma H} = p_{A\otimes H} \circ (\eta_A \otimes \eta_H).$$

If σ satisfies the twisted and the cocycle conditions the object $A \times H$ admits an associative product defined by

$$\mu_{A\times_\sigma H} = p_{A\otimes H} \circ \mu_{A\otimes_\sigma H} \circ (i_{A\otimes H} \otimes i_{A\otimes H})$$

where

$$\mu_{A\otimes_\sigma H} = (\mu_A \otimes H) \circ (\mu_A \otimes \sigma_H^A) \circ (A \otimes \psi_H^A \otimes H).$$

Moreover, if the normal condition holds $A \times H$ is an algebra with unit

$$\eta_{A\times_\sigma H} = p_{A\otimes H} \circ (\eta_A \otimes \eta_H).$$

In what follows we denote this algebra by

$$A \times_\sigma H$$

Proposition

Let H be a cocommutative weak Hopf algebra and (φ_A, σ) a crossed system for H over A. Then, the algebra $A \times_\sigma H$ is a right H-comodule algebra for the coaction

$$\rho_{A\times_\sigma H} = (p_{A\otimes H} \otimes H) \circ (A \otimes \delta_H) \circ i_{A\otimes H}.$$

Moreover, $(A \times_\sigma H)_H = A$.

Proposition

Let H be a cocommutative weak Hopf algebra and (φ_A, σ) a crossed system for H over A. Then, the algebra $A \times_\sigma H$ is a right H-comodule algebra for the coaction

$$\rho_{A\times_\sigma H} = (p_{A\otimes H} \otimes H) \circ (A \otimes \delta_H) \circ i_{A\otimes H}.$$

Moreover, $(A \times_\sigma H)_H = A$.

Proposition

Let H be a cocommutative weak Hopf algebra and (φ_A, σ) a crossed system for H over A. Then $A \hookrightarrow A \times_\sigma H$ is an H-cleft extension.

Proposition

Let H be a cocommutative weak Hopf algebra and (φ_A, σ) a crossed system for H over A. Then, the algebra $A \times_\sigma H$ is a right H-comodule algebra for the coaction

$$\rho_{A\times_\sigma H} = (p_{A\otimes H} \otimes H) \circ (A \otimes \delta_H) \circ i_{A\otimes H}.$$

Moreover, $(A \times_\sigma H)_H = A$.

Proposition

Let H be a cocommutative weak Hopf algebra and (φ_A, σ) a crossed system for H over A. Then $A \hookrightarrow A \times_\sigma H$ is an H-cleft extension.

Proposition

Let H be a cocommutative weak Hopf algebra and let A be an algebra. If (φ_A, α) and (ϕ_A, β) are two equivalent crossed systems, so are the associated H-cleft extensions $A \hookrightarrow A \times_\alpha H$ and $A \hookrightarrow A \times_\beta H$.

Proposition

Let H be a cocommutative weak Hopf algebra. If $A_H \hookrightarrow A$ is an H-cleft extension with cleaving morphism f, the morphism

$$\sigma_A = (\mu_A \circ (f \otimes f)) \wedge (f^{-1} \circ \mu_H) : H^2 \to A,$$

factors through the equalizer i_A. Moreover, σ_{A_H}, the factorization of σ_A, is a morphism in $Reg_{\varphi_{A_H}}(H^2, A_H)$ satisfying the normal condition with $\sigma_{A_H}^{-1}$ the factorization through the equalizer i_A of the morphism

$$\sigma_A^{-1} = (f \circ \mu_H) \wedge (\mu_A \circ c_{A,A} \circ (f^{-1} \otimes f^{-1})).$$

Proposition

Let H be a cocommutative weak Hopf algebra. If $A_H \hookrightarrow A$ is an H-cleft extension with cleaving morphism f, the morphism

$$\sigma_A = (\mu_A \circ (f \otimes f)) \wedge (f^{-1} \circ \mu_H) : H^2 \to A,$$

factors through the equalizer i_A. Moreover, σ_{A_H}, the factorization of σ_A, is a morphism in $Reg_{\varphi_{A_H}}(H^2, A_H)$ satisfying the normal condition with $\sigma_{A_H}^{-1}$ the factorization through the equalizer i_A of the morphism

$$\sigma_A^{-1} = (f \circ \mu_H) \wedge (\mu_A \circ c_{A,A} \circ (f^{-1} \otimes f^{-1})).$$

Proposition

Let H be a cocommutative weak Hopf algebra and let $A_H \hookrightarrow A$ be an H-cleft extension with cleaving morphism f. Then, the pair $(\varphi_{A_H}, \sigma_{A_H})$ is a crossed system for H over A_H. Moreover, the H-cleft extensions $A_H \hookrightarrow A$ and $A_H \hookrightarrow A_H \times_{\sigma_{A_H}} H$ are equivalent.

Proposition

Let H be a cocommutative weak Hopf algebra and let (φ_A, σ) be a crossed system for H over A. Let $A \hookrightarrow A \times_\sigma H$ be the associated H-cleft extension. Then, if (ϕ_A, τ) is the crossed system associated to the H-cleft extension $A \hookrightarrow A \times_\sigma H$, we have that $(\phi_A, \tau) = (\varphi_A, \sigma)$.

Proposition

Let H be a cocommutative weak Hopf algebra and let (φ_A, σ) be a crossed system for H over A. Let $A \hookrightarrow A \times_\sigma H$ be the associated H-cleft extension. Then, if (ϕ_A, τ) is the crossed system associated to the H-cleft extension $A \hookrightarrow A \times_\sigma H$, we have that $(\phi_A, \tau) = (\varphi_A, \sigma)$.

Proposition

Let H be a cocommutative weak Hopf algebra and let $A_H \hookrightarrow A$ be an H-cleft extension with cleaving morphism f . Assume that $g : H \to A$ is other cleaving morphism with associated crossed system (ϕ_{A_H}, τ_{A_H}). Then the crossed systems $(\varphi_{A_H}, \sigma_{A_H})$ and (ϕ_{A_H}, τ_{A_H}) are equivalent.

Proposition

Let H be a cocommutative weak Hopf algebra and let (φ_A, σ) be a crossed system for H over A. Let $A \hookrightarrow A \times_\sigma H$ be the associated H-cleft extension. Then, if (ϕ_A, τ) is the crossed system associated to the H-cleft extension $A \hookrightarrow A \times_\sigma H$, we have that $(\phi_A, \tau) = (\varphi_A, \sigma)$.

Proposition

Let H be a cocommutative weak Hopf algebra and let $A_H \hookrightarrow A$ be an H-cleft extension with cleaving morphism f . Assume that $g : H \to A$ is other cleaving morphism with associated crossed system (ϕ_{A_H}, τ_{A_H}). Then the crossed systems $(\varphi_{A_H}, \sigma_{A_H})$ and (ϕ_{A_H}, τ_{A_H}) are equivalent.

Corollary

Let H be a cocommutative weak Hopf algebra and let $A_H \hookrightarrow A$, $A_H \hookrightarrow B$ two equivalent H-cleft extensions with cleaving morphisms f and g respectively. Then the corresponding crossed systems $(\varphi_{A_H}, \sigma_{A_H})$ and (ϕ_{A_H}, τ_{A_H}) are equivalent.

Theorem

Let H be a cocommutative weak Hopf algebra. Two H-cleft extensions $A_H \hookrightarrow A$, $A_H \hookrightarrow B$ are equivalent if and only if so are their respective associated crossed systems.

Theorem

Let H be a cocommutative weak Hopf algebra. Two H-cleft extensions $A_H \hookrightarrow A$, $A_H \hookrightarrow B$ are equivalent if and only if so are their respective associated crossed systems.

Theorem

Let H be a cocommutative weak Hopf algebra and (A, ρ_A) a right H-comodule algebra. There exists a bijective correspondence between the equivalence classes of H-cleft extensions $A_H \hookrightarrow B$ and the equivalence classes of crossed systems for H over A_H.

Theorem

Let H be a cocommutative weak Hopf algebra. Two H-cleft extensions $A_H \hookrightarrow A$, $A_H \hookrightarrow B$ are equivalent if and only if so are their respective associated crossed systems.

Theorem

Let H be a cocommutative weak Hopf algebra and (A, ρ_A) a right H-comodule algebra. There exists a bijective correspondence between the equivalence classes of H-cleft extensions $A_H \hookrightarrow B$ and the equivalence classes of crossed systems for H over A_H.

$$F : CS(H, A_H) \to Cleft(A_H), \quad G : Cleft(A_H) \to CS(H, A_H)$$

$$F([(\varphi_{A_H}, \sigma_{A_H})]) = [A_H \hookrightarrow A_H \times_{\sigma_{A_H}} H]$$

$$G([A_H \hookrightarrow B]) = [(\phi_{A_H}, \tau_{A_H})].$$

Crossed systems and cohomology

- Alonso Álvarez J.N., Fernández Vilaboa J.M., González Rodríguez R.
  Cohomology of algebras over weak Hopf algebras (2012). arXiv:1206.3850

- Alonso Álvarez J.N., Fernández Vilaboa J.M., González Rodríguez R. Cohomology of algebras over weak Hopf algebras (2012). arXiv:1206.3850

Let H be a cocommutative weak Hopf algebra. If (A, φ_A) is a left H-module algebra, the groups $Reg_{\varphi_A}(H^n, A)$, $n \geq 0$ are the objects of a semicosimplicial complex of groups with coface operators defined by

$$\partial_{0,i} : Reg_{\varphi_A}(H_L, A) \to Reg_{\varphi_A}(H, A), \quad i \in \{0, 1\}$$

$$\partial_{0,0}(g) = \varphi_A \circ (H \otimes (g \circ p_L \circ \Pi_H^R)) \circ \delta_H, \quad \partial_{0,1}(g) = g \circ p_L.$$

- Alonso Álvarez J.N., Fernández Vilaboa J.M., González Rodríguez R. Cohomology of algebras over weak Hopf algebras (2012). arXiv:1206.3850

Let H be a cocommutative weak Hopf algebra. If (A, φ_A) is a left H-module algebra, the groups $Reg_{\varphi_A}(H^n, A)$, $n \geq 0$ are the objects of a semicosimplicial complex of groups with coface operators defined by

$$\partial_{0,i} : Reg_{\varphi_A}(H_L, A) \to Reg_{\varphi_A}(H, A), \quad i \in \{0, 1\}$$

$$\partial_{0,0}(g) = \varphi_A \circ (H \otimes (g \circ p_L \circ \Pi_H^R)) \circ \delta_H, \quad \partial_{0,1}(g) = g \circ p_L.$$

$$\partial_{1,i} : Reg_{\varphi_A}(H, A) \to Reg_{\varphi_A}(H^2, A), \quad i \in \{0, 1, 2\}$$

$$\partial_{1,0}(h) = \varphi_A \circ (H \otimes h), \quad \partial_{1,1}(h) = h \circ \mu_H, \quad \partial_{1,2}(h) = h \circ \mu_H \circ (H \otimes \Pi_H^L).$$

$$\partial_{k-1,i} : Reg_{\varphi_A}(H^{k-1}, A) \to Reg_{\varphi_A}(H^k, A), \quad k > 2, \quad i \in \{0, 1, \cdots, k\}$$

$$\partial_{k-1,i}(\sigma) = \begin{cases} \partial_{k-1,0}(\sigma) = \varphi_A \circ (H \otimes \sigma), \\ \partial_{k-1,i}(\sigma) = \sigma \circ (H^{i-1} \otimes \mu_H \otimes H^{k-i-1}), \quad i \in \{1, \cdots, k-1\} \\ \partial_{k-1,k}(\sigma) = \sigma \circ (H^{k-2} \otimes (\mu_H \circ (H \otimes \Pi_H^L))), \end{cases}$$

For this complex the codegeneracy operators are defined by

$$s_{1,0} : Reg_{\varphi_A}(H, A) \to Reg_{\varphi_A}(H_L, A),$$

$$s_{1,0}(h) = h \circ i_L,$$

For this complex the codegeneracy operators are defined by

$$s_{1,0} : Reg_{\varphi_A}(H, A) \to Reg_{\varphi_A}(H_L, A),$$

$$s_{1,0}(h) = h \circ i_L,$$

$$s_{2,i} : Reg_{\varphi_A}(H^2, A) \to Reg_{\varphi_A}(H, A), \quad i \in \{0, 1\}$$

$$s_{2,0}(\sigma) = \sigma \circ (\eta_H \otimes H), \quad s_{2,1}(\sigma) = \sigma \circ (H \otimes \eta_H),$$

$$s_{k+1,i} : Reg_{\varphi_A}(H^{k+1}, A) \to Reg_{\varphi_A}(H^k, A), \quad k \geq 2, \quad i \in \{0, 1, \cdots, k\}$$

$$s_{k+1,i}(\sigma) = \begin{cases} s_{k+1,0}(\sigma) = \sigma \circ (\eta_H \otimes H^k), \\ s_{k+1,i}(\sigma) = \sigma \circ (H^i \otimes \eta_H \otimes H^{k-i}), \quad i \in \{1, \cdots, k-1\} \\ s_{k+1,k}(\sigma) = \sigma \circ (H^k \otimes \eta_H). \end{cases}$$

Let

$$D^k_{\varphi_A} = \partial_{k,0} \wedge \partial^{-1}_{k,1} \wedge \cdots \wedge \partial^{(-1)^{k+1}}_{k,k+1}$$

be the coboundary morphisms of the cochain complex

$$Reg_{\varphi_A}(H_L, A) \xrightarrow{D^0_{\varphi_A}} Reg_{\varphi_A}(H, A) \xrightarrow{D^1_{\varphi_A}} Reg_{\varphi_A}(H^2, A) \xrightarrow{D^2_{\varphi_A}} \cdots$$

$$\cdots \xrightarrow{D^{k-1}_{\varphi_A}} Reg_{\varphi_A}(H^k, A) \xrightarrow{D^k_{\varphi_A}} Reg_{\varphi_A}(H^{k+1}, A) \xrightarrow{D^{k+1}_{\varphi_A}} \cdots$$

associated to the cosimplicial complex $Reg_{\varphi_A}(H^\bullet, A)$.
Then, when (A, φ_A) is a commutative left H-module algebra, $(Reg_{\varphi_A}(H^\bullet, A), D^\bullet_{\varphi_A})$ gives the Sweedler cohomology of H in (A, φ_A). Therefore, the kth group, will be defined by

$$\frac{Ker(D^k_{\varphi_A})}{Im(D^{k-1}_{\varphi_A})}$$

for $k \geq 1$ and $Ker(D^0_{\varphi_A})$ for $k = 0$. We will denote it by $H^k_{\varphi_A}(H, A)$.

Proposition

Let H be a cocommutative weak Hopf algebra and let $A_H \hookrightarrow A$ be an H-cleft extension. We denote by $(\varphi_{A_H}, \sigma_{A_H})$ the corresponding crossed system defined by the convolution invertible total integral $f : H \to A$. Then $(Z(A_H), \varphi_{Z(A_H)})$ is a left H-module algebra, where $\varphi_{Z(A_H)}$ is the factorization through the morphism $i_{Z(A_H)}$ of the morphism $\varphi_{A_H} \circ (H \otimes i_{Z(A_H)})$.

Theorem.

Let H be a cocommutative weak Hopf algebra and let $A_H \hookrightarrow A$ be an H-cleft extension. We denote by $(\varphi_{A_H}, \sigma_{A_H})$ the corresponding crossed system defined by the cleaving morphism $f : H \to A$. Then there is a bijective correspondence between the second cohomology group $H^2_{\varphi_{Z(A_H)}}(H, Z(A_H))$ and the equivalence classes of crossed systems for H over A_H having φ_{A_H} as weak H-module algebra structure.

Theorem.

Let H be a cocommutative weak Hopf algebra and let $A_H \hookrightarrow A$ be an H-cleft extension. We denote by $(\varphi_{A_H}, \sigma_{A_H})$ the corresponding crossed system defined by the cleaving morphism $f : H \to A$. Then there is a bijective correspondence between the second cohomology group $H^2_{\varphi_{Z(A_H)}}(H, Z(A_H))$ and the equivalence classes of crossed systems for H over A_H having φ_{A_H} as weak H-module algebra structure.

References:

- Alonso Álvarez, J.N., Fernández Vilaboa, J.M. y González Rodríguez, R.: Integrals and crossed products over weak Hopf algebras (2012). arXiv:1207.5363