Cleft extensions, integrals and crossed products in a weak setting

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Based in a joint work with J.N. Alonso Álvarez and J.M. Fernández Vilaboa
Research supported by Ministerio de Ciencia e Innovación: MTM2010-15634.

Recent Trends in Rings and Algebras
Murcia, 3-7 June 2013
Outline

1. Cleft extensions for weak Hopf algebras
2. Crossed systems for weak Hopf algebras
3. Crossed systems and cohomology
Some notation and conventions.

- From now on $C$ denotes a strict symmetric category with tensor product denoted by $\otimes$ and unit object $K$. With $c$ we will denote the natural isomorphism of symmetry and we also assume that every idempotent morphism $q : Y \to Y$ splits, i.e., there exist an object $Z$ and morphisms $i : Z \to Y$ and $p : Y \to Z$ such that $q = i \circ p$ and $p \circ i = id_Z$. 

\begin{align*}
\text{(A, } \eta_A, \mu_A) &\text{ is an associative algebra with multiplication } \mu_A \text{ and unit } \eta_A. \\
\text{(C, } \epsilon_C, \delta_C) &\text{ is a coassociative coalgebra with comultiplication } \delta_C \text{ and counit } \epsilon_C.
\end{align*}

For simplicity of notation, given three objects $V, U, B$ in $C$ and a morphism $f : V \to U$, we write $B \otimes f$ for $\text{id}_B \otimes f$ and $f \otimes B$ for $f \otimes \text{id}_B$. 

For an algebra $A$ we denote by $Z(A)$ the center of $A$ and by $i_{Z(A)}$ the inclusion morphism of $Z(A)$ on $A$. 

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1. Cleft extensions for weak Hopf algebras

2. Crossed systems for weak Hopf algebras

3. Crossed systems and cohomology
Definition.

A weak Hopf algebra in $C$ is an object in $C$ with an algebra structure $(H, \eta_H, \mu_H)$ and a coalgebra structure $(H, \varepsilon_H, \delta_H)$ satisfying:

1. $\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H)$.

2. $\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = ((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes \delta_H \otimes H)$
   $$= ((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes (c_{H,H} \circ \delta_H) \otimes H).$$

3. $(\delta_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \mu_H \otimes H) \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H))$
   $$= (H \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H)).$$

4. There exists a morphism $\lambda_H : H \rightarrow H$ in $C$ (called the antipode of $H$) satisfying:

   4-1. $id_H \wedge \lambda_H = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$.

   4-2. $\lambda_H \wedge id_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$.

   4-3. $\lambda_H \wedge id_H \wedge \lambda_H = \lambda_H$. 

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Cleft extensions, integrals and crossed products in a weak setting
If $H$ is a weak Hopf algebra in $C$, the antipode $\lambda_H$ is unique, antimultiplicative, antico-
multiplicative and leaves the unit and the counit invariant:

$$\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}, \quad \delta_H \circ \lambda_H = c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H,$$

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$$

$$
\lambda_H \circ \eta_H = \eta_H, \quad \varepsilon_H \circ \lambda_H = \varepsilon_H.
$$

If we define the morphisms $\Pi^L_H$ (target), $\Pi^R_H$ (source), by

$$
\Pi^L_H = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H),
$$

$$
\Pi^R_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)),
$$

they are idempotent and we denote by $H_L$, $p_L$ and $i_L$ the object and the morphisms such that $i_L \circ p_L = \Pi^L_H$ and $p_L \circ i_L = id_{H_L}$. 

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Definition.

A weak Hopf algebra $H$ is cocommutative if $\delta_H = c_{H,H} \circ \delta_H$. 

Example. Let $G$ be a finite groupoid and $R$ a commutative ring. Let $G_0$ be the set of objects and $G_1$ the set of morphisms. The groupoid algebra is the direct product $RG = \bigoplus_{\sigma \in G_1} R\sigma$ with the product of two morphisms being equal to their composition if the latter is defined and 0 in otherwise, i.e. $\sigma \tau = \sigma \circ \tau$ if $s(\sigma) = t(\tau)$ and $\sigma \tau = 0$ if $s(\sigma) \neq t(\tau)$. The unit element is $1_{RG} = \sum_{x \in G_0} id_x$. $RG$ is a cocommutative weak Hopf algebra, with $\delta_{RG}(\sigma) = \sigma \otimes \sigma$, $\varepsilon_{RG}(\sigma) = 1$, $\lambda_{RG}(\sigma) = \sigma - \dot{1}$. The morphisms target and source are $\Pi_L^{RG}(\sigma) = id_{t(\sigma)}$, $\Pi_R^{RG}(\sigma) = id_{s(\sigma)}$. 

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$$\delta_{RG}(\sigma) = \sigma \otimes \sigma, \quad \varepsilon_{RG}(\sigma) = 1, \quad \lambda_{RG}(\sigma) = \sigma^{-1}.$$ 

The morphisms target and source are $\Pi_{RG}^L(\sigma) = id_{t(\sigma)}$, $\Pi_{RG}^R(\sigma) = id_{s(\sigma)}$. 

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Definition.

Let $H$ be a weak Hopf algebra. We will say that a right $H$-comodule $(A, \rho_A)$ is a right $H$-comodule algebra if it satisfies

$$\rho_A \circ \mu_A = (\mu_A \otimes \mu_H) \circ (A \otimes c_{H,A} \otimes H) \circ (\rho_A \otimes \rho_A)$$

and any of the following equivalent conditions hold:

(1) $(A \otimes \Pi^L_H) \circ \rho_A = (\mu_A \otimes H) \circ (A \otimes c_{H,A}) \circ ((\rho_A \circ \eta_A) \otimes A)$.

(2) $(A \otimes \Pi^R_H) \circ \rho_A = (\mu_A \otimes H) \circ (A \otimes (\rho_A \circ \eta_A))$.

(3) $(A \otimes \Pi^L_H) \circ \rho_A \circ \eta_A = \rho_A \circ \eta_A$.

(4) $(A \otimes \Pi^R_H) \circ \rho_A \circ \eta_A = \rho_A \circ \eta_A$.

(5) $(\rho_A \otimes H) \circ \rho_A \circ \eta_A = (A \otimes \mu_H \otimes H) \circ (\rho_A \otimes \delta_H) \circ (\eta_A \otimes \eta_H)$.

(6) $(\rho_A \otimes H) \circ \rho_A \circ \eta_A = (A \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ (\rho_A \otimes \delta_H) \circ (\eta_A \otimes \eta_H)$. 
If \((A, \rho_A)\) is a right \(H\)-comodule algebra, the triple \((A, H, \Psi)\) is a weak entwining structure where the entwining morphism is

\[
\Psi = (A \otimes \mu_H) \circ (c_{H,A} \otimes H) \circ (H \otimes \rho_A) : H \otimes A \to A \otimes H
\]

Therefore the following identities hold:

1. \(\Psi \circ (H \otimes \mu_A) = (\mu_A \otimes H) \circ (A \otimes \Psi) \circ (\Psi \otimes A)\),
2. \((A \otimes \delta_H) \circ \Psi = (\Psi \otimes H) \circ (H \otimes \Psi) \circ (\delta_H \otimes A)\),
3. \(\Psi \circ (H \otimes \eta_A) = (e_A \otimes H) \circ \delta_H\),
4. \((A \otimes \varepsilon_H) \circ \Psi = \mu_A \circ (e_A \otimes A)\),

where

\[
e_A = (A \otimes \varepsilon_H) \circ \Psi \circ (H \otimes \eta_A) : H \to A.
\]
We denote by $\mathcal{M}_A^H(\Psi)$ the category of weak entwined modules, i.e., the objects $M$ in $\mathcal{C}$ together with two morphisms $\phi_M : M \otimes A \to A$ and $\rho_M : M \to M \otimes H$ such that $(M, \phi_M)$ is a right $A$-module, $(M, \rho_M)$ is a right $H$-comodule and such that the following equality

$$\rho_M \circ \phi_M = (\phi_M \otimes H) \circ (M \otimes \Psi) \circ (\rho_M \otimes A)$$

holds.

Then, if $(A, \rho_A)$ is a right $H$-comodule algebra, $(A, \mu_A, \rho_A)$ is an object of $\mathcal{M}_A^H(\Psi)$. 

If \((A, \rho_A)\) is a right \(H\)-comodule algebra, we define the \textit{subalgebra of coinvariants} of \(A\) as the equalizer:

![Equalizer Diagram]

where

\[
\zeta_A = (\mu_A \otimes H) \circ (A \otimes c_{H,A}) \circ ((\rho_A \circ \eta_A) \otimes A).
\]

Note that

\[
\zeta_A = (A \otimes \Pi^L_H) \circ \rho_A.
\]
If \((A, \rho_A)\) is a right \(H\)-comodule algebra, we define the subalgebra of coinvariants of \(A\) as the equalizer:

\[
A_H \xrightarrow{i_A} A \xrightarrow{\rho_A} A \otimes H
\]

where

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Also

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A_H \xrightarrow{i_A} A \xrightarrow{\rho_A} A \otimes H
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is an equalizer diagram.
The triple

\((A_H, \eta_{A_H}, \mu_{A_H})\)

is an algebra, being \(\eta_{A_H}\) and \(\mu_{A_H}\) the factorizations through the equalizer \(i_A\) of the morphisms \(\eta_A\) and \(\mu_A \circ (i_A \otimes i_A)\), respectively.
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For example, the weak Hopf algebra \(H\) is a right \(H\)-comodule algebra with right comodule structure giving by \(\rho_H = \delta_H\) and subalgebra of coinvariants \(H_H\), the image of the idempotent morphism \(\Pi^L_H\), which we denoted by \(H_L\).
### Definition

Let $H$ be a weak Hopf algebra and let $(A, \rho_A)$ be a right $H$-comodule algebra. We define an **integral** as a morphism of right $H$-comodules $f : H \to A$. If moreover $f \circ \eta_H = \eta_A$ we will say that the integral is **total**.

An integral $f : H \to A$ is **convolution invertible** if there exists a morphism $f^{-1} : H \to A$ (called the convolution inverse of $f$) such that

1. $f^{-1} \wedge f = e_A$.
2. $f \wedge f^{-1} = (A \otimes (\varepsilon_H \circ \mu_H)) \circ ((\rho_A \circ \eta_A) \otimes H)$.
3. $f^{-1} \wedge f \wedge f^{-1} = f^{-1}$.

Trivially, $f^{-1}$ is unique and by (1), if $f$ is an integral convolution invertible, we get that $f \wedge f^{-1} \wedge f^{-1} = f^{-1}$. Finally, when $f$ is a total integral we can rewrite equality (1) as $f^{-1} \wedge f = f \circ \Pi_R^H$ and (2) as $f \wedge f^{-1} = f \circ \Pi_L^H$. 

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Definition

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Trivially, $f^{-1}$ is unique and by (1), if $f$ is an integral convolution invertible, we get that

$$ f \wedge f^{-1} \wedge f = f. $$

Finally, when $f$ is a total integral we can rewrite equality (1) as

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and (2) as

$$ f \wedge f^{-1} = f \circ \Pi^L_H. $$
Example

Let $H$ be a weak Hopf algebra such that $\Pi_L^H = \overline{\Pi}_L^H$ (equivalently, $\Pi_R^H = \overline{\Pi}_R^H$). Then the identity $id_H$ is a total integral convolution invertible with inverse $\lambda_H$. Note that this equality is always true in the Hopf algebra setting. In our case it holds, for example, if $H$ is a cocommutative weak Hopf algebra.
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Definition

Let $H$ be a weak Hopf algebra and $(A, \rho_A)$ a right $H$-comodule algebra. We say that $A_H \hookrightarrow A$ is a $H$-cleft extension if there exists an integral $f : H \to A$ convolution invertible and such that the morphism $f \wedge f^{-1}$ factorizes through the equalizer $i_A$. In what follows, the morphism $f$ will be called a cleaving morphism associated to the $H$-cleft extension $A_H \hookrightarrow A$. 
Definition

Two $H$-cleft extensions $A_H \hookrightarrow A$ and $B_H \hookrightarrow B$ are equivalent

$$A_H \hookrightarrow A \sim B_H \hookrightarrow B$$

if $A_H = B_H$ and there is a morphism of right $H$-comodule algebras $T : A \rightarrow B$ such that $T \circ i_A = i_B$. 
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Remark

If $A_H \hookrightarrow A$ and $B_H \hookrightarrow B$ are equivalent, the morphism $T$ is an isomorphism.
Proposition

Let $H$ be a weak Hopf algebra and $(A, \rho_A)$ a right $H$-comodule algebra such that $A_H \hookrightarrow A$ is a $H$-cleft extension with cleaving morphism $f$. Then the equality

$$\rho_A \circ f^{-1} = (f^{-1} \otimes \lambda_H) \circ c_{H,H} \circ \delta_H$$

holds.
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Let $H$ be a weak Hopf algebra and $(A, \rho_A)$ a right $H$-comodule algebra such that $A_H \hookrightarrow A$ is a $H$-cleft extension with cleaving morphism $f$. Then the equality

$$\rho_A \circ f^{-1} = (f^{-1} \otimes \lambda_H) \circ c_{H,H} \circ \delta_H$$

holds.

Proposition

Let $H$ be a cocommutative weak Hopf algebra and let $(A, \rho_A)$ be a right $H$-comodule algebra. If there exists a convolution invertible integral $f : H \rightarrow A$, then $A_H \hookrightarrow A$ is an $H$-cleft extension.
Let $H$ be a weak Hopf algebra and let $(A, \rho_A)$ be a right $H$-comodule algebra. In


we introduce the set $\text{Reg}^{WR}(H, A)$ as the one whose elements are the morphisms

$$h : H \rightarrow A$$

such that there exists a morphism $h^{-1} : H \rightarrow A$, called the left weak inverse of $h$, such that

$$h^{-1} \wedge h = e_A$$

where $e_A$ is the morphism associated to the right-right weak entwining structure $\Psi$ associated to $(A, \rho_A)$. 
Definition

Let $H$ be a weak Hopf algebra and let $(A, \rho_A)$ be a right $H$-comodule algebra. We say that $A_H \hookrightarrow A$ is a weak $H$-cleft extension if there exists a morphism $h : H \to A$ in $\text{Reg}^{WR}(H, A)$ of right $H$-comodules such that

$$\psi \circ (H \otimes h^{-1}) \circ \delta_H = \zeta_A \circ h^{-1}.$$
Definition

Let $H$ be a weak Hopf algebra and let $(A, \rho_A)$ be a right $H$-comodule algebra. We say that $A_H \hookrightarrow A$ is a weak $H$-cleft extension if there exists a morphism $h : H \to A$ in $\text{Reg}^{WR}(H, A)$ of right $H$-comodules such that

$$\Psi \circ (H \otimes h^{-1}) \circ \delta_H = \zeta_A \circ h^{-1}.$$ 

Proposition

Let $H$ be a weak Hopf algebra and let $(A, \rho_A)$ be a right $H$-comodule algebra. If there exists $h \in \text{Reg}^{WR}(H, A)$ of right $H$-comodules such that $e_A \wedge h^{-1} = h^{-1}$, the following assertions are equivalent:

(i) The morphism $h \wedge h^{-1}$ factorizes through the equalizer $i_A$ and $h^{-1}$ satisfies

$$\rho_A \circ f^{-1} = (f^{-1} \otimes \lambda_H) \circ c_{H,H} \circ \delta_H.$$ 

(ii) The equality

$$\Psi \circ (H \otimes h^{-1}) \circ \delta_H = \zeta_A \circ h^{-1}$$

holds.
Corollary

Let $H$ be a weak Hopf algebra and let $(A, \rho_A)$ be a right $H$-comodule algebra. If $A_H \hookrightarrow A$ is an $H$-cleft extension then it is a weak $H$-cleft extension.
Proposition

Let $H$ be a weak Hopf algebra with invertible antipode. If $A_H \hookrightarrow A$ is an $H$-cleft extension with cleaving morphism $f$, then $h = \mu_A \circ (f \otimes (f^{-1} \circ \eta_H))$ is a total integral. Moreover, if $H$ is cocommutative $h$ is convolution invertible.
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Let $H$ be a weak Hopf algebra with invertible antipode. If $A_H \hookrightarrow A$ is an $H$-cleft extension with cleaving morphism $f$, then $h = \mu_A \circ (f \otimes (f^{-1} \circ \eta_H))$ is a total integral. Moreover, if $H$ is cocommutative $h$ is convolution invertible.

Remark

As a consequence of the previous proposition, in the cocommutative setting we can assume that the cleaving morphism is a total integral.
Definition

Let $H$ be a weak Hopf algebra. We will say that $A$ is a weak left $H$-module algebra if there exists a morphism $\varphi_A : H \otimes A \to A$ satisfying:

1. $\varphi_A \circ (\eta_H \otimes A) = id_A$.
2. $\varphi_A \circ (H \otimes \mu_A) = \mu_A \circ (\varphi_A \otimes \varphi_A) \circ (H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes A \otimes A)$.
3. $\varphi_A \circ (\mu_H \otimes \eta_A) = \varphi_A \circ (H \otimes (\varphi_A \circ (H \otimes \eta_A)))$.

and any of the following equivalent conditions hold:

4. $\varphi_A \circ (\Pi^L_H \otimes A) = \mu_A \circ ((\varphi_A \circ (H \otimes \eta_A)) \otimes A)$.
5. $\varphi_A \circ (\overline{\Pi}^L_H \otimes A) = \mu_A \circ c_{A,A} \circ ((\varphi_A \circ (H \otimes \eta_A)) \otimes A)$.
6. $\varphi_A \circ (\Pi^L_H \otimes \eta_A) = \varphi_A \circ (H \otimes \eta_A)$.
7. $\varphi_A \circ (\overline{\Pi}^L_H \otimes \eta_A) = \varphi_A \circ (H \otimes \eta_A)$.
8. $\varphi_A \circ (H \otimes (\varphi_A \circ (H \otimes \eta_A))) = ((\varphi_A \circ (H \otimes \eta_A)) \otimes (\varepsilon_H \circ \mu_H)) \circ (\delta_H \otimes H)$.
9. $\varphi_A \circ (H \otimes (\varphi_A \circ (H \otimes \eta_A))) = ((\varepsilon_H \circ \mu_H) \otimes (\varphi_A \circ (H \otimes \eta_A))) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)$.

If we replace (3) by

(3-1) $\varphi_A \circ (\mu_H \otimes A) = \varphi_A \circ (H \otimes \varphi_A)$

we will say that $(A, \varphi_A)$ is a left $H$-module algebra.
Proposition

Let $H$ be a cocommutative weak Hopf algebra. If $A_H \hookrightarrow A$ is an $H$-cleft extension with cleaving morphism $f$, the pair $(A_H, \varphi_{A_H})$ is a weak left $H$-module algebra, being $\varphi_{A_H}$ the factorization of the morphism

$$\varphi_A = \mu_A \circ (A \otimes (\mu_A \circ c_{A,A})) \circ (((f \otimes f^{-1}) \circ \delta_H) \otimes i_A)$$

through the equalizer $i_A$. 
1 Cleft extensions for weak Hopf algebras

2 Crossed systems for weak Hopf algebras

3 Crossed systems and cohomology
**Definition**

Let $H$ be a cocommutative weak Hopf algebra and $(A, \varphi_A)$ be a weak left $H$-module algebra. By $\text{Reg}_{\varphi_A}(H_L, A)$ we denote the set of morphisms $g : H_L \to A$ such that there exists a morphism $g^{-1} : H_L \to A$ satisfying

$$g \land g^{-1} = g^{-1} \land g = u_0, \quad g \land g^{-1} \land g = g, \quad g^{-1} \land g \land g^{-1} = g^{-1}$$

where $u_0 = u_1 \circ i_L$ where $u_1 = \varphi_A \circ (H \otimes \eta_A)$.

By $\text{Reg}_{\varphi_A}(H, A)$, as the set of morphisms $h : H \to A$ such that there exists a morphism $h^{-1} : H \to A$ satisfying the following equalities:

1. $h \land h^{-1} = h^{-1} \land h = u_1$,
2. $h \land h^{-1} \land h = h$,
3. $h^{-1} \land h \land h^{-1} = h^{-1}$,

Note that

$$u_1 = u_0 \circ p_L.$$
For $n > 1$, we denote by $H^n$ the $n$-fold tensor power $H \otimes \cdots \otimes H$. $\text{Reg}_{\varphi_A}(H^n, A)$ is the set of morphisms $\sigma : H^n \to A$ such that there exists a morphism $\sigma^{-1} : H^n \to A$ satisfying:

1. $\sigma \wedge \sigma^{-1} = \sigma^{-1} \wedge \sigma = u_n$
2. $\sigma \wedge \sigma^{-1} \wedge \sigma = \sigma$.
3. $\sigma^{-1} \wedge \sigma \wedge \sigma^{-1} = \sigma^{-1}$.

where $u_n = \varphi_A \circ (H \otimes u_{n-1})$. 
For $n > 1$, we denote by $H^n$ the $n$-fold tensor power $H \otimes \cdots \otimes H$. $\text{Reg}_{\varphi_A}(H^n, A)$ is the set of morphisms $\sigma : H^n \to A$ such that there exists a morphism $\sigma^{-1} : H^n \to A$ satisfying:

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3. $\sigma^{-1} \wedge \sigma \wedge \sigma^{-1} = \sigma^{-1}$.

where $u_n = \varphi_A \circ (H \otimes u_{n-1})$.

If we denote by $H^0$ the object $H_L$, $u_n \in \text{Reg}_{\varphi_A}(H^n, A)$ and $\text{Reg}_{\varphi_A}(H^n, A)$ is a group with neutral element $u_n$ for all $n \geq 0$. Also, if $A$ is commutative, we have that $\text{Reg}_{\varphi_A}(H^n, A)$ is an abelian group for all $n \geq 0$. 
Definition

Let $H$ be a cocommutative weak Hopf algebra, $(A, \varphi_A)$ a weak left $H$-module algebra and $\sigma \in \text{Reg}_{\varphi_A}(H^2, A)$. We say that

$$(\varphi_A, \sigma)$$

is a crossed system for $H$ over $A$ if the following conditions hold:

1. **Twisted condition**

$$\mu_A \circ (A \otimes \varphi_A) \circ (\sigma \otimes \mu_H \otimes A) \circ (\delta_{H \otimes H} \otimes A)$$

$$= \mu_A \circ ((\varphi_A \circ (H \otimes \varphi_A)) \otimes A) \circ (H \otimes H \otimes c_{A,A}) \circ (H \otimes H \otimes \sigma \otimes A) \circ (\delta_{H \otimes H} \otimes A).$$

2. **Two cocycle condition**

$$(\varphi_A \circ (H \otimes \sigma)) \wedge (\sigma \circ (H \otimes \mu_H)) = (\sigma \circ (H \otimes (\mu_H \circ (H \otimes \Pi^L_H)))) \wedge (\sigma \circ (\mu_H \otimes H)).$$

3. **Normal condition**

$$\sigma \circ (H \otimes \eta_H) = \sigma \circ (\eta_H \otimes H) = u_1.$$
Two crossed systems for $H$ over $A$, $(\varphi_A, \sigma)$ and $(\phi_A, \tau)$ are said to be equivalent, denoted by

$$(\varphi_A, \sigma) \approx (\phi_A, \tau),$$

if

$$\varphi_A \circ (H \otimes \eta_A) = \phi_A \circ (H \otimes \eta_A)$$

and there exists $h$ in $\text{Reg}_{\varphi_A}(H, A) \cap \text{Reg}_{\phi_A}(H, A)$ with $h \circ \eta_H = \eta_A$ and such that

$$\varphi_A = \mu_A \circ (\mu_A \otimes A) \circ (h \otimes \phi_A \otimes h^{-1}) \circ (\delta_H \otimes c_{H, A}) \circ (\delta_H \otimes A),$$

$$\sigma = \mu_A \circ (\mu_A \otimes h^{-1}) \circ (\mu_A \otimes \tau \otimes \mu_H) \circ (h \otimes \phi_A \otimes \delta_{H \otimes H}) \circ (\delta_H \otimes h \otimes H \otimes H) \circ \delta_{H \otimes H}.$$
Two crossed systems for $H$ over $A$, $(\varphi_A, \sigma)$ and $(\phi_A, \tau)$ are said to be equivalent, denoted by

$$(\varphi_A, \sigma) \approx (\phi_A, \tau),$$

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$$\varphi_A = \mu_A \circ (\mu_A \otimes A) \circ (h \otimes \phi_A \otimes h^{-1}) \circ (\delta_H \otimes c_{H, A}) \circ (\delta_H \otimes A),$$

$$\sigma = \mu_A \circ (\mu_A \otimes h^{-1}) \circ (\mu_A \otimes \tau \otimes \mu_H) \circ (h \otimes \phi_A \otimes \delta_{H \otimes H}) \circ (\delta_H \otimes h \otimes H \otimes H) \circ \delta_{H \otimes H}.$$

**Proposition**

Let $H$ be a cocommutative weak Hopf algebra. Then $\approx$ is an equivalence relation.
Proposition

Let $H$ be a cocommutative weak Hopf algebra, $(A, \varphi_A)$ a weak left $H$-module algebra and $\sigma \in \text{Reg}_{\varphi_A}(H^2, A)$ satisfying the twisted condition. The following assertions are equivalent:

(i) $(A, \varphi_A)$ is a left $H$-module algebra.

(ii) The morphism $\sigma$ factors through the center of $A$. 

Proposition

Let $H$ be a cocommutative weak Hopf algebra, $(A, \varphi_A)$ a weak left $H$-module algebra and $\sigma \in \text{Reg}_{\varphi_A}(H^2, A)$ satisfying the twisted condition. The following assertions are equivalent:

(i) $(A, \varphi_A)$ is a left $H$-module algebra.

(ii) The morphism $\sigma$ factors through the center of $A$.

Corollary

Let $H$ be a cocommutative weak Hopf algebra and $(A, \varphi_A)$ a weak left $H$-module algebra. The following assertions are equivalent:

(i) $(A, \varphi_A)$ is a left $H$-module algebra.

(ii) $(\varphi_A, u_2)$ is a crossed system for $H$ over $A$. 

Ramón González Rodríguez

Cleft extensions, integrals and crossed products in a weak setting
Let $H$ be a weak Hopf algebra, $(A, \varphi_A)$ a weak left $H$-module algebra and $\sigma : H^2 \to A$ a morphism. We define the morphisms

$$\psi^A_H : H \otimes A \to A \otimes H, \quad \sigma^A_H : H \otimes H \to A \otimes H,$$

by

$$\psi^A_H = (\varphi_A \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A) : H \otimes A \to A \otimes H$$

and

$$\sigma^A_H = (\sigma \otimes \mu_H) \circ (H \otimes c_{H,H \otimes H}) \circ (\delta_H \otimes \delta_H) : H \otimes H \to A \otimes H$$

The morphism $\nabla_{A \otimes H} : A \otimes H \to A \otimes H$ defined by

$$\nabla_{A \otimes H} = (\mu_A \otimes H) \circ (A \otimes \psi^A_H) \circ (A \otimes H \otimes \eta_A)$$

is idempotent.

With $A \times H$, $i_{A \otimes H} : A \times H \to A \otimes H$ and $p_{A \otimes H} : A \otimes H \to A \times H$ we denote the object, the injection and the projection associated to the factorization of $\nabla_{A \otimes H}$. 
If \( \sigma \) satisfies the twisted and the cocycle conditions the object \( A \times H \) admits an associative product defined by

\[
\mu_{A \times \sigma H} = p_{A \otimes H} \circ \mu_{A \otimes \sigma H} \circ (i_{A \otimes H} \otimes i_{A \otimes H})
\]

where

\[
\mu_{A \otimes \sigma H} = (\mu_A \otimes H) \circ (\mu_A \otimes \sigma^A_H) \circ (A \otimes \psi^A_H \otimes H).
\]

Moreover, if the normal condition holds \( A \times H \) is an algebra with unit

\[
\eta_{A \times \sigma H} = p_{A \otimes H} \circ (\eta_A \otimes \eta_H).
\]
If $\sigma$ satisfies the twisted and the cocycle conditions the object $A \times H$ admits an associative product defined by

$$\mu_{A \times \sigma H} = p_{A \otimes H} \circ \mu_{A \otimes \sigma H} \circ (i_{A \otimes H} \otimes i_{A \otimes H})$$

where

$$\mu_{A \otimes \sigma H} = (\mu_A \otimes H) \circ (\mu_A \otimes \sigma^A_H) \circ (A \otimes \psi^A_H \otimes H).$$

Moreover, if the normal condition holds $A \times H$ is an algebra with unit

$$\eta_{A \times \sigma H} = p_{A \otimes H} \circ (\eta_A \otimes \eta_H).$$

In what follows we denote this algebra by

$$A \times_{\sigma} H$$
Proposition

Let $H$ be a cocommutative weak Hopf algebra and $(\varphi_A, \sigma)$ a crossed system for $H$ over $A$. Then, the algebra $A \times \sigma H$ is a right $H$-comodule algebra for the coaction

$$\rho_{A \times \sigma H} = (p_{A \otimes H} \otimes H) \circ (A \otimes \delta_H) \circ i_{A \otimes H}.$$ 

Moreover, $(A \times \sigma H)_H = A$. 
**Proposition**

Let $H$ be a cocommutative weak Hopf algebra and $(\varphi_A, \sigma)$ a crossed system for $H$ over $A$. Then, the algebra $A \times \sigma H$ is a right $H$-comodule algebra for the coaction

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Moreover, $(A \times \sigma H)_H = A$.

**Proposition**

Let $H$ be a cocommutative weak Hopf algebra and $(\varphi_A, \sigma)$ a crossed system for $H$ over $A$. Then $A \hookrightarrow A \times \sigma H$ is an $H$-cleft extension.
Proposition

Let $H$ be a cocommutative weak Hopf algebra and $(\varphi_A, \sigma)$ a crossed system for $H$ over $A$. Then, the algebra $A \times_\sigma H$ is a right $H$-comodule algebra for the coaction

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Proposition

Let $H$ be a cocommutative weak Hopf algebra and $(\varphi_A, \sigma)$ a crossed system for $H$ over $A$. Then $A \hookrightarrow A \times_\sigma H$ is an $H$-cleft extension.

Proposition

Let $H$ be a cocommutative weak Hopf algebra and let $A$ be an algebra. If $(\varphi_A, \alpha)$ and $(\phi_A, \beta)$ are two equivalent crossed systems, so are the associated $H$-cleft extensions $A \hookrightarrow A \times_\alpha H$ and $A \hookrightarrow A \times_\beta H$. 
Proposition

Let $H$ be a cocommutative weak Hopf algebra. If $A_H \hookrightarrow A$ is an $H$-cleft extension with cleaving morphism $f$, the morphism

$$\sigma_A = (\mu_A \circ (f \otimes f)) \wedge (f^{-1} \circ \mu_H) : H^2 \to A,$$

factors through the equalizer $i_A$. Moreover, $\sigma_{A_H}$, the factorization of $\sigma_A$, is a morphism in $\text{Reg}_{\varphi_{A_H}}(H^2, A_H)$ satisfying the normal condition with $\sigma_{A_H}^{-1}$ the factorization through the equalizer $i_A$ of the morphism

$$\sigma_A^{-1} = (f \circ \mu_H) \wedge (\mu_A \circ c_{A,A} \circ (f^{-1} \otimes f^{-1})).$$
Proposition

Let $H$ be a cocommutative weak Hopf algebra. If $A_H \hookrightarrow A$ is an $H$-cleft extension with cleaving morphism $f$, the morphism

$$\sigma_A = (\mu_A \circ (f \otimes f)) \wedge (f^{-1} \circ \mu_H) : H^2 \rightarrow A,$$

factors through the equalizer $i_A$. Moreover, $\sigma_{A_H}$, the factorization of $\sigma_A$, is a morphism in $\text{Reg}_{\varphi_{A_H}}(H^2, A_H)$ satisfying the normal condition with $\sigma_{A_H}^{-1}$ the factorization through the equalizer $i_A$ of the morphism

$$\sigma_{A_H}^{-1} = (f \circ \mu_H) \wedge (\mu_A \circ c_{A,H} \circ (f^{-1} \otimes f^{-1})).$$

Proposition

Let $H$ be a cocommutative weak Hopf algebra and let $A_H \hookrightarrow A$ be an $H$-cleft extension with cleaving morphism $f$. Then, the pair $(\varphi_{A_H}, \sigma_{A_H})$ is a crossed system for $H$ over $A_H$. Moreover, the $H$-cleft extensions $A_H \hookrightarrow A$ and $A_H \hookrightarrow A_H \times \sigma_{A_H} H$ are equivalent.
Proposition

Let $H$ be a cocommutative weak Hopf algebra and let $(\varphi_A, \sigma)$ be a crossed system for $H$ over $A$. Let $A \hookrightarrow A \times_\sigma H$ be the associated $H$-cleft extension. Then, if $(\phi_A, \tau)$ is the crossed system associated to the $H$-cleft extension $A \hookrightarrow A \times_\sigma H$, we have that $(\phi_A, \tau) = (\varphi_A, \sigma)$. 
Proposition

Let \( H \) be a cocommutative weak Hopf algebra and let \((\varphi_A, \sigma)\) be a crossed system for \( H \) over \( A \). Let \( A \xleftarrow{\phi} A \times_\sigma H \) be the associated \( H \)-cleft extension. Then, if \((\phi_A, \tau)\) is the crossed system associated to the \( H \)-cleft extension \( A \xleftarrow{\phi} A \times_\sigma H \), we have that \((\phi_A, \tau) = (\varphi_A, \sigma)\).

Proposition

Let \( H \) be a cocommutative weak Hopf algebra and let \( A_H \xleftarrow{\phi} A \) be an \( H \)-cleft extension with cleaving morphism \( f \). Assume that \( g : H \to A \) is other cleaving morphism with associated crossed system \((\phi_{A_H}, \tau_{A_H})\). Then the crossed systems \((\varphi_{A_H}, \sigma_{A_H})\) and \((\phi_{A_H}, \tau_{A_H})\) are equivalent.
Proposition

Let $H$ be a cocommutative weak Hopf algebra and let $(\varphi_A, \sigma)$ be a crossed system for $H$ over $A$. Let $A \hookrightarrow A \times_{\sigma} H$ be the associated $H$-cleft extension. Then, if $(\phi_A, \tau)$ is the crossed system associated to the $H$-cleft extension $A \hookrightarrow A \times_{\sigma} H$, we have that $(\phi_A, \tau) = (\varphi_A, \sigma)$.

Proposition

Let $H$ be a cocommutative weak Hopf algebra and let $A_H \hookrightarrow A$ be an $H$-cleft extension with cleaving morphism $f$. Assume that $g : H \to A$ is other cleaving morphism with associated crossed system $(\phi_{A_H}, \tau_{A_H})$. Then the crossed systems $(\varphi_{A_H}, \sigma_{A_H})$ and $(\phi_{A_H}, \tau_{A_H})$ are equivalent.

Corollary

Let $H$ be a cocommutative weak Hopf algebra and let $A_H \hookrightarrow A$, $A_H \hookrightarrow B$ two equivalent $H$-cleft extensions with cleaving morphisms $f$ and $g$ respectively. Then the corresponding crossed systems $(\varphi_{A_H}, \sigma_{A_H})$ and $(\phi_{A_H}, \tau_{A_H})$ are equivalent.
Theorem

Let $H$ be a cocommutative weak Hopf algebra. Two $H$-cleft extensions $A_H \hookrightarrow A$, $A_H \hookrightarrow B$ are equivalent if and only if so are their respective associated crossed systems.
Theorem

Let $H$ be a cocommutative weak Hopf algebra. Two $H$-cleft extensions $A_H \hookrightarrow A$, $A_H \hookrightarrow B$ are equivalent if and only if so are their respective associated crossed systems.

Theorem

Let $H$ be a cocommutative weak Hopf algebra and $(A, \rho_A)$ a right $H$-comodule algebra. There exists a bijective correspondence between the equivalence classes of $H$-cleft extensions $A_H \hookrightarrow B$ and the equivalence classes of crossed systems for $H$ over $A_H$. 
Cleft extensions for weak Hopf algebras
Crossed systems for weak Hopf algebras
Crossed systems and cohomology

Theorem
Let $H$ be a cocommutative weak Hopf algebra. Two $H$-cleft extensions $A_H \hookrightarrow A$, $A_H \hookrightarrow B$ are equivalent if and only if so are their respective associated crossed systems.

Theorem
Let $H$ be a cocommutative weak Hopf algebra and $(A, \rho_A)$ a right $H$-comodule algebra. There exists a bijective correspondence between the equivalence classes of $H$-cleft extensions $A_H \hookrightarrow B$ and the equivalence classes of crossed systems for $H$ over $A_H$.

\[
F : CS(H, A_H) \rightarrow Cleft(A_H), \quad G : Cleft(A_H) \rightarrow CS(H, A_H)
\]

\[
F([((\varphi_{A_H}, \sigma_{A_H})])] = [A_H \hookrightarrow A_H \times_{\sigma_{A_H}} H]
\]

\[
G([A_H \hookrightarrow B]) = [((\phi_{A_H}, \tau_{A_H})] .
\]
Crossed systems and cohomology
Let $H$ be a cocommutative weak Hopf algebra. If $(A, \varphi_A)$ is a left $H$-module algebra, the groups $\text{Reg}_{\varphi_A}(H^n, A)$, $n \geq 0$ are the objects of a semicosimplicial complex of groups with coface operators defined by

\[
\partial_{0,i} : \text{Reg}_{\varphi_A}(H^n, A) \rightarrow \text{Reg}_{\varphi_A}(H^{n-i}, A), \quad i \in \{0, 1\}
\]

\[
\partial_{1,0} : g \mapsto \varphi_A \circ (H \otimes (g \circ p_L \circ \Pi_R H)) \circ \delta_H,
\]

\[
\partial_{1,1} : h \mapsto h \circ \mu_H,
\]

\[
\partial_{1,2} : h \mapsto h \circ \mu_H \circ (H \otimes \Pi_L H).
\]
Let $H$ be a cocommutative weak Hopf algebra. If $(A, \varphi_A)$ is a left $H$-module algebra, the groups $\text{Reg}_{\varphi_A}(H^n, A), \quad n \geq 0$ are the objects of a semicosimplicial complex of groups with coface operators defined by

$$\partial_{0,i} : \text{Reg}_{\varphi_A}(H_L, A) \to \text{Reg}_{\varphi_A}(H, A), \quad i \in \{0, 1\}$$

$$\partial_{0,0}(g) = \varphi_A \circ (H \otimes (g \circ p_L \circ \Pi_R)) \circ \delta, \quad \partial_{0,1}(g) = g \circ p_L.$$
Let $H$ be a cocommutative weak Hopf algebra. If $(A, \varphi_A)$ is a left $H$-module algebra, the groups $\text{Reg}_{\varphi_A}(H^n, A)$, $n \geq 0$ are the objects of a semicosimplicial complex of groups with coface operators defined by

$$\partial_{0,i} : \text{Reg}_{\varphi_A}(H_L, A) \rightarrow \text{Reg}_{\varphi_A}(H, A), \quad i \in \{0, 1\}$$

$$\partial_{0,0}(g) = \varphi_A \circ (H \otimes (g \circ p_L \circ \Pi^R_H)) \circ \delta_H, \quad \partial_{0,1}(g) = g \circ p_L.$$ 

$$\partial_{1,i} : \text{Reg}_{\varphi_A}(H, A) \rightarrow \text{Reg}_{\varphi_A}(H^2, A), \quad i \in \{0, 1, 2\}$$

$$\partial_{1,0}(h) = \varphi_A \circ (H \otimes h), \quad \partial_{1,1}(h) = h \circ \mu_H, \quad \partial_{1,2}(h) = h \circ \mu_H \circ (H \otimes \Pi^L_H).$$
\[ \partial_{k-1,i} : \text{Reg}_{\varphi_A}(H^{k-1}, A) \to \text{Reg}_{\varphi_A}(H^k, A), \quad k > 2, \quad i \in \{0, 1, \cdots, k\} \]

\[ \partial_{k-1,i}(\sigma) = \begin{cases} 
\partial_{k-1,0}(\sigma) = \varphi_A \circ (H \otimes \sigma), \\
\partial_{k-1,i}(\sigma) = \sigma \circ (H^{i-1} \otimes \mu_H \otimes H^{k-i-1}), \quad i \in \{1, \cdots, k-1\} \\
\partial_{k-1,k}(\sigma) = \sigma \circ (H^{k-2} \otimes (\mu_H \circ (H \otimes \Pi^L_H))),
\end{cases} \]
For this complex the codegeneracy operators are defined by

\[ s_{1,0} : \text{Reg}_{\varphi_A}(H, A) \to \text{Reg}_{\varphi_A}(H_L, A), \]

\[ s_{1,0}(h) = h \circ i_L, \]
For this complex the codegeneracy operators are defined by

\[ s_{1,0} : \text{Reg}_{\varphi_A}(H, A) \rightarrow \text{Reg}_{\varphi_A}(H_L, A), \]
\[ s_{1,0}(h) = h \circ i_L, \]

\[ s_{2,i} : \text{Reg}_{\varphi_A}(H^2, A) \rightarrow \text{Reg}_{\varphi_A}(H, A), \quad i \in \{0, 1\} \]
\[ s_{2,0}(\sigma) = \sigma \circ (\eta_H \otimes H), \quad s_{2,1}(\sigma) = \sigma \circ (H \otimes \eta_H), \]
\[ s_{k+1,i} : \text{Reg}_{\varphi_A}(H^{k+1}, A) \to \text{Reg}_{\varphi_A}(H^k, A), \quad k \geq 2, \quad i \in \{0, 1, \cdots, k\} \]

\[
\begin{align*}
\sigma & 
\end{align*} \]

\[
\begin{align*}
\sigma \circ (\eta_H \otimes H^k), \\
\sigma \circ (H^i \otimes \eta_H \otimes H^{k-i}), \quad i \in \{1, \cdots, k-1\} \\
\sigma \circ (H^k \otimes \eta_H).
\end{align*}
\]
Let

\[ D^k \varphi_A = \partial_{k,0} \wedge \partial_{k,1}^{-1} \wedge \cdots \wedge \partial_{k,k+1}^{(-1)^{k+1}} \]

be the coboundary morphisms of the cochain complex

\[
\begin{align*}
    &\text{Reg}_{\varphi_A}(H_L, A) \xrightarrow{D^0 \varphi_A} \text{Reg}_{\varphi_A}(H, A) \xrightarrow{D^1 \varphi_A} \text{Reg}_{\varphi_A}(H^2, A) \xrightarrow{D^2 \varphi_A} \cdots \\
    &\cdots \xrightarrow{D^{k-1} \varphi_A} \text{Reg}_{\varphi_A}(H^k, A) \xrightarrow{D^k \varphi_A} \text{Reg}_{\varphi_A}(H^{k+1}, A) \xrightarrow{D^{k+1} \varphi_A} \cdots
\end{align*}
\]

associated to the cosimplicial complex \( \text{Reg}_{\varphi_A}(H^\bullet, A) \).

Then, when \((A, \varphi_A)\) is a commutative left \(H\)-module algebra, \((\text{Reg}_{\varphi_A}(H^\bullet, A), D^\bullet_{\varphi_A})\) gives the Sweedler cohomology of \(H\) in \((A, \varphi_A)\). Therefore, the \(k\)th group, will be defined by

\[
\frac{\text{Ker}(D^k_{\varphi_A})}{\text{Im}(D^{k-1}_{\varphi_A})}
\]

for \(k \geq 1\) and \(\text{Ker}(D^0_{\varphi_A})\) for \(k = 0\). We will denote it by \(H^k_{\varphi_A}(H, A)\).
Let $H$ be a cocommutative weak Hopf algebra and let $A_H \hookrightarrow A$ be an $H$-cleft extension. We denote by $(\varphi_A H, \sigma_A H)$ the corresponding crossed system defined by the convolution invertible total integral $f : H \to A$. Then $(Z(A_H), \varphi_Z(A_H))$ is a left $H$-module algebra, where $\varphi_Z(A_H)$ is the factorization through the morphism $i_Z(A_H)$ of the morphism $\varphi_A H \circ (H \otimes i_Z(A_H))$. 

Proposition
Theorem.

Let $H$ be a cocommutative weak Hopf algebra and let $A_H \hookrightarrow A$ be an $H$-cleft extension. We denote by $(\varphi_{A_H}, \sigma_{A_H})$ the corresponding crossed system defined by the cleaving morphism $f : H \rightarrow A$. Then there is a bijective correspondence between the second cohomology group $H^2_{\varphi_{Z(A_H)}}(H, Z(A_H))$ and the equivalence classes of crossed systems for $H$ over $A_H$ having $\varphi_{A_H}$ as weak $H$-module algebra structure.
Theorem.

Let $H$ be a cocommutative weak Hopf algebra and let $A_H \hookrightarrow A$ be an $H$-cleft extension. We denote by $(\varphi_{A_H}, \sigma_{A_H})$ the corresponding crossed system defined by the cleaving morphism $f : H \rightarrow A$. Then there is a bijective correspondence between the second cohomology group $H^2_{\varphi_Z(A_H)}(H, Z(A_H))$ and the equivalence classes of crossed systems for $H$ over $A_H$ having $\varphi_{A_H}$ as weak $H$-module algebra structure.

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