Quantum groupoids with projection

J.N. Alonso Álvarez[1], J.M. Fernández Vilaboa[2], R. González Rodríguez[3]

[1] Departamento de Matemáticas, Universidad de Vigo, Campus Universitario Lagoas-Marcosende, E-36280 Vigo, Spain (e-mail: jnalonso@ uvigo.es)
[2] Departamento de Álxebra, Universidad de Santiago de Compostela. E-15771 Santiago de Compostela, Spain (e-mail: alvila@usc.es)
[3] Departamento de Matemática Aplicada II, Universidad de Vigo, Campus Universitario Lagoas-Marcosende, E-36310 Vigo, Spain (e-mail: rgon@dma.uvigo.es)

Abstract

In this survey we explain in detail how Radford's ideas and results about Hopf algebras with projection can be generalized to quantum groupoids in a strict symmetric monoidal category with split idempotents.

Introduction

Let H be a Hopf algebra over a field K and let A be a K-algebra. A well-known result of Radford [23] gives equivalent conditions for an object $A \otimes H$ equipped with smash product algebra and coalgebra to be a Hopf algebra and characterizes such objects via bialgebra projections. Majid in [16] interpreted this result in the modern context of Yetter-Drinfeld modules and stated that there is a correspondence between Hopf algebras in this category, denoted by ${}_H^H\mathcal{YD}$, and Hopf algebras B with morphisms of Hopf algebras $f : H \to B$, $g : B \to H$ such that $g \circ f = id_H$. Later, Bespalov proved the same result for braided categories with split idempotents in [5]. The key point in Radford-Majid-Bespalov's theorem is to define an object B_H, called the algebra of coinvariants, as the equalizer of $(B \otimes g) \circ \delta_B$ and $B \otimes \eta_H$. This object is a Hopf algebra in the category ${}_H^H\mathcal{YD}$ and there exists a Hopf algebra isomorphism between B and $B_H \bowtie H$ (the smash (co)product of B_H and H). It is important to point out that in the construction of $B_H \bowtie H$ they use that B_H is the image of the idempotent morphism $q_H^B = \mu_B \circ (B \otimes (f \circ \lambda_H \circ g)) \circ \delta_B$.

In [11], Bulacu and Nauwelaerts generalize Radford's theorem about Hopf algebras with projection to the quasi-Hopf algebra setting. Namely, if H and B are quasi-Hopf algebras with bijective antipode and with morphisms of quasi-Hopf algebras $f : H \to B$, $g : B \to H$ such that $g \circ f = id_H$, then they define a subalgebra B^i (the generalization of B_H to this setting) and with some additional structures B^i becomes, a Hopf algebra in the category of left-left Yetter-Drinfeld modules ${}_H^H\mathcal{YD}$ defined by Majid in [17]. Moreover, as the main result in [11], Bulacu and Nauwelaerts state that $B^i \times H$ is isomorphic to B as quasi-Hopf algebras where the algebra structure of $B^i \times H$ is the smash product defined in [10] and the quasi-coalgebra structure is the one introduced in [11].

The basic motivation of this survey is to explain in detail how the above ideas and results can be generalized to quantum groupoids in a strict symmetric monoidal category with split idempotents. Quantum groupoids or weak Hopf algebras have been introduced by Böhm, Nill and Szlachányi [7] as a new generalization of Hopf algebras and groupoid algebras. Roughly speaking, a weak Hopf algebra H in a symmetric monoidal category is an object that has both algebra and coalgebra structures with some relations between them and that possesses an antipode λ_H which does not necessarily verify $\lambda_H \wedge id_H = id_H \wedge \lambda_H = \varepsilon_H \otimes \eta_H$ where ε_H, η_H are the counity and unity morphisms respectively and $\wedge$ denotes the usual convolution product. The main differences with other Hopf algebraic constructions, such as quasi-Hopf algebras and rational Hopf algebras, are the following: weak Hopf algebras are coassociative but the coproduct is not required to preserve the unity η_H or, equivalently, the counity is not an algebra morphism. Some motivations to study weak Hopf algebras come from their connection with the theory of algebra extensions, the important applications in the study of dynamical twists of Hopf algebras and their link with quantum field theories and operator algebras (see [20]).

The survey is organized as follows.

In Section 1 we give basis definitions and examples of quantum groupoids without finiteness conditions. Also we introduce the category of left-left Yetter-Drinfeld modules defined by Böhm for a quantum groupoid with invertible antipode. As in the case of Hopf algebras this category is braided monoidal but in this case is not strict.

The exposition of the theory of crossed products associated to projections of quantum groupoids in Section 2 follows [2] and is the good generalization of the classical theory developed by Blattner, Cohen and Montgomery in [6]. The main theorem in this section generalizes a well know result, due to Blattner, Cohen and Montgomery, which shows that if $B \xrightarrow{\pi} H \to 0$ is an exact sequence of Hopf algebras with coalgebra splitting then $B \approx A\sharp_{\sigma}H$, where A is the left Hopf kernel of π and σ is a suitable cocycle (see Theorem (4.14) of [6]). In this section we show that if $g : B \to H$ is a morphism of quantum groupoids and there exists a morphism of coalgebras $f : H \to B$ such that $g \circ f = id_H$ and $f \circ \eta_H = \eta_B$, using the idempotent morphism $q_H^B = \mu_B \circ (B \otimes (\lambda_B \circ f \circ g)) \circ \delta_B : B \to B$ it is possible to construct an equalizer diagram and an algebra B_H, i.e, the algebra of coinvariants or the Hopf kernel of g, and morphisms $\varphi_{B_H} : H \otimes B_H \to B_H$ (the weak measuring), $\sigma_{B_H} : H \otimes H \to B_H$ (the weak cocycle) such that there exists an idempotent endomorphism of $B_H \otimes H$ which image, denoted by $B_H \times H$, is isomorphic with B as algebras being the algebra structure (crossed product algebra)

$$\eta_{B_H \times H} = r_B \circ (\eta_{B_H} \otimes \eta_H),$$

$$\mu_{B_H \times H} = r_B \circ (\mu_{B_H} \otimes H) \circ (\mu_{B_H} \otimes \sigma_{B_H} \otimes \mu_H) \circ (B_H \otimes \varphi_{B_H} \otimes \delta_{H \otimes H}) \circ$$
$$(B_H \otimes H \otimes c_{H,B_H} \otimes H) \circ (B_H \otimes \delta_H \otimes B_H \otimes H) \circ (s_B \otimes s_B),$$

where s_B is the inclusion of $B_H \times H$ in $B_H \otimes H$ and r_B the projection of $B_H \otimes H$ on $B_H \times H$. Of course, when H, B are Hopf algebras we recover the result of Blattner, Cohen and Montgomery. For this reason, we denote the algebra $B_H \times H$ by $B_H \sharp_{\sigma_{B_H}} H$. If moreover f is an algebra morphism, the cocycle is trivial in a weak sense and then we obtain that $\mu_{B_H \times H}$ is the weak version of the smash product used by Radford in the Hopf algebra setting. Also, we prove the dual results using similar arguments but passing to the opposite category, for a morphism of quantum groupoids $h : H \to B$ and an algebra morphism $t : B \to H$ such that $t \circ h = id_H$ and $\varepsilon_H \circ t = \varepsilon_B$.

Finally, in Section 3, linking the information of section 2 with the results of [1], [2], [3] and [4], we obtain our version of Radford's Theorem for quantum groupoids with projection. In this section we prove that the algebra of coinvariants B_H associated to a quantum groupoid projection (i.e. a pair of morphisms of quantum groupoids $f : H \to B$, $g : B \to H$ such that

$g \circ f = id_H$) can be obtained as an equalizer or, by duality, as a coequalizer (in this case the classical theory developed in Section 2 and the dual one provide the same object B_H with dual algebraic structures, algebra-coalgebra, module-comodule, etc...). Therefore, it is possible to find an algebra coalgebra structure for B_H and morphisms $\varphi_{B_H} = p_H^B \circ \mu_B \circ (f \otimes i_H^B) : H \otimes B_H \rightarrow B_H$ and $\varrho_{B_H} = (g \otimes p_H^B) \circ \delta_B \circ i_H^B : B_H \rightarrow H \otimes B_H$ such that (B_H, φ_{B_H}) is a left H-module and (B_H, ϱ_{B_H}) is a left H-comodule. We show that B_H is a Hopf algebra in the category of left-left Yetter-Drinfeld modules ${}_H^H\mathcal{YD}$ and, using the the the weak smash product and the weak smash coproduct of B_H and H we give a good weak Hopf algebra interpretation of the theorems proved by Radford [23] and Majid [16] in the Hopf algebra setting, obtaining an isomorphism of quantum groupoids between $B_H \times H$ and B.

1 Quantum groupoids in monoidal categories

In this section we give definitions and discuss basic properties of quantum groupoids in monoidal categories.

Let $\mathcal{C}$ be a category. We denote the class of objects of $\mathcal{C}$ by $|\mathcal{C}|$ and for each object $X \in |\mathcal{C}|$, the identity morphism by $id_X : X \rightarrow X$.

A monoidal category $(\mathcal{C}, \otimes, K, a, l, r)$ is a category $\mathcal{C}$ which is equipped with a tensor product $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, with an object K, called the unit of the monoidal category, with a natural isomorphism $a : \otimes(id \times \otimes) \rightarrow \otimes(\otimes \times id)$, called the associativity constrain, and with natural isomorphisms $l : \otimes(K \times id) \rightarrow id$, $r : \otimes(id \times K) \rightarrow id$, called left unit constraint and right unit constraint respectively, such that the Pentagon Axiom

$$(a_{U,V,W} \otimes id_X) \circ a_{U,V \otimes W,X} \circ (id_U \otimes a_{V,W,X}) = a_{U \otimes V,W,X} \circ a_{U,V,W \otimes X}$$

and the Triangle Axiom

$$id_V \otimes l_W = (r_V \otimes id_W) \circ a_{V,K,W}$$

are satisfied.

The monoidal category is said to be strict if the associativity and the unit constraints a, l, r are all identities of the category.

Let $\Psi : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$ be the flip functor defined by $\Psi(V, W) = (W, V)$ on any pair of objects of $\mathcal{C}$. A commutativity constrain is a natural isomorphism $c : \otimes \rightarrow \otimes\Psi$. If $(\mathcal{C}, \otimes, K, a, l, r)$ is a monoidal category, a braiding in $\mathcal{C}$ is a commutativity constraint satisfying the Hexagon Axiom

$$a_{W,U,V} \circ c_{U \otimes V,W} \circ a_{U,V,W} = (c_{U,W} \otimes id_V) \circ a_{U,W,V} \circ (id_U \otimes c_{V,W}),$$

$$a^{-1}_{V,W,U} \circ c_{U,V \otimes W} \circ a^{-1}_{U,V,W} = (id_V \otimes c_{U,W}) \circ a^{-1}_{V,U,W} \circ (c_{U,V} \otimes id_W).$$

A braided monoidal category is a monoidal category with a braiding c. These categories generalizes the classical notion of symmetric monoidal category introduced earlier by category theorists. A braided monoidal category is symmetric if the braiding satisfies $c_{W,V} \circ c_{V,W} = id_{V \otimes W}$ for all $V, W \in |\mathcal{C}|$.

From now on we assume that $\mathcal{C}$ is strict symmetric and admits split idempotents, i.e., for every morphism $\nabla_Y : Y \rightarrow Y$ such that $\nabla_Y = \nabla_Y \circ \nabla_Y$ there exist an object Z and morphisms $i_Y : Z \rightarrow Y$ and $p_Y : Y \rightarrow Z$ such that $\nabla_Y = i_Y \circ p_Y$ and $p_Y \circ i_Y = id_Z$. There is not loss of generality in assuming the strict character for $\mathcal{C}$ because it is well know that given a monoidal category we can construct a strict monoidal category $\mathcal{C}^{st}$ which is tensor equivalent to $\mathcal{C}$ (see

[15] for the details). For simplicity of notation, given objects M, N, P in $\mathcal{C}$ and a morphism $f : M \to N$, we write $P \otimes f$ for $id_P \otimes f$ and $f \otimes P$ for $f \otimes id_P$.

Definition 1.1 An algebra in $\mathcal{C}$ is a triple $A = (A, \eta_A, \mu_A)$ where A is an object in $\mathcal{C}$ and $\eta_A : K \to A$ (unit), $\mu_A : A \otimes A \to A$ (product) are morphisms in $\mathcal{C}$ such that $\mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A)$, $\mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A)$. Given two algebras $A = (A, \eta_A, \mu_A)$ and $B = (B, \eta_B, \mu_B)$, $f : A \to B$ is an algebra morphism if $\mu_B \circ (f \otimes f) = f \circ \mu_A$, $f \circ \eta_A = \eta_B$. Also, if A, B are algebras in $\mathcal{C}$, the object $A \otimes B$ is an algebra in $\mathcal{C}$ where $\eta_{A\otimes B} = \eta_A \otimes \eta_B$ and $\mu_{A\otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B)$.

A coalgebra in $\mathcal{C}$ is a triple $D = (D, \varepsilon_D, \delta_D)$ where D is an object in $\mathcal{C}$ and $\varepsilon_D : D \to K$ (counit), $\delta_D : D \to D \otimes D$ (coproduct) are morphisms in $\mathcal{C}$ such that $(\varepsilon_D \otimes D) \circ \delta_D = id_D = (D \otimes \varepsilon_D) \circ \delta_D$, $(\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D$. If $D = (D, \varepsilon_D, \delta_D)$ and $E = (E, \varepsilon_E, \delta_E)$ are coalgebras, $f : D \to E$ is a coalgebra morphism if $(f \otimes f) \circ \delta_D = \delta_E \circ f$, $\varepsilon_E \circ f = \varepsilon_D$. When D, E are coalgebras in $\mathcal{C}$, $D \otimes E$ is a coalgebra in $\mathcal{C}$ where $\varepsilon_{D\otimes E} = \varepsilon_D \otimes \varepsilon_E$ and $\delta_{D\otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E)$.

If A is an algebra, B is a coalgebra and $\alpha : B \to A$, $\beta : B \to A$ are morphisms, we define the convolution product by $\alpha \wedge \beta = \mu_A \circ (\alpha \otimes \beta) \circ \delta_B$.

By quantum groupoids or weak Hopf algebras we understand the objects introduced in [7], as a generalization of ordinary Hopf algebras. Here, for the convenience of the reader, we recall the definition of these objects and some relevant results from [7] without proof, thus making our exposition self-contained.

Definition 1.2 A quantum groupoid H is an object in $\mathcal{C}$ with an algebra structure (H, η_H, μ_H) and a coalgebra structure $(H, \varepsilon_H, \delta_H)$ such that the following axioms hold:

(a1) $\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ \delta_{H\otimes H}$,

(a2) $\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes \delta_H \otimes H)$

$= (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes (c_{H,H} \circ \delta_H) \otimes H)$,

(a3) $(\delta_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \mu_H \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$

$= (H \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$.

(a4) There exists a morphism $\lambda_H : H \to H$ in $\mathcal{C}$ (called the antipode of H) verifiying:

(a4-1) $id_H \wedge \lambda_H = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$,

(a4-2) $\lambda_H \wedge id_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$,

(a4-3) $\lambda_H \wedge id_H \wedge \lambda_H = \lambda_H$.

Note that, in this definition, the conditions (a2), (a3) weaken the conditions of multiplicativity of the counit, and comultiplicativity of the unit that we can find in the Hopf algebra definition. On the other hand, axioms (a4-1), (a4-2) and (a4-3) weaken the properties of the antipode in a Hopf algebra. Therefore, a quantum groupoid is a Hopf algebra if an only if the morphism δ_H (comultiplication) is unit-preserving and if and only if the counit is a homomorphism of algebras.

1.3 If H is a quantum groupoid in $\mathcal{C}$, the antipode λ_H is unique, antimultiplicative, anticomultiplicative and leaves the unit η_H and the counit ε_H invariant:

$$\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}, \quad \delta_H \circ \lambda_H = c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H,$$

$$\lambda_H \circ \eta_H = \eta_H, \quad \varepsilon_H \circ \lambda_H = \varepsilon_H.$$

If we define the morphisms Π_H^L (target morphism), Π_H^R (source morphism), $\overline{\Pi}_H^L$ and $\overline{\Pi}_H^R$ by

$$\Pi_H^L = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H),$$

$$\Pi_H^R = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)),$$

$$\overline{\Pi}_H^L = (H \otimes (\varepsilon_H \circ \mu_H)) \circ ((\delta_H \circ \eta_H) \otimes H),$$

$$\overline{\Pi}_H^R = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)).$$

it is straightforward to show that they are idempotent and Π_H^L, Π_H^R satisfy the equalities

$$\Pi_H^L = id_H \wedge \lambda_H, \quad \Pi_H^R = \lambda_H \wedge id_H.$$

Moreover, we have that

$$\Pi_H^L \circ \overline{\Pi}_H^L = \Pi_H^L, \quad \Pi_H^L \circ \overline{\Pi}_H^R = \overline{\Pi}_H^R, \quad \Pi_H^R \circ \overline{\Pi}_H^L = \overline{\Pi}_H^L, \quad \Pi_H^R \circ \overline{\Pi}_H^R = \Pi_H^R,$$

$$\overline{\Pi}_H^L \circ \Pi_H^L = \overline{\Pi}_H^L, \quad \overline{\Pi}_H^L \circ \Pi_H^R = \Pi_H^R, \quad \overline{\Pi}_H^R \circ \Pi_H^L = \Pi_H^L, \quad \overline{\Pi}_H^R \circ \Pi_H^R = \overline{\Pi}_H^R.$$

Also it is easy to show the formulas

$$\Pi_H^L = \overline{\Pi}_H^R \circ \lambda_H = \lambda_H \circ \overline{\Pi}_H^L, \quad \Pi_H^R = \overline{\Pi}_H^L \circ \lambda_H = \lambda_H \circ \overline{\Pi}_H^R,$$

$$\Pi_H^L \circ \lambda_H = \Pi_H^L \circ \Pi_H^R = \lambda_H \circ \Pi_H^R, \quad \Pi_H^R \circ \lambda_H = \Pi_H^R \circ \Pi_H^L = \lambda_H \circ \Pi_H^L.$$

If λ_H is an isomorphism (for example, when H is finite), we have the equalities:

$$\overline{\Pi}_H^L = \mu_H \circ (H \otimes \lambda_H^{-1}) \circ c_{H,H} \circ \delta_H, \quad \overline{\Pi}_H^R = \mu_H \circ (\lambda_H^{-1} \otimes H) \circ c_{H,H} \circ \delta_H.$$

If the antipode of H is an isomorphism, the opposite operator and the coopposite operator produce quantum groupoids from quantum groupoids. In the first one the product μ_H is replaced by the opposite product $\mu_{H^{op}} = \mu_H \circ c_{H,H}$ while in the second the coproduct δ_H is replaced by $\delta_{H^{coop}} = c_{H,H} \circ \delta_H$. In both cases the antipode λ_H is replaced by λ_H^{-1}.

A morphism between quantum groupoids H and B is a morphism $f : H \to B$ which is both algebra and coalgebra morphism. If $f : H \to B$ is a weak Hopf algebra morphism, then $\lambda_B \circ f = f \circ \lambda_H$ (see Proposition 1.4 of [1]).

Examples 1.4 (i) As group algebras and their duals are the natural examples of Hopf algebras, groupoid algebras and their duals provide examples of quantum groupoids. Recall that a groupoid G is simply a category in which every morphism is an isomorphism. In this example, we consider finite groupoids, i.e. groupoids with a finite number of objects. The set of objects of G will be denoted by G_0 and the set of morphisms by G_1. The identity morphism on $x \in G_0$ will also be denoted by id_x and for a morphism $\sigma : x \to y$ in G_1, we write $s(\sigma)$ and $t(\sigma)$, respectively for the source and the target of σ.

Let G be a groupoid, and R a commutative ring. The groupoid algebra is the direct product

$$RG = \bigoplus_{\sigma \in G_1} R\sigma$$

with the product of two morphisms being equal to their composition if the latter is defined and 0 in otherwise, i.e. $\sigma\tau = \sigma \circ \tau$ if $s(\sigma) = t(\tau)$ and $\sigma\tau = 0$ if $s(\sigma) \neq t(\tau)$. The unit element is $1_{RG} = \sum_{x \in G_0} id_x$. The algebra RG is a cocommutative quantum groupoid, with coproduct δ_{RG}, counit ε_{RG} and antipode λ_{RG} given by the formulas:

$$\delta_{RG}(\sigma) = \sigma \otimes \sigma, \quad \varepsilon_{RG}(\sigma) = 1, \quad \lambda_{RG}(\sigma) = \sigma^{-1}.$$

For the quantum groupoid RG the morphisms target and source are respectively,

$$\Pi^L_{RG}(\sigma) = id_{t(\sigma)}, \quad \Pi^R_{RG}(\sigma) = id_{s(\sigma)}$$

and $\lambda_{RG} \circ \lambda_{RG} = id_{RG}$, i.e. the antipode is involutive.

If G_1 is finite, then RG is free of a finite rank as a R-module, hence $GR = (RG)^* = Hom_R(RG, R)$ is a commutative quantum groupoid with involutory antipode. As R-module

$$GR = \bigoplus_{\sigma \in G_1} Rf_\sigma$$

with $\langle f_\sigma, \tau \rangle = \delta_{\sigma,\tau}$. The algebra structure is given by the formulas $f_\sigma f_\tau = \delta_{\sigma,\tau} f_\sigma$ and $1_{GR} = \sum_{\sigma \in G_1} f_\sigma$. The coalgebra structure is

$$\delta_{GR}(f_\sigma) = \sum_{\tau\rho = \sigma} f_\tau \otimes f_\rho = \sum_{\rho \in G_1} f_{\sigma\rho^{-1}} \otimes f_\rho, \quad \varepsilon_{GR}(f_\sigma) = \delta_{\sigma, id_{t(\sigma)}}.$$

The antipode is given by $\lambda_{GR}(f_\sigma) = f_{\sigma^{-1}}$.

(ii) It is known that any group action on a set gives rise to a groupoid (see [24]). In [20] Nikshych and Vainerman extend this construction associating a quantum groupoid with any action of a Hopf algebra on a separable algebra.

(iii) It was shown in [19] that any inclusion of type Π_1 factors with finite index and depth give rise to a quantum groupoid describing the symmetry of this inclusion. In [20] can be found an example of this construction applied to the case of Temperley-Lieb algebras (see [13]).

(iv) In [22] Nill proved that Hayashi's face algebras [14] are examples of quantum groupoids whose counital subalgebras, i.e., the images of Π^L_H and Π^R_H, are commutative. Also, in [22] we can find that Yamanouchi's generalized Kac algebras (see [25]) are exactly C^*-quantum groupoids with involutive antipode.

1.5 Let H be a quantum groupoid. We say that (M, φ_M) is a left H-module if M is an object in $\mathcal{C}$ and $\varphi_M : H \otimes M \to M$ is a morphism in $\mathcal{C}$ satisfying $\varphi_M \circ (\eta_H \otimes M) = id_M$, $\varphi_M \circ (H \otimes \varphi_M) = \varphi_M \circ (\mu_H \otimes M)$. Given two left H-modules (M, φ_M) and (N, φ_N), $f : M \to N$ is a morphism of left H-modules if $\varphi_N \circ (H \otimes f) = f \circ \varphi_M$. We denote the category of right H-modules by ${}_H\mathcal{C}$. In an analogous way we define the category of right H-modules and we denote it by $\mathcal{C}_H$.

If (M, φ_M) and (N, φ_N) are left H-modules we denote by $\varphi_{M\otimes N}$ the morphism $\varphi_{M\otimes N} : H \otimes M \otimes N \to M \otimes N$ defined by

$$\varphi_{M\otimes N} = (\varphi_M \otimes \varphi_N) \circ (H \otimes c_{H,M} \otimes N) \circ (\delta_H \otimes M \otimes N).$$

We say that (M, ϱ_M) is a left H-comodule if M is an object in $\mathcal{C}$ and $\varrho_M : M \to H \otimes M$ is a morphism in $\mathcal{C}$ satisfying $(\varepsilon_H \otimes M) \circ \varrho_M = id_M$, $(H \otimes \varrho_M) \circ \varrho_M = (\delta_H \otimes M) \circ \varrho_M$. Given two left H-comodules (M, ϱ_M) and (N, ϱ_N), $f : M \to N$ is a morphism of left H-comodules if $\varrho_N \circ f = (H \otimes f) \circ \varrho_M$. We denote the category of left H-comodules by ${}^H\mathcal{C}$. Analogously, $\mathcal{C}^H$ denotes the category of right H-comodules.

For two left H-comodules (M, ϱ_M) and (N, ϱ_N), we denote by $\varrho_{M\otimes N}$ the morphism $\varrho_{M\otimes N} : M \otimes N \to H \otimes M \otimes N$ defined by

$$\varrho_{M\otimes N} = (\mu_H \otimes M \otimes N) \circ (H \otimes c_{M,H} \otimes N) \circ (\varrho_M \otimes \varrho_N).$$

Let (M, φ_M), (N, φ_N) be left H-modules. Then the morphism

$$\nabla_{M\otimes N} = \varphi_{M\otimes N} \circ (\eta_H \otimes M \otimes N) : M \otimes N \to M \otimes N$$

is idempotent. In this setting we denote by $M \times N$ the image of $\nabla_{M\otimes N}$ and by $p_{M,N} : M \otimes N \to M \times N$, $i_{M,N} : M \times N \to M \otimes N$ the morphisms such that $i_{M,N} \circ p_{M,N} = \nabla_{M\otimes N}$ and $p_{M,N} \circ i_{M,N} = id_{M\times N}$. Using the definition of $\times$ we obtain that the object $M \times N$ is a left H-module with action $\varphi_{M\times N} = p_{M,N} \circ \varphi_{M\otimes N} \circ (H \otimes i_{M,N}) : H \otimes (M \times N) \to M \times N$ (see [20]). Note that, if $f : M \to M'$ and $g : N \to N'$ are morphisms of left H-modules then $(f \otimes g) \circ \nabla_{M\otimes N} = \nabla_{M'\otimes N'} \circ (f \otimes g)$.

In a similar way, if (M, ϱ_M) and (N, ϱ_N) are left H-comodules the morphism

$$\nabla'_{M\otimes N} = (\varepsilon_H \otimes M \otimes N) \circ \varrho_{M\otimes M} : M \otimes N \to M \otimes N$$

is idempotent. We denote by $M \odot N$ the image of $\nabla'_{M\otimes N}$ and by $p'_{M,N} : M \otimes N \to M \odot N$, $i'_{M,N} : M \odot N \to M \otimes N$ the morphisms such that $i'_{M,N} \circ p'_{M,N} = \nabla'_{M\otimes N}$ and $p'_{M,N} \circ i'_{M,N} = id_{M\odot N}$. Using the definition of $\odot$ we obtain that the object $M \odot N$ is a left H-comodule with coaction $\varrho_{M\odot N} = (H \otimes p'_{M,N}) \circ \varrho_{M\otimes N} \circ i'_{M,N} : M \odot N \to H \otimes (M \odot N)$. If $f : M \to M'$ and $g : N \to N'$ are morphisms of left H-comodules then $(f \otimes g) \circ \nabla'_{M\otimes N} = \nabla'_{M'\otimes N'} \circ (f \otimes g)$.

Let (M, φ_M), (N, φ_N), (P, φ_P) be left H-modules. Then the following equalities hold (Lemma 1.7 of [3]):

$$\varphi_{M\otimes N} \circ (H \otimes \nabla_{M\otimes N}) = \varphi_{M\otimes N},$$

$$\varphi_{M\otimes N} = \varphi_{M\otimes N} \circ \nabla_{M\otimes N},$$

$$(i_{M,N} \otimes P) \circ \nabla_{(M\times N)\otimes P} \circ (p_{M,N} \otimes P) = (M \otimes i_{N,P}) \circ \nabla_{M\otimes(N\times P)} \circ (M \otimes p_{N,P}),$$

$$(M\otimes i_{N,P})\circ\nabla_{M\otimes(N\times P)}\circ(M\otimes p_{N,P}) = (\nabla_{M\otimes N}\otimes P)\circ(M\otimes\nabla_{N\otimes P}) = (M\otimes\nabla_{N\otimes P})\circ(\nabla_{M\otimes N}\otimes P).$$

Furthermore, by a similar calculus, if (M, ϱ_M), (N, ϱ_N), (P, ϱ_P) be left H-comodules we have

$$(H \otimes \nabla'_{M\otimes N}) \circ \varrho_{M\otimes N} = \varrho_{M\otimes N},$$

$$\varrho_{M\otimes N} \circ \nabla'_{M\otimes N} = \varrho_{M\otimes N},$$

$$(i'_{M,N} \otimes P) \circ \nabla'_{(M\odot N)\otimes P} \circ (p'_{M,N} \otimes P) = (M \otimes i'_{N,P}) \circ \nabla'_{M\otimes(N\odot P)} \circ (M \otimes p'_{N,P}),$$

$$(M\otimes i'_{N,P})\circ\nabla'_{M\otimes(N\odot P)}\circ(M\otimes p'_{N,P}) = (\nabla'_{M\otimes N}\otimes P)\circ(M\otimes\nabla'_{N\otimes P}) = (M\otimes\nabla'_{N\otimes P})\circ(\nabla'_{M\otimes N}\otimes P).$$

Yetter-Drinfeld modules over finite dimensional weak Hopf algebras over fields have been introduced by Böhm in [9]. It is shown in [9] that the category of finite dimensional Yetter-Drinfeld modules is monoidal and in [18] it is proved that this category is isomorphic to the category of finite dimensional modules over the Drinfeld double. In [12], the results of [18] are generalized, using duality results between entwining structures and smash product structures, and more properties are given.

Definition 1.6 Let H be a weak Hopf algebra. We shall denote by ${}^{H}_{H}\mathcal{YD}$ the category of left-left Yetter-Drinfeld modules over H. That is, $M=(M,\varphi_M,\varrho_M)$ is an object in ${}^{H}_{H}\mathcal{YD}$ if (M,φ_M) is a left H-module, (M,ϱ_M) is a left H-comodule and

(b1) $\quad (\mu_H\otimes M)\circ(H\otimes c_{M,H})\circ((\varrho_M\circ\varphi_M)\otimes H)\circ(H\otimes c_{H,M})\circ(\delta_H\otimes M)$

$$=(\mu_H\otimes\varphi_M)\circ(H\otimes c_{H,H}\otimes M)\circ(\delta_H\otimes\varrho_M).$$

(b2) $\quad (\mu_H\otimes\varphi_M)\circ(H\otimes c_{H,H}\otimes M)\circ((\delta_H\circ\eta_H)\otimes\varrho_M)=\varrho_M.$

Let M, N in ${}^{H}_{H}\mathcal{YD}$. The morphism $f:M\to N$ is a morphism of left-left Yetter-Drinfeld modules if $f\circ\varphi_M=\varphi_N\circ(H\otimes f)$ and $(H\otimes f)\circ\varrho_M=\varrho_N\circ f$.

Note that if (M,φ_M,ϱ_M) is a left-left Yetter-Drinfeld module then (b2) is equivalent to

(b3) $\quad ((\varepsilon_H\circ\mu_H)\otimes\varphi_M)\circ(H\otimes c_{H,H}\otimes M)\circ(\delta_H\otimes\varrho_M)=\varphi_M.$

and we have the identity $\varphi_M\circ(\Pi_H^L\otimes M)\circ\varrho_M=id_M$.

The conditions (b1) and (b2) of the last definition can also be restated (see Proposition 2.2 of [12]) in the following way: suppose that $(M,\varphi_M)\in|\,{}_{H}\mathcal{C}|$ and $(M,\varrho_M)\in|\,{}^{H}\mathcal{C}|$, then M is a left-left Yetter-Drinfeld module if and only if

$$\varrho_M\circ\varphi_M=(\mu_H\otimes M)\circ(H\otimes c_{M,H})\circ$$

$$(((\mu_H\otimes\varphi_M)\circ(H\otimes c_{H,H}\otimes M)\circ(\delta_H\otimes\varrho_M))\otimes\lambda_H)\circ(H\otimes c_{H,M})\circ(\delta_H\otimes M).$$

Moreover, the following Proposition, proved in [4], guaranties the equality between the morphisms $\nabla_{M\otimes N}$ and $\nabla'_{M\otimes N}$ defined in 1.5 for all $M,N\in|\,{}^{H}_{H}\mathcal{YD}|$.

Proposition 1.7 *Let H be a weak Hopf algebra. Let (M,φ_M,ϱ_M) and (N,φ_N,ϱ_N) be left-left Yetter-Drinfeld modules over H. Then we have the following assertions.*

(i) $\nabla_{M\otimes N}=((\varphi_M\circ(\overline{\Pi}_H^L\otimes M)\circ c_{M,H})\otimes N)\circ(M\otimes\varrho_N)$.

(ii) $\nabla'_{M\otimes N}=(M\otimes\varphi_N)\circ(((M\otimes\overline{\Pi}_H^R)\circ c_{H,M}\circ\varrho_M)\otimes N)$.

(iii) $\nabla_{M\otimes N}=\nabla'_{M\otimes N}$.

(iv) $\nabla_{M\otimes H}=((\varphi_M\circ(\overline{\Pi}_H^L\otimes M)\circ c_{M,H})\otimes H)\circ(M\otimes\delta_H)$.

(v) $\nabla'_{M\otimes H}=(M\otimes\mu_H)\circ(((M\otimes\overline{\Pi}_H^R)\circ c_{H,M}\circ\varrho_M)\otimes H)$.

(vi) $\nabla_{M\otimes H}=\nabla'_{M\otimes H}$.

1.8 It is a well know fact that, if the antipode of a weak Hopf algebra H is invertible, ${}^{H}_{H}\mathcal{YD}$ is a non-strict braided monoidal category. In the following lines we give a brief resume of the braided monoidal structure that we can construct in the category ${}^{H}_{H}\mathcal{YD}$ (see Proposition 2.7 of [18] for modules over a field K or Theorem 2.6 of [12] for modules over a commutative ring).

For two left-left Yetter-Drinfeld modules $(M, \varphi_M, \varrho_M)$, $(N, \varphi_N, \varrho_N)$ the tensor product is defined as object as the image of $\nabla_{M\otimes N}$ (see 1.5). As a consequence, by (iii) of Proposition 1.7, $M \times N = M \odot N$ and this object is a left-left Yetter-Drinfeld module with the following action and coaction:

$$\varphi_{M\times N} = p_{M,N} \circ \varphi_{M\otimes N} \circ (H \otimes i_{M,N}), \quad \varrho_{M\times N} = (H \otimes p_{M,N}) \circ \varrho_{M\otimes N} \circ i_{M,N}.$$

The base object is $H_L = Im(\Pi_H^L)$ or, equivalently, the equalizer of δ_H and $\zeta_H^1 = (H \otimes \Pi_H^L) \circ \delta_H$ (see (9)) or the equalizer of δ_H and $\zeta_H^2 = (H \otimes \overline{\Pi}_H^R) \circ \delta_H$. The structure of left-left Yetter-Drinfeld module for H_L is the one derived of the following morphisms

$$\varphi_{H_L} = p_L \circ \mu_H \circ (H \otimes i_L), \quad \varrho_{H_L} = (H \otimes p_L) \circ \delta_H \circ i_L.$$

where $p_L : H \to H_L$ and $i_L : H_L \to H$ are the morphism such that $\Pi_H^L = i_L \circ p_L$ and $p_L \circ i_L = id_{H_L}$.

The unit constrains are:

$$l_M = \varphi_M \circ (i_L \otimes M) \circ i_{H_L,M} : H_L \times M \to M,$$

$$r_M = \varphi_M \circ c_{M,H} \circ (M \otimes (\overline{\Pi}_H^L \circ i_L)) \circ i_{M,H_L} : M \times H_L \to M.$$

These morphisms are isomorphisms with inverses:

$$l_M^{-1} = p_{H_L,M} \circ (p_L \otimes \varphi_M) \circ ((\delta_H \circ \eta_H) \otimes M) : M \to H_L \times M,$$

$$r_M^{-1} = p_{M,H_L} \circ (\varphi_M \otimes p_L) \circ (H \otimes c_{H,M}) \circ ((\delta_H \circ \eta_H) \otimes M) : M \to M \times H_L.$$

If M, N, P are objects in the category ${}^{H}_{H}\mathcal{YD}$, the associativity constrains are defined by

$$a_{M,N,P} = p_{(M\times N),P} \circ (p_{M,N} \otimes P) \circ (M \otimes i_{N,P}) \circ i_{M,(N\times P)} : M \times (N \times P) \to (M \times N) \times P$$

where the inverse is the morphism

$$a_{M,N,P}^{-1} = a_{M,N,P} = p_{M,(N\times P)} \circ (M \otimes p_{N,P}) \circ (i_{M,N} \otimes P) \circ i_{(M\times N),P} : (M \times N) \times P \to M \times (N \times P).$$

If $\gamma : M \to M'$ and $\phi : N \to N'$ are morphisms in the category, then

$$\gamma \times \phi = p_{M'\times N'} \circ (\gamma \otimes \phi) \circ i_{M,N} : M \times N \to M' \times N'$$

is a morphism in ${}^{H}_{H}\mathcal{YD}$ and $(\gamma' \times \phi') \circ (\gamma \times \phi) = (\gamma' \circ \gamma) \times (\phi' \circ \phi)$, where $\gamma' : M' \to M''$ and $\phi' : N' \to N''$ are morphisms in ${}^{H}_{H}\mathcal{YD}$.

Finally, the braiding is

$$\tau_{M,N} = p_{N,M} \circ t_{M,N} \circ i_{M,N} : M \times N \to N \times M$$

where $t_{M,N} = (\varphi_N \otimes M) \circ (H \otimes c_{M,N}) \circ (\varrho_M \otimes N) : M \otimes N \to N \otimes M$. The morphism $\tau_{M,N}$ is a natural isomorphism with inverse:

$$\tau_{M,N}^{-1} = p_{M,N} \circ t'_{M,N} \circ i_{N,M} : N \times M \to M \times N$$

where $t'_{M,N} = c_{N,M} \circ (\varphi_N \otimes M) \circ (c_{N,H} \otimes M) \circ (N \otimes \lambda_H^{-1} \otimes M) \circ (N \otimes \varrho_M)$.

2 Projections, quantum groupoids and crossed products

In this section we give basic properties of quantum groupoids with projection. The material presented here can be found in [1] and [2]. For example, in Theorem 2.2 we will show that if H, B are quantum groupoids in $\mathcal{C}$ and $g : B \to H$ is a quantum groupoid morphism such that there exist a coalgebra morphism $f : H \to B$ verifiying $g \circ f = id_H$ and $f \circ \eta_H = \eta_B$ then, it is possible to find an object B_H, defined by an equalizer diagram an called the algebra of coinvariants, morphisms $\varphi_{B_H} : H \otimes B_H \to B_H$, $\sigma_{B_H} : H \otimes H \to B_H$ and an isomorphism of algebras and comodules $b_H : B \to B_H \times H$ being $B_H \times H$ a subobject of $B_H \otimes H$ with its algebra structure twisted by the morphism σ_{B_H}. Of course, the multiplication in $B_H \times H$ is a generalization of the crossed product and in the Hopf algebra case the Theorem 2.2 is the classical and well know result obtained by Blattner, Cohen and Montgomery in [6].

The following Proposition is a generalization to the quantum groupoid setting of classic result obtained by Radford in [23].

Proposition 2.1 *Let H, B be quantum groupoids in $\mathcal{C}$. Let $g : B \to H$ be a morphism of quantum groupoids and $f : H \to B$ be a morphism of coalgebras such that $g \circ f = id_H$. Then the following morphism is an idempotent in $\mathcal{C}$:*

$$q_H^B = \mu_B \circ (B \otimes (\lambda_B \circ f \circ g)) \circ \delta_B : B \to B.$$

Proof. See Proposition 2.1 of [2].

As a consequence of this proposition, we obtain that there exist an epimorphism p_H^B, a monomorphism i_H^B and an object B_H such that the diagram

$$\begin{tikzcd} B \arrow[rr, "q_H^B"] \arrow[dr, "p_H^B"'] & & B \\ & B_H \arrow[ur, "i_H^B"'] & \end{tikzcd}$$

commutes and $p_H^B \circ i_H^B = id_{B_H}$. Moreover, we have that

$$\begin{tikzcd} B_H \arrow[r, "i_H^B"] & B \arrow[r, shift left, "(B \otimes g) \circ \delta_B"] \arrow[r, shift right, "(B \otimes (\Pi_H^L \circ g)) \circ \delta_B"'] & B \otimes H \end{tikzcd}$$

is an equalizer diagram.

Now, let η_{B_H} and μ_{B_H} be the factorizations, through the equalizer i_H^B, of the morphisms η_B and $\mu_B \circ (i_H^B \otimes i_H^B)$. Then $(B_H, \eta_{B_H} = p_H^B \circ \eta_B, \mu_{B_H} = p_H^B \circ \mu_B \circ (i_H^B \otimes i_H^B))$ is an algebra in $\mathcal{C}$.

On the other hand, by Proposition 2.4 of [2] we have that there exists an unique morphism $\varphi_{B_H} : H \otimes B_H \to B_H$ such that $i_H^B \circ \varphi_{B_H} = y_B$ where $y_B : H \otimes B_H \to B$ is the morphism defined by $y_B = \mu_B \circ (B \otimes (\mu_B \circ c_{B,B})) \circ (f \otimes (\lambda_B \circ f) \otimes B) \circ (\delta_H \otimes i_H^B)$. The morphism φ_{B_H} satisfies:

$$\varphi_{B_H} = p_H^B \circ \mu_B \circ (f \otimes i_H^B),$$

$$\varphi_{B_H} \circ (\eta_H \otimes B_H) = id_{B_H},$$

$$\varphi_{B_H} \circ (H \otimes \eta_{B_H}) = \varphi_{B_H} \circ (\Pi_H^L \otimes \eta_{B_H}),$$

$$\mu_{B_H}\circ(\varphi_{B_H}\otimes B_H)\circ(H\otimes\eta_{B_H}\otimes B_H)=\varphi_{B_H}\circ(\Pi_H^L\otimes B_H),$$

$$\varphi_{B_H}\circ(H\otimes\mu_{B_H})=\mu_{B_H}\circ(\varphi_{B_H}\otimes\varphi_{B_H})\circ(H\otimes c_{H,B_H}\otimes B_H)\circ(\delta_H\otimes B_H\otimes B_H),$$

$$\mu_{B_H}\circ c_{B_H,B_H}\circ((\varphi_{B_H}\circ(H\otimes\eta_{B_H}))\otimes B_H)=\varphi_{B_H}\circ(\overline{\Pi}_H^L\otimes B_H).$$

and, if f is an algebra morphism, (B_H,φ_{B_H}) is a left H-module (Proposition 2.5 of [1]).

Moreover, in this setting, there exists an unique morphism $\sigma_{B_H}:H\otimes H\to B_H$ such that $i_H^B\circ\sigma_{B_H}=\sigma_B$ where $\sigma_B:H\otimes H\to B$ is the morphism defined by:

$$\sigma_B=\mu_B\circ((\mu_B\circ(f\otimes f))\otimes(\lambda_B\circ f\circ\mu_H))\circ\delta_{H\otimes H}.$$

Then, as a consequence, we have the equality $\sigma_{B_H}=p_H^B\circ\sigma_B$ (Proposition 2.6, [2]).

Now let $\omega_B:B_H\otimes H\to B$ be the morphism defined by $\omega_B=\mu_B\circ(i_H^B\otimes f)$. If we define $\omega'_B:B\to B_H\otimes H$ by $\omega'_B=(p_H^B\otimes g)\circ\delta_B$ we have $\omega_B\circ\omega'_B=id_B$. Then, the morphism $\Omega_B=\omega'_B\circ\omega_B:B_H\otimes H\to B_H\otimes H$ is idempotent and there exists a diagram

$$\begin{tikzcd}
& B \arrow[dr, "\omega'_B"] \arrow[dd, "b_B"] & \\
B_H\otimes H \arrow[ur, "\omega_B"] \arrow[rr, "\Omega_B"] \arrow[dr, "r_B"'] & & B_H\otimes H \\
& B_H\times H \arrow[ur, "s_B"'] &
\end{tikzcd}$$

where $s_B\circ r_B=\Omega_B,\ r_B\circ s_B=id_{B_H\times H},\ b_B=r_B\circ\omega'_B$.

It is easy to prove that the morphism b_B is an isomorphism with inverse $b_B^{-1}=\omega_B\circ s_B$. Therefore, the object $B_H\times H$ is an algebra with unit and product defined by $\eta_{B_H\times H}=b_B\circ\eta_B$, $\mu_{B_H\times H}=b_B\circ\mu_B\circ(b_B^{-1}\otimes b_B^{-1})$ respectively. Also, $B_H\times H$ is a right H-comodule where $\rho_{B_H\times H}=(b_B\otimes H)\circ(B\otimes g)\circ\delta_B\circ b_B^{-1}$. Of course, with these structures b_B is an isomorphism of algebras and right H-comodules being $\rho_B=(B\otimes g)\circ\delta_B$.

On the other hand, we can define the following morphisms:

$$\eta_{B_H\sharp_{\sigma_{B_H}}H}:K\to B_H\times H,\ \mu_{B_H\sharp_{\sigma_{B_H}}H}:B_H\times H\otimes B_H\times H\to B_H\times H,\ \rho_{B_H\sharp_{\sigma_{B_H}}H}:B_H\to B_H\times H\otimes H$$

where

$$\eta_{B_H\sharp_{\sigma_{B_H}}H}=r_B\circ(\eta_{B_H}\otimes\eta_H),$$

$$\mu_{B_H\sharp_{\sigma_{B_H}}H}=r_B\circ(\mu_{B_H}\otimes H)\circ(\mu_{B_H}\otimes\sigma_{B_H}\otimes\mu_H)\circ(B_H\otimes\varphi_{B_H}\otimes\delta_{H\otimes H})\circ$$
$$(B_H\otimes H\otimes c_{H,B_H}\otimes H)\circ(B_H\otimes\delta_H\otimes B_H\otimes H)\circ(s_B\otimes s_B),$$

$$\rho_{B_H\sharp_{\sigma_{B_H}}H}=(r_B\otimes H)\circ(B_H\otimes\delta_H)\circ s_B.$$

Finally, if we denote by $B_H\sharp_{\sigma_{B_H}}H$ (the crossed product of B_H and H) the triple

$$(B_H\times H,\eta_{B_H\sharp_{\sigma_{B_H}}H},\mu_{B_H\sharp_{\sigma_{B_H}}H})$$

we have the following theorem.

Theorem 2.2 *Let H, B be quantum groupoids in $\mathcal{C}$. Let $g : B \to H$ be a morphism of quantum groupoids and $f : H \to B$ be a morphism of coalgebras such that $g \circ f = id_H$ and $f \circ \eta_H = \eta_B$. Then, $B_H \sharp_{\sigma_{B_H}} H$ is an algebra, $(B_H \times H, \rho_{B_H \sharp_{\sigma_{B_H}} H})$ is a right H-comodule and $b_B : B \to B_H \sharp_{\sigma_{B_H}} H$ is an isomorphism of algebras and right H-comodules.*

Proof: The proof of this Theorem is a consequence of the following identities (see Theorem 2.8 of [2] for the complete details)

$$\eta_{B_H \sharp_{\sigma_{B_H}} H} = \eta_{B_H \times H}, \quad \mu_{B_H \sharp_{\sigma_{B_H}} H} = \mu_{B_H \times H}, \quad \rho_{B_H \sharp_{\sigma_{B_H}} H} = \rho_{B_H \times H}.$$

Remark 2.3 We point out that if H and B are Hopf algebras, Theorem 2.2 is the result obtained by Blattner, Cohen and Montgomery in [6]. Moreover, if f is an algebra morphism, we have $\sigma_{B_H} = \varepsilon_H \otimes \varepsilon_H \otimes \eta_{B_H}$ and then $B_H \sharp_{\sigma_{B_H}} H$ is the smash product of B_H and H, denoted by $B_H \sharp H$. Observe that the product of $B_H \sharp H$ is

$$\mu_{B_H \sharp H} = (\mu_{B_H} \otimes \mu_H) \circ (B_H \otimes ((\varphi_{B_H} \otimes H) \circ (H \otimes c_{H,B_H}) \circ (\delta_H \otimes B_H)) \otimes H)$$

Let H, B be quantum groupoids in $\mathcal{C}$. Let $g : B \to H$, $f : H \to B$ be morphisms of quantum groupoids such that $g \circ f = id_H$. In this case $\sigma_B = \Pi_L^B \circ f \circ \mu_H$ and then, using $\mu_B \circ (\Pi_B^L \otimes B) \circ \delta_B = id_B$, we obtain

$$\mu_{B_H \sharp_{\sigma_{B_H}} H} = r_B \circ (\mu_{B_H} \otimes \mu_H) \circ (B_H \otimes ((\varphi_{B_H} \otimes H) \circ (H \otimes c_{H,B_H}) \circ (\delta_H \otimes B_H)) \otimes H) \circ (s_B \otimes s_B)$$

As a consequence, for analogy with the Hopf algebra case, when $\sigma_B = \Pi_L^B \circ f \circ \mu_H$, we will denote the triple $B_H \sharp_{\sigma_{B_H}} H$ by $B_H \sharp H$ (the smash product of B_H and H).

Therefore, if f and g are morphisms of quantum groupoids, we have the following particular case of 2.2.

Corollary 2.4 *Let H, B be quantum groupoids in $\mathcal{C}$. Let $g : B \to H$, $f : H \to B$ be morphisms of quantum groupoids such that $g \circ f = id_H$. Then $B_H \sharp H$ is an algebra, $(B_H \times H, \rho_{B_H \sharp H})$ is a right H-comodule and $b_B : B \to B_H \sharp H$ is an isomorphism of algebras and right H-comodules.*

In a similar way we can obtain a dual theory. The arguments are similar to the ones used previously in this section, but passing to the opposite category. Let H, B be quantum groupoids in $\mathcal{C}$. Let $h : H \to B$ be a morphism of quantum groupoids and $t : B \to H$ be a morphism of algebras such that $t \circ h = id_H$ and $\varepsilon_H \circ t = \varepsilon_B$. The morphism $k_H^B : B \to B$ defined by

$$k_H^B = \mu_B \circ (B \otimes (h \circ t \circ \lambda_B)) \circ \delta_B$$

is idempotent in $\mathcal{C}$ and, as a consequence, we obtain that there exist an epimorphism l_H^B, a monomorphism n_H^B and an object B^H such that the diagram

$$\begin{tikzcd} B \arrow[rr, "k_H^B"] \arrow[dr, "l_H^B"'] & & B \\ & B^H \arrow[ur, "n_H^B"'] & \end{tikzcd}$$

commutes and $l_H^B \circ n_H^B = id_{B^H}$. Moreover, using the next coequalizer diagram in $\mathcal{C}$

$$\begin{tikzcd} B\otimes H \arrow[r, shift left, "\mu_B\circ (B\otimes h)"] \arrow[r, shift right, "\mu_B\circ (B\otimes (h\circ \Pi_H^L))"'] & B \arrow[r, "l_H^B"] & B^H \end{tikzcd}$$

it is possible to obtain a coalgebra structure for B^H. This structure is given by

$$(B^H, \varepsilon_{B^H} = \varepsilon_B \circ n_H^B, \delta_{B^H} = (l_H^B \otimes l_H^B) \circ \delta_B \circ n_H^B)).$$

Let $y^B : B \to H \otimes B^H$ be the morphism defined by:

$$y^B = (\mu_H \otimes l_H^B) \circ (t \otimes (t \circ \lambda_B) \otimes B) \circ (B \otimes (c_{B,B} \circ \delta_B)) \circ \delta_B.$$

The morphism y^B verifies that $y^B \circ \mu_B \circ (B \otimes h) = y^B \circ \mu_B \circ (B \otimes (\Pi_B^L \circ h))$ and then, there exists an unique morphism $r_{B^H} : B^H \to H \otimes B^H$ such that $r_{B^H} \circ l_H^B = y^B$.

Moreover the morphism ϱ_{B^H} satisfies:

$$\varrho_{B^H} = (t \otimes l_H^B) \circ \delta_B \circ n_H^B,$$

$$(\varepsilon_H \otimes B^H) \circ \varrho_{B^H} = id_{B^H},$$

$$(H \otimes \varepsilon_{B^H}) \circ \varrho_{B^H} = (\Pi_H^L \otimes \varepsilon_{B^H}) \circ \varrho_{B^H},$$

$$(H \otimes \varepsilon_{B^H} \otimes B^H) \circ (\varrho_{B^H} \otimes B^H) \circ \delta_{B_H} = (\Pi_H^L \otimes B^H) \circ \varrho_{B^H}$$

$$(H \otimes \delta_{B^H}) \circ \varrho_{B^H} = (\mu_H \otimes B^H \otimes B^H) \circ (H \otimes c_{B^H,H} \otimes B^H) \circ (\varrho_{B^H} \otimes \varrho_{B^H}) \circ \delta_{B^H},$$

$$(((H \otimes \varepsilon_{B^H}) \circ \varrho_{B^H}) \otimes B^H) \circ c_{B^H,B^H} \circ \delta_{B^H} = (\overline{\Pi}_H^L \otimes B^H) \circ \varrho_{B^H},$$

and, if t is a morphism of quantum groupoids, (B^H, ϱ_{B^H}) is a left H-comodule. Let $\gamma_B : B \to H \otimes H$ be the morphism defined by

$$\gamma_B = \mu_{H\otimes H} \circ (((t \otimes t) \circ \delta_B) \otimes (\delta_H \circ t \circ \lambda_B)) \circ \delta_B.$$

The morphism γ_B verifies that $\gamma_B \circ \mu_B \circ (B \otimes h) = \gamma_B \circ \mu_B \circ (B \otimes (\Pi_B^L \circ h))$ and then, there exists an unique morphism $\gamma_{B^H} : B^H \to H \otimes H$ such that $\gamma_{B^H} \circ l_H^B = \gamma_B$.

It is not difficult to see that the morphism $\Upsilon_B : B^H \otimes H \to B^H \otimes H$ defined by

$$\Upsilon_B = \varpi'_B \circ \varpi_B,$$

being $\varpi_B = \mu_B \circ (n_H^B \otimes h)$ and $\varpi'_B = (l_H^B \otimes t) \circ \delta_B$, is idempotent and there exists a diagram

$$\begin{tikzcd} & B \arrow[dr, "\varpi'_B"] \arrow[dd, "d_B"] & \\ B^H \otimes H \arrow[ur, "\varpi_B"] \arrow[rr, "\Upsilon_B"] \arrow[dr, "u_B"'] & & B^H \otimes H \\ & B^H \boxdot H \arrow[ur, "v_B"'] & \end{tikzcd}$$

where $v_B \circ u_B = \Upsilon_B$, $u_B \circ v_B = id_{B^H \boxdot H}$, $d_B = u_B \circ \varpi'_B$. Moreover, d_B is an isomorphism with inverse $d_B^{-1} = \varpi_B \circ v_B$ and the object $B^H \boxdot H$ is a coalgebra with counit and coproduct defined by

$$\varepsilon_{B^H \boxdot H} = \varepsilon_B \circ d_B^{-1}, \quad \delta_{B^H \boxdot H} = (d_B \otimes d_B) \circ \delta_B \circ d_B^{-1}$$

respectively.

Also, $B^H \boxdot H$ is a right H-module where

$$\psi_{B^H \boxdot H} = d_B \circ \mu_B \circ (d_B^{-1} \otimes h).$$

With these structures d_B is an isomorphism of coalgebras and right H-modules being $\psi_B = \mu_B \circ (B \otimes h)$. Finally, we define the morphisms:

$$\varepsilon_{B^H \ominus_{\gamma_{B^H}} H} : B^H \boxdot H \to K, \;\; \delta_{B^H \ominus_{\gamma_{B^H}} H} : B^H \boxdot H \to B^H \boxdot H \otimes B^H \boxdot H,$$

$$\psi_{B^H \ominus_{\gamma_{B^H}} H} : B^H \boxdot H \otimes H \to B^H \boxdot H$$

where

$$\begin{aligned}
&\varepsilon_{B^H \ominus_{\gamma_{B^H}} H} = (\varepsilon_{B^H} \otimes \varepsilon_H) \circ v_B,\\
&\delta_{B^H \ominus_{\gamma_{B^H}} H} = (u_B \otimes u_B) \circ (B^H \otimes \mu_H \otimes B^H \otimes H) \circ (B^H \otimes H \otimes c_{B^H,H} \otimes H) \circ\\
&\qquad (B^H \otimes \varrho_{B^H} \otimes \mu_{H\otimes H}) \circ (\delta_{B^H} \otimes \gamma_{B^H} \otimes \delta_H) \circ (\delta_{B^H} \otimes H) \circ v_B,\\
&\psi_{B^H \ominus_{\gamma_{B^H}} H} = u_B \circ (B^H \otimes \mu_H) \circ (v_B \otimes H).
\end{aligned}$$

If we denote by $B^H \ominus_{\gamma_{B^H}} H$ (the crossed coproduct of B^H and H) the triple

$$(B^H \boxdot H, \varepsilon_{B^H \ominus_{\gamma_{B^H}} H}, \delta_{B^H \ominus_{\gamma_{B^H}} H}),$$

we have the following theorem:

Theorem 2.5 *Let H, B be quantum groupoids in $\mathcal{C}$. Let $h : H \to B$ be a morphism of quantum groupoids and $t : B \to H$ be a morphism of algebras such that $t \circ h = id_H$ and $\varepsilon_H \circ t = \varepsilon_B$. Then, $B^H \ominus_{\gamma_{B^H}} H$ is a coalgebra, $(B^H \boxdot H, \psi_{B^H \ominus_{\gamma_{B^H}} H})$ is a right H-module and $d_B : B \to B^H \ominus_{\gamma_{B^H}} H$ is an isomorphism of coalgebras and right H-modules.*

Remark 2.6 In the Hopf algebra case (H and B Hopf algebras) Theorem 2.5 is the dual of the result obtained by Blattner, Cohen and Montgomery. In this case, if t is an algebra-coalgebra morphism, we have $\gamma_{B^H} = \varepsilon_{B^H} \otimes \eta_H \otimes \eta_H$ and then $B^H \ominus_{\gamma_{B^H}} H$ is the smash coproduct of B^H and H, denoted by $B^H \ominus H$. In $B^H \ominus H$ the coproduct is

$$\delta_{B^H \ominus H} = (B^H \otimes ((\mu_H \otimes B^H) \circ (H \otimes c_{B^H,H}) \circ (\varrho_{B^H} \otimes H)) \otimes H) \circ (\delta_{B^H} \otimes \delta_H).$$

If t is a morphism of quantum groupoids we have $\gamma_B = \delta_H \circ \Pi_H^L \circ t$ and then the expression of $\delta_{B^H \ominus_{\gamma_{B^H}} H}$ is:

$$\delta_{B^H \ominus_{\gamma_{B^H}} H} = (u_B \otimes u_B) \circ (B^H \otimes ((\mu_H \otimes B^H) \circ (H \otimes c_{B^H,H}) \circ (\varrho_{B^H} \otimes H)) \otimes H) \circ (\delta_{B^H} \otimes \delta_H) \circ v_B.$$

As a consequence, for analogy with the Hopf algebra case, when $\gamma_B = \delta_H \circ \Pi_L^H \circ t$, we will denote the triple $B^H \ominus_{\gamma_{B^H}} H$ by $B^H \ominus H$ (the smash coproduct of B^H and H).

Therefore, if h and t are morphisms of quantum groupoids, we have:

Corollary 2.7 *Let H, B be quantum groupoids in $\mathcal{C}$. Let $t : B \to H$, $h : H \to B$ be morphisms of quantum groupoids such that $t \circ h = id_H$. Then, $B^H \ominus H$ is a coalgebra, $(B^H \boxdot H, \psi_{B^H \ominus H})$ is a right H-module and $d_B : B \to B^H \ominus H$ is an isomorphism of coalgebras and right H-modules.*

3 Quantum groupoids, projections and Hopf algebras in ${}^H_H\mathcal{YD}$

In this section we give the connection between projection of quantum groupoids an Hopf algebras in the category ${}^H_H\mathcal{YD}$. The results presented here can be found in [3].

Suppose that $g : B \to H$ and $f : H \to B$ are morphisms of weak Hopf algebras such that $g \circ f = id_H$. Then $q_H^B = k_H^B$ and therefore $B_H = B^H$, $p_H^B = l_H^B$ and $i_H^B = n_H^B$. Thus

$$B_H \xrightarrow{i_H^B} B \underset{(B\otimes(\Pi_H^L\circ g))\circ\delta_B}{\overset{(B\otimes g)\circ\delta_B}{\rightrightarrows}} B\otimes H$$

is an equalizer diagram and

$$B\otimes H \underset{\mu_B\circ(B\otimes(f\circ\Pi_H^L))}{\overset{\mu_B\circ(B\otimes f)}{\rightrightarrows}} B \xrightarrow{p_B^H} B_H$$

is a coequalizer diagram.

Then $(B_H, \eta_{B_H} = p_H^B \circ \eta_B, \mu_{B_H} = p_H^B \circ \mu_B \circ (i_H^B \otimes i_H^B))$ is an algebra in $\mathcal{C}$, $(B_H, \varepsilon_{B_H} = \varepsilon_B \circ i_H^B, \delta_{B_H} = (p_H^B \otimes p_H^B) \circ \delta_B \circ i_H^B))$ is a coalgebra in $\mathcal{C}$, (B_H, φ_{B_H}) is a left H-module and (B_H, ϱ_{B_H}) is a left H-comodule.

Also, $\omega_B = \varpi_B$, $\omega'_B = \varpi'_B$ and then $B_H \times H = B^H \boxdot H$. Moreover, the morphism $\Omega_B = \omega'_B \circ \omega_B$ admits a new formulation. Note that by the usual arguments in the quantum groupoid calculus, we have

$$\begin{aligned}
\Omega_B &= (p_H^B \otimes \mu_H) \circ (\mu_B \otimes H \otimes g) \circ (B \otimes c_{H,B} \otimes B) \circ (((B \otimes g) \circ \delta_B \circ i_H^B) \otimes (\delta_B \circ f)) \\
&= (p_H^B\otimes\mu_H)\circ(\mu_B\otimes H\otimes H)\circ(B\otimes c_{H,B}\otimes H)\circ(((B\otimes(\overline{\Pi}_H^R\circ g))\circ\delta_B\circ i_H^B)\otimes((f\otimes H)\circ\delta_H))) \\
&= (p_H^B \otimes \varepsilon_H \otimes H) \circ (\mu_{B\otimes H} \otimes H) \circ (((B \otimes g) \circ \delta_B \circ i_H^B) \otimes ((f \otimes \delta_H) \circ \delta_H))) \\
&= (p_H^B \otimes (\varepsilon_H \circ g) \otimes H) \circ (\mu_{B\otimes B} \otimes H) \circ (\delta_B \otimes \delta_B \otimes H) \circ (i_H^B \otimes ((f \otimes H) \circ \delta_H)) \\
&= ((p_H^B \circ \mu_B) \otimes H) \circ (i_H^B \otimes ((f \otimes H) \circ \delta_H)) \\
&= ((p_H^B \circ \mu_B \circ (B \otimes q_H^B)) \otimes H) \circ (i_H^B \otimes ((f \otimes H) \circ \delta_H)) \\
&= (p_H^B \otimes H) \circ ((\mu_B \circ (B \otimes (\Pi_B^L \circ f)) \otimes H) \circ (i_H^B \otimes \delta_H) \\
&= (p_H^B \otimes H) \circ ((\mu_B \circ c_{B,B} \circ ((\Pi_B^L \circ f) \otimes i_H^B)) \otimes H) \circ (c_{B_H,H} \otimes H) \circ (B_H \otimes \delta_H) \\
&= ((p_H^B \circ i_H^B \circ \varphi_{B_H} \circ (\overline{\Pi}_H^L \otimes B_H)) \otimes H) \circ (c_{B_H,H} \otimes H) \circ (B_H \otimes \delta_H) \\
&= (\varphi_{B_H} \otimes H) \circ (c_{B_H,H} \otimes H) \circ (B_H \otimes \overline{\Pi}_H^L \otimes H) \circ (B_H \otimes \delta_H)
\end{aligned}$$

$$= (\varphi_{B_H} \otimes \mu_H) \circ (H \otimes c_{H,B_H} \otimes H) \circ ((\delta_H \circ \eta_H) \otimes B_H \otimes H).$$

$$= \nabla_{B_H \otimes H}.$$

Therefore, the object $B_H \times H$ is the tensor product of B_H and H in the representation category of H, i.e. the category of left H-modules, studied in [8] and [21].

Proposition 3.1 *Let $g : B \to H$ and $f : H \to B$ be morphisms of quantum groupoids such that $g \circ f = id_H$. Then, if the antipode of H is an isomorphism, $(B_H, \varphi_{B_H}, \varrho_{B_H})$ belongs to ${}^H_H\mathcal{YD}$.*

Proof: In Proposition 2.8 of [1] we prove that $(B_H, \varphi_{B_H}, \varrho_{B_H})$ satisfy

$$(\mu_H \otimes B_H) \circ (H \otimes c_{B_H,H}) \circ ((\varrho_{B_H} \circ \varphi_{B_H}) \otimes H) \circ (H \otimes c_{H,B_H}) \circ (\delta_H \otimes B_H)$$

$$= (\mu_H \otimes B_H) \circ (H \otimes c_{B_H,H}) \circ (\mu_H \otimes \varphi_{B_H} \otimes H) \circ (H \otimes c_{H,H} \otimes B_H \otimes H) \circ (\delta_H \otimes \varrho_{B_H} \otimes \Pi_H^R) \circ$$

$$(H \otimes c_{H,B_H}) \circ (\delta_H \otimes B_H).$$

Moreover, the following identity

$$(\mu_H \otimes B_H) \circ (H \otimes c_{B_H,H}) \circ (\mu_H \otimes \varphi_{B_H} \otimes H) \circ (H \otimes c_{H,H} \otimes B_H \otimes H) \circ (\delta_H \otimes \varrho_{B_H} \otimes \Pi_H^R) \circ$$

$$(H \otimes c_{H,B_H}) \circ (\delta_H \otimes B_H)$$

$$= (\mu_H \otimes \varphi_{B_H}) \circ (H \otimes c_{H,H} \otimes (\varphi_{B_H} \circ ((\overline{\Pi}_H^L \circ \overline{\Pi}_H^R) \otimes B_H) \circ \varrho_{B_H})) \circ (\delta_H \otimes \varrho_{B_H}).$$

is true because B_H is a left H-module and a left H-comodule. Then, using the identity

$$\varphi_{B_H} \circ ((\overline{\Pi}_H^L \circ \overline{\Pi}_H^R) \otimes B_H) \circ \varrho_{B_H} = id_{B_H}$$

we prove (b1). The prove for (b2) is easy and we leave the details to the reader.

3.2 As a consequence of the previous proposition we obtain $\nabla_{B_H \otimes B_H} = \nabla'_{B_H \otimes B_H}$ and $\nabla_{B_H \otimes H} = \nabla'_{B_H \otimes H} = \Omega_B$.

3.3 Let $g : B \to H$ and $f : H \to B$ be morphisms of quantum groupoids such that $g \circ f = id_H$. Put $u_{B_H} = p_H^B \circ f \circ i_L : H_L \to B_H$ and $e_{B_H} = p_L \circ g \circ i_H^B : B_H \to H_L$. This morphisms belong to ${}^H_H\mathcal{YD}$ and we have the same for $m_{B_H \times B_H} : B_H \times B_H \to B_H$ defined by

$$m_{B_H \times B_H} = \mu_{B_H} \circ i_{B_H,B_H}$$

and $\Delta_{B_H} : B_H \to B_H \times B_H$ defined by $\Delta_{B_H} = p_{B_H,B_H} \circ \delta_{B_H}$.

Then, we have the following result.

Proposition 3.4 *Let $g : B \to H$ and $f : H \to B$ be morphisms of quantum groupoids such that $g \circ f = id_H$. Then, if the antipode of H is an isomorphism, we have the following:*

(i) (B_H, u_{B_H}, m_{B_H}) is an algebra in ${}^H_H\mathcal{YD}$.

(ii) $(B_H, e_{B_H}, \Delta_{B_H})$ is a coalgebra in ${}^H_H\mathcal{YD}$.

Proof: See Proposition 2.6 in [3].

3.5 Let $g : B \to H$ and $f : H \to B$ be morphisms of weak Hopf algebras such that $g \circ f = id_H$. Let Θ_H^B be the morphism $\Theta_H^B = ((f \circ g) \wedge \lambda_B) \circ i_H^B : B_H \to B$. Following Proposition 2.9 of [1] we have that $(B \otimes g) \circ \delta_B \circ \Theta_H^B = (B \otimes (\Pi_H^L \circ g)) \circ \delta_B \circ \Theta_H^B$ and, as a consequence, there exists an unique morphism $\lambda_{B_H} : B_H \to B_H$ such that $i_H^B \circ \lambda_{B_H} = \Theta_H^B$. Therefore, $\lambda_{B_H} = p_H^B \circ \Theta_H^B$ and λ_{B_H} belongs to the category of left-left Yetter-Drinfeld modules.

The remainder of this section will be devoted to the proof of the main Theorem of this paper.

Theorem 3.6 *Let $g : B \to H$ and $f : H \to B$ be morphisms of weak Hopf algebras satisfying the equality $g \circ f = id_H$ and suppose that the antipode of H is an isomorphism. Let u_{B_H}, m_{B_H}, e_{B_H}, Δ_{B_H}, λ_{B_H} be the morphisms defined in 3.3 and 3.5 respectively. Then $(B_H, u_{B_H}, m_{B_H}, e_{B_H}, \Delta_{B_H}, \lambda_{B_H})$ is a Hopf algebra in the category of left-left Yetter-Drinfeld modules.*

Proof: By Proposition 3.4 we know that (B_H, u_{B_H}, m_{B_H}) is an algebra and $(B_H, e_{B_H}, \Delta_{B_H})$ is a coalgebra in ${}_H^H\mathcal{YD}$.

First we prove that m_{B_H} is a coalgebra morphism. That is:

(c1) $$\Delta_{B_H} \circ m_{B_H} = (m_{B_H} \times m_{B_H}) \circ a_{B_H,B_H,B_H \times B_H} \circ (B_H \times a^{-1}_{B_H,B_H,B_H}) \circ$$
$$(B_H \times (\tau_{B_H,B_H} \times B_H)) \circ (B_H \times a_{B_H,B_H,B_H}) \circ a^{-1}_{B_H,B_H,B_H \times B_H} \circ (\Delta_{B_H} \times \Delta_{B_H}),$$

(c2) $e_{B_H} \circ m_{B_H} = l_{H_L} \circ (e_{B_H} \times e_{B_H})$.

Indeed:

$$(m_{B_H} \times m_{B_H}) \circ a_{B_H,B_H,B_H \times B_H} \circ (B_H \times a^{-1}_{B_H,B_H,B_H}) \circ (B_H \times (\tau_{B_H,B_H} \times B_H)) \circ$$
$$(B_H \times a_{B_H,B_H,B_H}) \circ a^{-1}_{B_H,B_H,B_H \times B_H} \circ (\Delta_{B_H} \times \Delta_{B_H})$$
$$= p_{B_H,B_H} \circ (\mu_{B_H} \otimes \mu_{B_H}) \circ (B_H \otimes i_{B_H,B_H} \otimes B_H) \circ (\nabla_{B_H \otimes (B_H \times B_H)} \otimes B_H) \circ$$
$$(B_H \otimes \nabla_{(B_H \times B_H) \otimes B_H}) \circ (B_H \otimes (p_{B_H,B_H} \circ t_{B_H,B_H} \circ i_{B_H,B_H}) \otimes B_H) \circ (B_H \otimes \nabla_{(B_H \times B_H) \otimes B_H})$$
$$(\nabla_{B_H \otimes (B_H \times B_H)} \otimes B_H) \circ (B_H \otimes p_{B_H,B_H} \otimes B_H) \circ (\delta_{B_H} \otimes \delta_{B_H}) \circ i_{B_H,B_H}$$
$$= p_{B_H,B_H} \circ (\mu_{B_H} \otimes \mu_{B_H}) \circ (B_H \otimes (\nabla_{B_H \otimes B_H} \circ t_{B_H,B_H} \circ \nabla_{B_H \otimes B_H}) \otimes B_H) \circ (\delta_{B_H} \otimes \delta_{B_H}) \circ$$
$$i_{B_H,B_H}$$
$$= p_{B_H,B_H} \circ (\mu_{B_H} \otimes \mu_{B_H}) \circ (B_H \otimes t_{B_H,B_H} \otimes B_H) \circ (\delta_{B_H} \otimes \delta_{B_H}) \circ i_{B_H,B_H}$$
$$= p_{B_H,B_H} \circ \delta_{B_H} \circ \mu_{B_H} \circ i_{B_H,B_H}$$
$$= \Delta_{B_H} \circ m_{B_H}.$$

In the last computations, the first and the second equalities follow from Lemma 1.7 of [3] and by $\mu_{B_H} \circ \nabla_{B_H \otimes B_H} = \mu_{B_H}$, $\nabla_{B_H \otimes B_H} \circ \delta_{B_H} = \delta_{B_H}$. In the third one we use the following result: if M is a left-left Yetter-Drinfeld module then $t_{M,M} \circ \nabla_{M \otimes M} = t_{M,M}$, $\nabla_{M \otimes M} \circ t_{M,M} = t_{M,M}$. The fourth equality follows from Proposition 2.9 of [1] and, finally, the fifth one follows by definition.

On the other hand,

$$
\begin{aligned}
& l_{H_L} \circ (e_{B_H} \times e_{B_H}) \\
&= p_L \circ \mu_H \circ (i_L \otimes i_L) \circ \nabla_{H_L \otimes H_L} \circ (p_L \otimes p_L) \circ ((g \circ i_H^B) \otimes (g \circ i_H^B)) \circ i_{B_H,B_H} \\
&= p_L \circ \mu_H \circ ((\Pi_H^L \circ g \circ i_H^B) \otimes (\Pi_H^L \circ g \circ i_H^B)) \circ i_{B_H,B_H} \\
&= p_L \circ \mu_H \circ ((g \circ q_H^B \circ i_H^B) \otimes (g \circ q_H^B \circ i_H^B)) \circ i_{B_H,B_H} \\
&= p_L \circ \mu_H \circ ((g \circ i_H^B) \otimes (g \circ i_H^B)) \circ i_{B_H,B_H} \\
&= p_L \circ g \circ i_H^B \circ \mu_{B_H} \circ i_{B_H,B_H} \\
&= e_{B_H} \circ m_{B_H}.
\end{aligned}
$$

The first equality follows from definition, the second one from

$$p_L \circ \mu_H \circ (i_L \otimes i_L) \circ \nabla_{H_L \otimes H_L} \circ (p_L \otimes p_L) = p_L \circ \mu_H \circ (\Pi_H^L \otimes \Pi_H^L)$$

and the third one from $\Pi_H^L \circ g = g \circ q_H^B$. Finally, the fourth one follows from the idempotent character of q_H^B, the fifth one from the properties of g and the definition of μ_{B_H} and the sixth one from definition.

To finish the proof we only need to show

$$m_{B_H} \circ (\lambda_{B_H} \times B_H) \circ \Delta_{B_H} = l_{B_H} \circ (e_{B_H} \times u_{B_H}) \circ r_{B_H}^{-1} = m_{B_H} \circ (B_H \times \lambda_{B_H}) \circ \Delta_{B_H}.$$

We begin by proving $l_{B_H} \circ (e_{B_H} \times u_{B_H}) \circ r_{B_H}^{-1} = u_{B_H} \circ e_{B_H}$. Indeed:

$$
\begin{aligned}
& l_{B_H} \circ (e_{B_H} \times u_{B_H}) \circ r_{B_H}^{-1} \\
&= p_H^B \circ \mu_B \circ (f \otimes B) \circ (i_L \otimes i_H^B) \circ \nabla_{H_L \otimes B_H} \circ (p_L \otimes p_H^B) \circ (g \otimes f) \circ (i_H^B \otimes i_L) \circ \nabla_{B_H \otimes H_L} \circ (p_H^B \otimes p_L) \circ \\
& \quad ((\mu_B \circ (f \otimes i_H^B)) \otimes H) \circ (H \otimes c_{H,B_H}) \circ ((\delta_H \circ \eta_H) \otimes B_H) \\
&= p_H^B \circ \mu_B \circ ((\Pi_B^L \wedge \Pi_B^L) \otimes \Pi_B^L) \circ ((f \circ g \circ q_H^B) \otimes (\mu_B \circ (\Pi_B^L \otimes (f \circ g \circ \Pi_B^L)))) \circ (\delta_B \otimes B) \circ \\
& \quad \delta_B \circ i_H^B \\
&= p_H^B \circ \mu_B \circ ((\Pi_B^L \circ f \circ g \circ q_H^B) \otimes (f \circ \Pi_H^L \circ g \circ \Pi_B^L)) \circ \delta_B \circ i_H^B \\
&= p_H^B \circ f \circ \mu_H \circ (\Pi_H^L \otimes \Pi_H^L) \circ \delta_H \circ g \circ i_H^B \\
&= p_H^B \circ f \circ \Pi_H^L \circ g \circ i_H^B \\
&= u_{B_H} \circ e_{B_H}.
\end{aligned}
$$

The first equality follows from definition, the second one from

$$((\mu_B \circ (f \otimes i_H^B)) \otimes H) \circ (H \otimes c_{H,B_H}) \circ ((\delta_H \circ \eta_H) \otimes B_H) = (B \otimes (g \circ \Pi_B^L)) \circ \delta_B \circ i_H^B,$$

$$(i_H^B \otimes i_L) \circ \nabla_{B_H \otimes H_L} \circ (p_H^B \otimes p_L) = (q_H^B \otimes (\Pi_H^L \circ g \circ \mu_B)) \circ (B \otimes \Pi_B^L \otimes f) \circ (\delta_B \otimes H)$$

and

$$(i_L \otimes i_H^B) \circ \nabla_{H_L \otimes B_H} \circ (p_L \otimes p_H^B) = (\Pi_H^L \circ g) \otimes (q_H^B \circ \mu_B)) \circ (B \otimes \Pi_B^L \otimes B) \circ ((\delta_B \circ f) \otimes B).$$

In the third one we use $\Pi_B^L \wedge \Pi_B^L = \Pi_B^L$. The fourth one follows from $\Pi_H^L \circ g = g \circ q_H^B$ and from the idempotent character of Π_H^L. Finally, in the fifth one we apply (75) for $\Pi_H^L \wedge \Pi_H^L = \Pi_H^L$.

On the other hand,

$$
\begin{aligned}
& m_{B_H} \circ (\lambda_{B_H} \times B_H) \circ \Delta_{B_H} \\
&= \mu_{B_H} \circ \nabla_{B_H \otimes B_H} \circ (\lambda_{B_H} \otimes B_H) \circ \nabla_{B_H \otimes B_H} \circ \delta_{B_H} \\
&= \mu_{B_H} \circ (\lambda_{B_H} \otimes B_H) \circ \delta_{B_H} \\
&= ((\varepsilon_{B_H} \circ \mu_{B_H}) \otimes B_H) \circ (B_H \otimes t_{B_H,B_H}) \circ ((\delta_{B_H} \circ \eta_{B_H}) \otimes B_H) \\
&= ((\varepsilon_B \circ q_H^B \circ \mu_B) \otimes p_H^B) \circ ((\mu_B \circ (q_H^B \otimes (f \circ g)) \circ \delta_B) \otimes c_{B,B}) \circ ((\delta_B \circ q_H^B \circ \eta_B) \otimes i_H^B) \\
&= p_H^B \circ \Pi_B^L \circ i_H^B \\
&= p_H^B \circ f \circ \Pi_H^L \circ g \circ i_H^B \\
&= u_{B_H} \circ e_{B_H}.
\end{aligned}
$$

In these computations, the first equality follows from definition, the second one from $\mu_{B_H} \circ \nabla_{B_H \otimes B_H} = \mu_{B_H}$ and $\nabla_{B_H \otimes B_H} \circ \delta_{B_H} = \delta_{B_H}$, the third one from (4-1) of Proposition 2.9 of [1] and the fourth one is a consequence of the coassociativity of δ_B. The fifth equality follows from $\mu_B \circ (q_H^B \otimes (f \circ g)) \circ \delta_B = id_B$ and $q_H^B \circ \eta_B = \eta_B$, $\varepsilon_B \circ q_H^B = \varepsilon_B$. In the sixth one we use $f \circ \Pi_H^L \circ g = \Pi_B^L$ and the last one follows from definition.

Finally, using similar arguments and (4-2) of Proposition 2.9 of [1] we obtain:

$$
\begin{aligned}
& m_{B_H} \circ (B_H \times \lambda_{B_H}) \circ \Delta_{B_H} \\
&= \mu_{B_H} \circ \nabla_{B_H \otimes B_H} \circ (B_H \otimes \lambda_{B_H}) \circ \nabla_{B_H \otimes B_H} \circ \delta_{B_H} \\
&= \mu_{B_H} \circ (\lambda_{B_H} \otimes B_H) \circ \delta_{B_H} \\
&= (B_H \otimes (\varepsilon_{B_H} \circ \mu_{B_H})) \circ (t_{B_H,B_H} \otimes B_H) \circ (B_H \otimes (\delta_{B_H} \circ \eta_{B_H})) \\
&= p_H^B \circ \mu_B \circ ((f \circ g) \otimes \Pi_B^R) \circ \delta_B \circ i_H^B \\
&= p_H^B \circ \mu_B \circ ((f \circ g) \otimes (f \circ \Pi_H^R \circ g) \circ \delta_B \circ i_H^B \\
&= p_H^B \circ f \circ (id_H \wedge \Pi_H^R) \circ g \circ i_H^B \\
&= p_H^B \circ f \circ g \circ i_H^B \\
&= p_H^B \circ f \circ \Pi_H^L \circ g \circ i_H^B \\
&= u_{B_H} \circ e_{B_H}.
\end{aligned}
$$

Finally, using the last theorem and Theorem 4.1 of [2] we obtain the complete version of Radford's Theorem linking weak Hopf algebras with projection and Hopf algebras in the category of Yetter-Drinfeld modules over H.

Theorem 3.7 *Let H, B be weak Hopf algebras in $\mathcal{C}$. Let $g : B \to H$ and $f : H \to B$ be morphisms of weak Hopf algebras such that $g \circ f = id_H$ and suppose that the antipode of H is an isomorphism. Then there exists a Hopf algebra B_H living in the braided monoidal category ${}_H^H\mathcal{YD}$ such that B is isomorphic to $B_H \times H$ as weak Hopf algebras, being the (co)algebra structure in $B_H \times H$ the smash (co)product, that is the (co)product defined in 2.3, 2.6. The expression for the antipode of $B_H \times H$ is*

$$
\begin{aligned}
\lambda_{B_H\times H} := p_{B_H,H}\circ(\varphi_{B_H}\otimes H)\circ \\
&(H\otimes c_{H,B_H})\circ((\delta_H\circ\lambda_H\circ\mu_H)\otimes\lambda_{B_H})\circ(H\otimes c_{B_H,H})\circ \\
&(\varrho_{B_H}\otimes H)\circ i_{B_H,H}.
\end{aligned}
$$

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