Four theorems and one conjecture on the global asymptotic stability of delay differential equations

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(Dedicated to Prof. Jean Mawhin on the occasion of his first 60 years in Nonlinear Analysis)

Abstract

The aim of this notes is to bring together the results fruit of the labor of a group of mathematicians (including the author) during the last four years on the global asymptotic stability of a family of scalar delay differential equations with a unique equilibrium. We have obtained sharp conditions which generalize and unify many existing ones in the topic. Finally, we formulate one conjecture that also generalizes other classical ones.

1 Introduction

The object of study is a family of scalar functional delay differential equations

$$x'(t) = -\delta x(t) + f(t, x_t), \tag{1.1}$$

where $\delta \geq 0$, and, as usual, for every $t \geq 0$, x_t denotes the element of $\mathcal{C} \stackrel{def}{=} C[-h, 0]$, defined by $x_t(s) = x(t+s), s \in [-h, 0]$. Here $h > 0$ is the delay parameter. For the general theory of functional delay differential equations, we refer the reader to [6]. We will establish along the paper the conditions imposed on the functional $f : \mathbf{R} \times \mathcal{C} \to \mathbf{R}$, but we may anticipate some well-known examples which fall within our considerations.

If $f(t, \phi) = f(\phi(-h))$, Eq. (1.1) reads

$$x'(t) = -\delta x(t) + f(x(t-h)). \tag{1.2}$$

This equation is a general model in population dynamics known as *delayed recruitment model* [2, Section 3.3]; see also the classical reference [23] in mathematical biology for more discussions. Roughly speaking, $x(t)$ represents the number of adult (sexually mature) members in a population at time t, δ is the *per capita* death rate, and $f(x(t-h))$ is the rate at which new members are recruited into the population at time t (h is the age at which members mature, and it is assumed that the birth rate at a given time depends only of the adult population size). For different meanings, see the interesting list in [7, p. 78], including models in neurophysiology, metabolic regulation, and agricultural commodity markets. The most famous models of this type are:

- The Nicholson's blowflies equation proposed in [4] to explain the oscillatory population fluctuations observed by A. J. Nicholson in 1957 in his studies of the sheep blowfly *Lucilia cuprina*:

$$x'(t) = -\delta x(t) + px(t-h)e^{-\gamma x(t-h)}, \ \delta, p, \gamma, h > 0. \tag{1.3}$$

- The model for blood cell populations proposed by Mackey and Glass in [19]

$$x'(t) = -\delta x(t) + p\frac{x(t-h)}{1 + [x(t-h)]^n}, \ \delta, p, h > 0, n > 1. \tag{1.4}$$

- The model for the survival of red blood cells in an animal proposed by Wazewska-Czyzewska and Lasota in [32]

$$x'(t) = -\delta x(t) + pe^{-\gamma x(t-h)}, \ \delta, p, \gamma, h > 0. \tag{1.5}$$

When $\delta = 0$ and $f(t, \phi) = f(\phi(-1))$, we have the "prototype equation for delayed negative feedback" (see [3, Chapter XV]),

$$x'(t) = f(x(t-1)). \tag{1.6}$$

The most celebrated equation of this type is the Wright equation

$$x'(t) = p(e^{-x(t-1)} - 1), p > 0, \tag{1.7}$$

which comes after a change of variables from the delayed logistic equation

$$x'(t) = px(t)\left(1 - \frac{x(t-h)}{K}\right), p, h, K > 0. \tag{1.8}$$

Other examples for Eq. (1.6) are the "food-limitation" model [29] and the models exhibiting "Allee effect" [9, Section 4.6].

However, the first member of the family of equations (1.1) under our study was the functional differential equation with maxima

$$x'(t) = -\delta x(t) + a \max_{s\in[t-h,t]} x(s) + f(t), \tag{1.9}$$

where $\delta \geq 0$, $a \in \mathbf{R}$, and f is a continuous T-periodic real function. This equation appears as a model in automatic regulation (see [1] and references therein). In [25], the homogeneous case of (1.9) is studied in detail (see also [17]). In particular, the authors find the region of the plane of parameters (δ, a) for which the trivial solution is globally asymptotically stable. For the general case of $f \not\equiv 0$, some sufficient conditions were given in [1, 18, 25].

2 First theorem

Motivated by some results in [25], in the summer of 1999 we started to work in the problem of finding a necessary and sufficient condition for the existence of a globally asymptotically stable T-periodic solution to (1.9). This problem was completely solved in [8], where the following result was proved:

Theorem 2.1 *Assume that f is continuous and there exists $a < 0$ such that*

$$a \max_{s\in[-h,0]} \phi(s) \leq f(t, \phi) \leq a \min_{s\in[-h,0]} \phi(s), \; \forall \, \phi \in \mathcal{C}. \tag{2.1}$$

If $\delta > 0$ and

$$e^{-\delta h} > \frac{-a}{\delta} \ln\left(\frac{a^2 - a\delta}{a^2 + \delta^2}\right) \tag{2.2}$$

then the zero solution of (1.1) is globally asymptotically stable. Moreover, this condition is sharp: for every triple (a, δ, h) which does not *satisfy (2.2), there exists f satisfying (2.1) such that the equilibrium $x \equiv 0$ in (1.1) is not asymptotically stable.*

Remark 2.2 *It is easy to see that if (1.9) has a unique T-periodic solution $q(t)$, then the study of the global asymptotic stability of $q(t)$ is equivalent by a change of variables to the study of the global asymptotic stability of $x \equiv 0$ for an equation of the form (1.1) with f satisfying (2.1).*

Condition (2.1) was inspired by a celebrated work of J. A. Yorke [31]. In fact, modifying slightly our proofs, one can see that the conclusion of Theorem 2.1 is still valid if we replace condition (2.1) by the following weaker condition introduced in [31]:

$$a\mathcal{M}(\phi) \le f(t,\phi) \le -a\mathcal{M}(-\phi), \, \forall\, \phi \in \mathcal{C}, \tag{2.3}$$

where

$$\mathcal{M}(\phi) = \max\left\{ \max_{s\in[-h,0]} \phi(s), 0 \right\}. \tag{2.4}$$

Condition (2.3) is often referred to as the *Yorke condition* [9, Section 4.5]. The work of Yorke establishes that under condition (2.3) and an additional hypothesis which ensures that all monotone solutions converging to a constant in fact should converge to zero, the trivial solution of equation

$$x'(t) = f(t, x_t) \tag{2.5}$$

is globally asymptotically stable if

$$|a|h < 3/2. \tag{2.6}$$

In fact, the limit form of (2.2) when $\delta \to 0$ is precisely (2.6), and thus the Yorke result can be viewed as a limit case of Theorem 2.1.

Such Yorke's result is known as 3/2-theorem (see, for example, [6, Section 5.5] and [9, Section 4.5]), and it has two remarkable features:

- It is sharp: this character was established by A. Myshkis [24] in 1955 (see also the examples by J. Lillo in [10]), by studying the linear equation with variable delay

$$x'(t) = ax(t - h(t)), \, a < 0. \tag{2.7}$$

It is clear that if $0 \leq h(t) \leq h$, then $f(t,\phi) = a\phi(-h(t))$ satisfies (2.3). Myshkis showed that it is possible to find $h(t)$ such that $|a| \sup h(t) = 3/2$ and Eq. (2.7) has nontrivial periodic solutions (and hence $x \equiv 0$ is not globally asymptotically stable).

- Even in the linear autonomous case

$$x'(t) = ax(t-h),\ a < 0, \tag{2.8}$$

constant $3/2$ is very close to the constant $\pi/2$ for which Eq. (2.8) losses its stability.

In Theorem 2.1, we extended these features to the case $\delta > 0$: condition (2.2) is sharp, and approximates exceptionally well the exact stability domain for the linear equation with constant coefficients and fixed delay

$$x'(t) = -\delta x(t) + ax(t-h),\ \delta > 0, a < 0. \tag{2.9}$$

Indeed, in Fig. 1, we have depicted the local and global stability domains using new coordinates $(c,\theta) = (-a/\delta, \exp(-\delta h))$. Now, condition (2.2) reads

$$\theta > c \ln\left(\frac{c^2+c}{c^2+1}\right),$$

and the solid curve in Fig. 1 represents the graph of function

$$F(c) = c\ln((c^2+c)/(c^2+1)), \tag{2.10}$$

in such a way that Eq. (1.1) is globally asymptotically stable if $\theta > F(c)$.

On the other hand, the dashed line corresponds to the curve of asymptotic stability for the linear equation (2.9). From [6, p. 135] we know that the boundary of the region of stability of Eq. (2.9) is given by

$$\delta = a\cos(z)\ ;\ -a\sin(z) = z/h\,,\ z \in (\pi/2, \pi).$$

Hence we have $(c,\theta) = (-1/\cos(z), e^{z\cot(z)})$, $z \in (\pi/2, \pi)$, which is the parametric curve shown in Fig. 1 (dashed line).

The sharp nature of our condition is one remarkable aspect of Theorem 2.1, and it was established by studying the differential equation with maxima (1.9) (see [8, Section 4]).

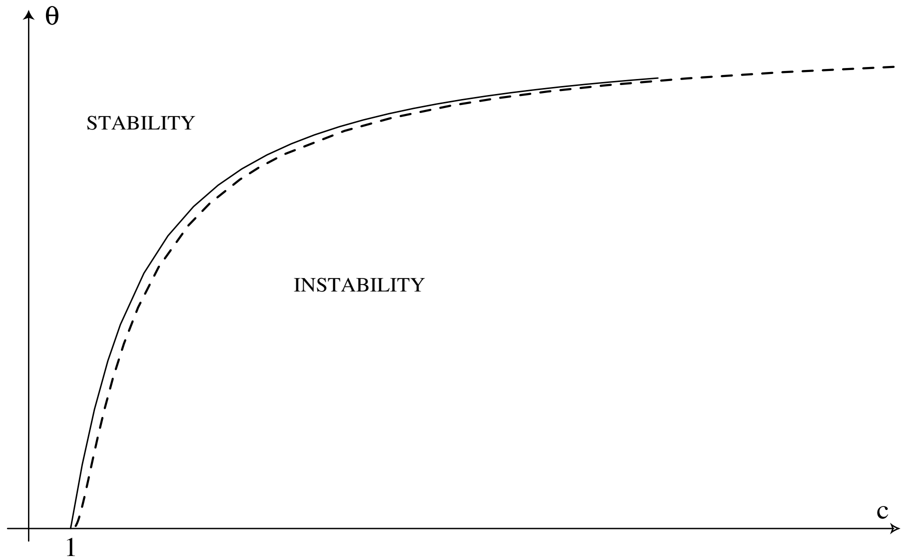

Figure 1: Domains of global and local stability.

3 Second theorem

In spite of the interest of the Yorke theorem and our Theorem 2.1, to be honest we have to notice that (2.3) is a kind of sublinearity condition (indeed, it implies that $|f(t,\phi)| \leq -a\|\phi\|$, for all $\phi \in \mathcal{C}$), and these results cannot be applied to the interesting cases of equation (1.1) mentioned in Section 1. Despite this fact, it is known from the work of E. Wright in 1955 [33] that the 3/2- stability condition for the global asymptotic stability applies to the delayed logistic equation (1.7). Wright proved that all solutions of this equation converge to zero as $t \to \infty$ if $p \leq 3/2$. Moreover, he affirms that, after considerable effort, his method can be used to obtain the same conclusion with the condition $p \leq 37/24 = 1.54\ldots$ (compare with the local asymptotic stability condition $p < \pi/2 = 1.57\ldots$), so that the 3/2 condition is not exact for the Wright equation. In fact, the famous *Wright conjecture* establishes that the zero solution in Eq. (1.7) is globally asymptotically stable for $p < \pi/2$. This conjecture still remains open (see [9, p. 125]).

However, we notice that, similarly to the Yorke result, the number 3/2 is exact for the nonautonomous case of the logistic equation (see

[22, 28]).

This "magic number" was very attractive for us, and during the year 2000 we devote some efforts to investigate the following question: *which are the properties of the Wright equation (1.7) that make possible this correlation between local and global asymptotic properties of the zero solution?* In other words, *can we extend the Wright* $3/2$*-theorem to a more general family of delay differential equations?*

We notice that Eq. (1.7) has motivated a number of papers by many specialists in the field: it is the most famous example of the family of scalar delay differential equations (1.6). Very frequently, Eq. (1.6) is investigated under the assumption of *negative feedback*:

(W1) $f : \mathbf{R} \to \mathbf{R}$ is continuous with $f(0) = 0$, differentiable at $x = 0$ with $f'(0) < 0$, and for all $x \neq 0$, $xf(x) < 0$.

Also, examples from applications satisfy certain boundedness conditions, and it is typically assumed that f is either bounded from above or from below. See, for example, Chapter XV in [3], where the reader can find very interesting historical notes and a good list of related references. We will assume

(W2) $f : \mathbf{R} \to \mathbf{R}$ is bounded from below, that is, there exists $k > 0$ such that $f(x) > -k$ for all $x \in \mathbf{R}$.

It is clear that (W1) and (W2) hold for the nonlinearity $f(x) = p(e^{-x} - 1)$ in the Wright equation. Returning to our question on the generalization of the Wright theorem, we note that the linearized equation of (1.6) at $x = 0$ is

$$x'(t) = f'(0)x(t-1), \tag{3.1}$$

which is asymptotically stable if $|f'(0)| < \pi/2$. Hence we are interested in finding sufficient conditions on f such that the equilibrium is globally attracting in (1.6) for $|f'(0)| < 3/2$. The recent paper by Walther [30] establishes, for every fixed $p > 0$, the existence of monotone nonlinearities f_p with $|f_p'(0)| = p$ in such a way that (W1) and (W2) hold and (1.6) has nontrivial periodic solutions.

Hence, we need an additional condition in order to have a $3/2$-type result. Motivated by the famous Singer results in one-dimensional dynamics (see [21, 26]), we consider unimodal functions with negative Schwarz derivative, that is, we assume that f satisfies condition

(W3) $f \in C^3(\mathbf{R}, \mathbf{R})$, has at most one critical point $x^* \in \mathbf{R}$ which is a local extremum, and $(Sf)(x) < 0$ for all $x \neq x^*$, where

$$(Sf)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

is the Schwarz derivative of f.

We note that the condition of the negative Schwarz derivative was already used in the setting of delay differential equations of the form (1.2) by Mallet-Paret and Nussbaum in [20]. Let us observe also that number 3/2 appears again in this definition!

In [12] we prove the following result:

Theorem 3.1 *Assume that f satisfies (W1), (W2), (W3), and $|f'(0)| \leq 3/2$. Then the steady state solution $x \equiv 0$ in (1.6) is globally attracting.*

Theorem 3.1 provides a new classification of delay differential equations of the form (1.6) for which the 3/2 theorem is valid. One of the interesting features of this result is that conditions (W1)-(W3) are satisfied not only by the Wright equation (and therefore the Wright theorem can be obtained as an immediate corollary) but also for other models used in population dynamics, where the nonlinearity is more complicated (see [12, Examples 1.4,1.5]).

4 Third theorem

There is a key contribution which is "hidden" in our paper [12]: one can realize that we use conditions (W1)-(W3) to show that the graph of the nonlinearity f in (1.6) is "dominated" by the graph of the rational function $r(x) = ax/(1+bx)$ which coincides with f, f' and f'' at $x = 0$, in the sense that

$$r(x) > f(x) > 0\,,\ x \in (-1/b, 0)\,,\ \text{ and }\ r(x) < f(x) < 0\,,\ x > 0. \quad (4.1)$$

Then, almost all arguments in the paper can be made using this property instead of conditions (W1)-(W3). This fact suggests that this is the essential characteristic of the nonlinearity f in the Wright equation which permits to extend the 3/2 result to other nonlinearities. Moreover, setting $\mu(x) = \max\{x, 0\}$, $x \in \mathbf{R}$, the inequalities in (4.1) can be written in the following form:

$$r(\mu(x)) = \frac{a\mu(x)}{1+b\mu(x)} < f(x) < \frac{-a\mu(-x)}{1-b\mu(-x)} = r(-\mu(-x)), \qquad (4.2)$$

where the first inequality holds for all $x \in \mathbf{R}$, and the second one for all $x > -1/b$.

Next, let us observe that μ is the scalar version of the Yorke functional $\mathcal{M}$ defined in (2.4), and hence for $b = 0$ condition (4.2) is very similar to the Yorke condition (2.3). This remark suggests the idea of unifying the Yorke and Wright 3/2–theorems by using a generalized form of (4.2). We did it in [14], where we proved the

Theorem 4.1 *Assume that f satisfies the following three hypotheses*

(H1) *$f : \mathbf{R}\times\mathcal{C} \to \mathbf{R}$ satisfies the Carathéodory condition (see [6, p. 58]). Moreover, for every $q \in \mathbf{R}$ there exists $g(q) \geq 0$ such that $f(t,\phi) \leq g(q)$ a.e. on $\mathbf{R}$ for every $\phi \in \mathcal{C}$ satisfying the inequality $\phi(s) \geq q, s \in [-h,0]$.*

(H2) *There exist $a < 0$, $b \geq 0$ such that*

$$\frac{a\mathcal{M}(\phi)}{1+b\mathcal{M}(\phi)} \leq f(t,\phi) \leq \frac{-a\mathcal{M}(-\phi)}{1-b\mathcal{M}(-\phi)}, \qquad (4.3)$$

where the first inequality holds for all $\phi \in \mathcal{C}$, and the second one for all ϕ such that $\min_{s\in[-h,0]}\phi(s) > -1/b$. Here, $\mathcal{M}(\phi)$ is the Yorke functional defined in (2.4).

(H3) *$\int_0^{+\infty} f(s,p_s)ds$ diverges for every continuous $p(s)$ having nonzero limit at infinity.*

If either $b > 0$ and $|a| \leq 3/2$, or $b = 0$ and $|a| < 3/2$ then all solutions of Eq. (2.5) converge to zero as $t \to \infty$.

Condition (H3) plays a similar role to the condition required by Yorke in his 3/2–theorem in order to ensure that the unique constant limit of the monotone solutions of (2.5) is zero. We have modified it a little bit since we allow f to be discontinuous. We notice that, in particular, it implies that $x \equiv 0$ is the unique equilibrium in (2.5). Next, for $b = 0$, condition (4.3) is the *Yorke condition* (2.3). In this case, (H1) is satisfied automatically with $g(q) = -a\mu(-q)$. Hence the Yorke theorem is a particular case of our Theorem 4.1. Moreover, Theorem 3.1 (and therefore the Wright theorem) is also a consequence of Theorem 4.1. Hence we succeed in unifying these two classical results.

Constant 3/2 in Theorem 4.1 is the best possible both for $b = 0$ (by the mentioned results of Myshkis), and for $b > 0$ (this can be shown by the example of the nonautonomous logistic equation mentioned in Section 3).

As it is shown in [14], Theorem 4.1 allows to obtain 3/2 stability conditions for the global asymptotic stability of very general forms of logistic-type equations, improving many previous results in the literature.

5 Fourth theorem

In our opinion, one of the key ideas in our paper [14] is the introduction of the generalized Yorke condition (4.3). If we consider Eq. (1.1) (and its particular case (1.2)), this condition is satisfied for the nonlinearity in the interesting models of Nicholson, Mackey-Glass, Wazewska-Lasota, etc., even allowing the consideration of variable delays and variable coefficients in these equations. Therefore, it makes sense trying to prove a generalization of Theorem 4.1 to the general case of Eq. (1.1). In [15] we complete our program showing that condition (2.2) introduced in Theorem 2.1 is valid to ensure the global asymptotic stability in a larger family of delay differential equations.

Theorem 5.1 *Assume that f satisfies conditions (H1) and (H2) in the statement of Theorem 4.1. If (2.2) holds, then all solutions of Eq. (1.1) converge to zero as $t \to \infty$.*

It is important to recall here that this result applies to Eq. (1.2) with f satisfying conditions (W1)-(W3). In this case, $a = f'(0) < 0$ and $b = -f''(0)/(2f'(0))$.

Remark 5.2 *Roughly speaking, the main tool in the proofs of our four theorems is the analysis of some related one-dimensional maps, obtained by the integration of some comparison equations. This idea was first developed by I. Györi and S. Trofimchuk in [5], where a first approximation to condition (2.2) was obtained for Eq. (1.2). The improvement of this condition up to (2.2) required the analysis of some complicated maps;*

for example, a function $G(x) = G(x, \delta, a, h)$ defined implicitly by

$$e^{-\delta h} = \int_{r(x)}^{G(x)} \frac{dt}{r^{-1}(t) - r(x)},$$

where $r(x) = ax/(1 + x)$, and $r^{-1}(x) = x/(a - x)$ is its inverse.

We also notice here that the ideas of [5] were adapted to investigate the global asymptotic stability of equations with infinite delay in [11].

6 The conjecture

In the particular case of Nicholson's blowflies equation (1.3), H. Smith [27] proposed the problem of studying if the positive equilibrium in (1.3) attracts all positive solutions for all values of the parameters for which it is locally asymptotically stable. By a change of variables, and using some estimations for the global attractor of (1.3) given in [5], we can apply Theorem 5.1 to obtain for this equation the surprisingly closeness between the regions of local and global asymptotic stability showed in Fig. 1 (see [15, Section 2] for more details). Hence, our theorem not only supports this conjecture, but also our estimation is the best possible for Eq. (1.3) with variable delays $h(t)$, $0 \leq h(t) \leq h$.

Since our results are valid for the general case of Eq. (1.2) with f satisfying conditions (W1)-(W3), we propose the following conjecture, which is more general than the Smith conjecture; note also that the celebrated Wright conjecture can be viewed as a limit case of it.

Conjecture 6.1 *Under conditions (W1)-(W3), the trivial solution of Eq. (1.2) is globally attracting if it is locally asymptotically stable.*

We formulated this conjecture in [13], before proving our Theorem 5.1, and, even with the support of this result, we have tried to disprove it. Indeed, looking more closely at Fig. 1, one notices a "qualitative discrepancy" between the curves of local and global stability near the point $(c, \theta) = (1, 0)$, which corresponds to the limit case when $h \to \infty$ and $c = |f'(0)/\delta| = 1$.

In order to have a "global vision" of Fig. 1 at this point, let us observe that Eq. (1.2) is transformed via the change of variables $y(s) = \delta x(ht)$ into the "singularly perturbed equation" (see [20])

$$\varepsilon y'(s) = -y(s) + \frac{1}{\delta} f(y(s - 1)), \tag{6.1}$$

where $\varepsilon = 1/h$. Hence, the limit case $\theta = 0$ corresponds to $\varepsilon = 0$, that is, to the difference equation

$$y(s) = \frac{1}{\delta} f(y(s-1)). \tag{6.2}$$

It is well known that under conditions (W1)-(W3) all solutions of (6.2) tend to zero as $s \to \infty$ if and only if

$$c = |f'(0)/\delta| \leq 1. \tag{6.3}$$

In fact this condition represents the "absolute stability condition" for Eq. (1.2) under conditions (W1)-(W3), that is, if (6.3) holds then the equilibrium $x = 0$ in (1.2) is globally attracting for all values of the delay h. Therefore, the curve of global stability given by function F defined in (2.10) is continued at $c = 1$ by the line $\theta = 0$, $0 \leq c \leq 1$. However, while the curve of local asymptotic stability has C^∞-contact with the horizontal line at $(1, 0)$, we can check that $F'(0) = 1/2$, and hence the contact with the axis is not C^1. This difference led us to question the veracity of Conjecture 6.1. However, as a support of this conjecture, we proved in [16] that for every function f satisfying (W1)-(W3) there exists $\eta = \eta(f) > 0$ such that Eq. (1.2) is globally asymptotically stable if $0 \leq c - 1 \leq \eta$ and $\delta h < K(c-1)^{-1/8}$, where $K = K(f) > 0$. In particular, this shows that in the plane (c, θ) the optimal curve of global stability for Equation (1.2) with f satisfying (W1)-(W3) has slope zero at $c = 1$, giving in this way an additional support to the above conjecture.

Acknowledgements

This research was supported in part by M.C.T. (Spain) and FEDER, under project BFM2001-3884-C02-02. The author is very thankful to the organizers of this conference for their hospitality and support.

References

[1] N. R. Bantsur, E. P. Trofimchuk, *Existence and stability of periodic and almost periodic solutions of quasilinear equations with maxima,* Ukrainian Math. J., **50**, 847–856, (1999).

[2] F. Brauer, C. Castillo-Chávez, *Mathematical models in population biology and epidemiology,* Springer-Verlag, 2001.

[3] O. Diekmann, S. A. Van Gils, S. M. Verduyn Lunel, H.-O. Walther, *Delay equations, functional-, complex-, and nonlinear analysis,* Applied Mathematical Sciences, Springer-Verlag, 1995.

[4] W. S. C. Gurney, S. P. Blythe, R. M. Nisbet, *Nicholson's blowflies revisited*, Nature, **287**, 17–21, (1980).

[5] I. Györi, S. Trofimchuk, *Global attractivity in* $x'(t) = -\delta x(t) + pf(x(t-\tau))$, Dynamic Syst. Appl., **8**, 197–210, (1999).

[6] J. K. Hale, S. M. Verduyn Lunel, *Introduction to functional differential equations,* Applied Mathematical Sciences, Springer-Verlag, 1993.

[7] U. an der Heiden, M. C. Mackey, *The dynamics of production and destruction: analytic insight into complex behavior,* J. Math. Biology, **16**, 75–101, (1982).

[8] A. Ivanov, E. Liz, S. Trofimchuk, *Halanay inequality, Yorke* 3/2 *stability criterion, and differential equations with maxima,* Tohoku Math. J., **54**, 277–295, (2002).

[9] Y. Kuang, *Delay differential equations with applications in population dynamics,* Academic Press, 1993.

[10] J. C. Lillo, *Oscillatory solutions of the equation* $y'(x) = m(x)y(x-n(x))$, J. Differential Equations, **6**, 1–36, (1969).

[11] E. Liz, C. Martínez, S. Trofimchuk, *Attractivity properties of infinite delay Mackey-Glass type equations,* Differential and Integral Equations, **15**, 875–896, (2002).

[12] E. Liz, M. Pinto, G. Robledo, V. Tkachenko, S. Trofimchuk, *Wright type delay differential equations with negative Schwarzian,* Discrete Contin. Dynam. Systems, **9**, 309–321, (2003).

[13] E. Liz, G. Robledo, S. Trofimchuk, *La conjetura de Wright: entre la distribución de los números primos y el crecimiento de la población*, Cubo Mat. Educ., **3**, 89–107, (2001), in Spanish.

[14] E. Liz, V. Tkachenko, S. Trofimchuk, *Yorke and Wright* 3/2*-stability theorems from a unified point of view,* Discrete Contin. Dynam. Systems, Supplement Volume, 580–589, (2003).

[15] E. Liz, V. Tkachenko, S. Trofimchuk, *A global stability criterion for scalar functional differential equations,* SIAM J. Math. Anal., **35**, 596–622, (2003).

[16] E. Liz, E. Trofimchuk, S. Trofimchuk, *Mackey-Glass type delay differential equations near the boundary of absolute stability*, J. Math. Anal. Appl., **275**, 747–760, (2002).

[17] E. Liz, S. Trofimchuk, *Análisis de estabilidad en ecuaciones diferenciales con máximo,* Actas del XVI CEDYA/VI CMA, 357–364, (1999), in Spanish.

[18] E. Liz, S. Trofimchuk, *Existence and stability of almost periodic solutions for quasilinear delay systems and the Halanay inequality,* J. Math. Anal. Appl., **248**, 625–644 (2000).

[19] M. C. Mackey, L. Glass, *Oscillation and chaos in physiological control systems*, Science, **197**, 287–289, (1977).

[20] J. Mallet-Paret, R. Nussbaum, *A differential-delay equation arising in optics and physiology,* SIAM J. Math. Anal., **20**, 249–292, (1989).

[21] W. de Melo, S. van Strien, *One-dimensional dynamics,* Springer, 1993.

[22] H. Matsunaga, R. Miyazaki, T. Hara, *Global attractivity results for nonlinear delay differential equations,* J. Math. Anal. Appl., **234**, 77–90, (1999).

[23] J.D. Murray, *Mathematical Biology,* Springer, 1993.

[24] A. D. Myshkis, *Lineare Differentialgleichungen mit Nacheilendem Argument,* Deutscher Verlag Wiss., Berlin, 1955.

[25] M. Pinto, S. Trofimchuk, *Stability and existence of multiple periodic solutions for a quasilinear differential equation with maxima,* Proc. Roy. Soc. Edinburgh Sect. A, **130**, 1103–1118, (2000).

[26] D. Singer, *Stable orbits and bifurcation of maps of the interval,* SIAM J. Appl. Math., **35**, 260–267, (1978).

[27] H. L. Smith, *Monotone Dynamical Systems. An Introduction to the Theory of Competitive and Cooperative systems,* AMS, Providence, RI, 1995.

[28] J. W.-H. So, J. S. Yu, *Global attractivity for a population model with time delay,* Proc. Amer. Math. Soc., **123**, 2687–2694, (1995).

[29] J. W.-H. So, J. S. Yu, *On the uniform stability for a 'food-limited' population model with time delay,* Proc. Roy. Soc. Edinburgh Sect. A, **125**, 991–1002, (1995).

[30] H.-O. Walther, *Contracting return maps for monotone delayed feedback,* Discrete Contin. Dynam. Systems, **7**, 259–274, (2001).

[31] J. A. Yorke, *Asymptotic stability for one dimensional differential-delay equations,* J. Differential Equations, **7**, 189–202, (1970).

[32] M. Wazewska-Czyzewska, A. Lasota, *Matematyczne problemy dynamiki ukladu krwinek czerwonych,* Matematyka Stosowana, **VI**, 23–40, (1976), in Polish.

[33] E. M. Wright, *A nonlinear difference-differential equation,* J. Reine Angew. Math., **194**, 66–87, (1955).