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How to control chaotic behaviour and population size with proportional feedback

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ABSTRACT

We study the control of chaos in one-dimensional discrete maps as they often occur in modelling population dynamics. For managing the population, we seek to suppress any possible chaotic behavior, leading the system to a stable equilibrium. In this Letter, we make a rigorous analysis of the proportional feedback method under certain conditions fulfilled by a wide family of maps. We show that it is possible to stabilize the chaotic dynamics towards a globally stable positive equilibrium, that can be chosen among a broad range of possible values. In particular, the size of the population can be enhanced by control in form of population reduction. This paradoxical phenomenon is known as the *hydra effect*, and it has important implications in the design of strategies in such areas as fishing, pest management, and conservation biology.

1. Introduction

Mechanisms of control of chaos have attracted much interest in different areas such as mechanics, electronics, chemistry, and biology (see, e.g., [1,2] and references therein). The aim of most of the proposed methods consists in stabilizing one of the unstable periodic orbits embedded in a chaotic attractor either by adjusting a parameter of the model or modifying the state variable. The latter method is in general implemented by introducing an external parameter that one can control to drive the chaotic system to a stable situation. These techniques are easier to implement, especially if we try to apply them to biological systems, since the intrinsic parameters as the growth rate are in general difficult to modify, while external parameters such as migration or fishing are more easily controllable.

We are interested in the stabilization of the system into a positive equilibrium. While in engineering it is desirable to use only small control because large perturbations of the system may be costly, this is not so important in biology, where the control problem can be seen as a strategy for sustainable development or control of plagues, and thus it can be necessary to apply larger perturbations to lead the population size to stabilization into a desirable size. For a further discussion and related references, see the nice introduction in [3].

In this Letter, we focus our attention on the proportional feedback (PF) method, introduced by Güémez and Matías [4,5]. Based on analysis, we identify control parameter values that allow to stabilize chaos to unique stable equilibria. Furthermore, it is possible to choose suitable values of the parameter control either to enrich or to reduce the size of the state variable. This aspect has important implications in ecology. For example, the phenomenon of a population increasing in response to an increase in its per-capita mortality rate has to be taken into consideration to design strategies in fisheries and pest management. This paradoxical phenomenon is known as the *hydra effect*, see the recent survey of Abrams [6]. Thus, on the one hand, we improve the knowledge about the PF method, since, as far as we are aware, previous studies were often based on simulations, not indicating how the control parameter needs to be chosen. On the other hand, our result illustrates the potentially paradoxical effect of control previously observed for the limiter control method [7].

Since it is important to drive most of the trajectories to the desirable attractor, another important aspect of our studies is the size of the basin of attraction of the equilibrium. In this direction,

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the proof of our main result shows that the basin of attraction of the controlled state is the whole interval, and, in consequence, the control is more robust in the presence of noise.

Section 2 is devoted to prove our main result and show an application to the Ricker model [8]. In Section 3, we discuss the main conclusions, including comparison of the PF method with other related control methods such as the *constant feedback method* (CF) [9] and the *limiter control* (LC) [10,11].

2. Main result

As it is well known, the development of the theory of chaos was strongly stimulated by the doubling-period bifurcations route found for one-dimensional discrete equations of the form $x_{n+1} = f(x_n)$ in the pioneering work of May and others [12–14]. In the usual models of population dynamics, f is a unimodal function such as the quadratic map $f(x) = rx(1-x)$ and the exponential (Ricker) map $f(x) = x\exp[r(1-x)]$, where r means the intrinsic growth rate of the population. Accordingly, most of the parametric control techniques for suppressing chaos were inspired by models in population dynamics governed by the above mentioned one-dimensional maps. Moreover, the new parameter that helps to stabilize the dynamics has a clear biological meaning.

We consider the proportional feedback method (PF). It consists in multiplying the state variable by a constant factor $\gamma > 0$ every p iterations, where p is an integer greater than zero. Here we only consider the case $p = 1$. Thus, the map f becomes $f(\gamma x)$ after the control. The biological interpretation is that a percentage of the population is removed (by migration, or harvesting) if $\gamma < 1$, whereas a factor $\gamma > 1$ means a population supplement at each generation.

The main result of this Letter states that, under some mild assumptions on the function f (met by the most usual one-dimensional maps used in mathematical modelling), it is possible to stabilize a chaotic system generated by f into a globally attracting equilibrium using the PF method. This is proved rigorously; moreover, we show that any target between zero and the maximum value of f can be reached. This makes a big difference with the CF and LC methods, as we discuss later. Our study is motivated by the gap in the literature concerning analytic results proving that stabilization is achievable when the control is applied, and how to choose an appropriate value of the parameter control. Two exceptions are the papers of Gueron [15] and Wieland [16], regarding the CF method. We prove a result in this direction using the PF method, and show that it is more versatile than the CF and the LC methods, especially if one is interested in using the control to achieve a desired population size. In accordance with the paradox of limiter control [7], it is shown that a reduction in the size of the state variable may lead the system to shift its mean value. Moreover, all admissible values of the population size can be achieved. We prove more: contrary to what happens with the limiter control approach [7], the enrichment effect is observed for a wide class of functions, including the exponential model.

Other important aspect of our result is that, whereas the equilibrium in a model controlled by constant feedback is only locally stable [15] (and thus the control is less robust against noise), we prove that the application of proportional feedback leads the system to a global attractor (that is, all solutions converge to it, regardless of the initial condition). Another consequence of this property, with important implications in ecology, is that the system is less prone to extinction, avoiding the Allee effect [17] induced by the introduction of negative constant feedback [18].

Now we formulate our main result. The hypotheses are motivated by the applications to well known systems as the quadratic and the exponential ones, but are met by many other functions.

Theorem 1. *Denote by $I = [0, b]$ a real interval ($b = +\infty$ is allowed). Let $f : I \to I$ be a C^1 function with only two fixed points: $x = 0$ and $x = K > 0$, with $f'(0) > 1$ and $f'(K) < -1$. Assume that f has a unique critical point $c < K$ in such a way that $f'(x) > 0$ for all $x \in (0, c)$, $f'(x) < 0$ for all $x > c$, and $f''(x) < 0$ on $(0, c)$. Then there exist $\gamma_1, \gamma_2 \in (0, 1)$ such that, for any $\gamma \in (\gamma_1, \gamma_2)$, the map $g_\gamma(x) = f(\gamma x)$ has a unique positive equilibrium K_γ that is asymptotically stable. Moreover, when γ ranges from γ_1 to γ_2, the equilibrium K_γ takes all values between zero and $f(c) = \max\{f(x):\ x \in I\}$. If, in addition, $f \in C^3(I)$ and $f'''(x)/f'(x) < (3/2)(f''(x)/f'(x))^2$ for all $x \neq c$, then the positive equilibrium K_γ is globally attracting for all $\gamma \in (\gamma_1, \gamma_2)$.*

Some remarks are in order: first, we notice that the technical assumption to ensure global stability is the so-called property of negative Schwarzian derivative, which is a well-known tool in discrete dynamical systems [19]. It is easy to check that the quadratic and the exponential maps, among other usual functions, satisfy this condition (see [20]). The other assumptions are also important; for example, if the sign of f'' is not constant on $(0, c)$, then the conclusion of Theorem 1 does not hold in general, since the controlled map can have more than one positive equilibrium, and therefore global stability is not possible. In particular, Theorem 1 is not applicable to Allee type population models, in which initial values of the population size below a critical threshold are driven to extinction.

A second remark is that the key values in the statement of the theorem are easy to determine (this is derived from the proof below):

- $\gamma_1 = 1/f'(0)$.
- γ_2 is determined by solving the system of equations $f(\gamma_2 x) = x$, $\gamma_2 f'(\gamma_2 x) = -1$.
- The maximum value $f(c)$ is attained for $\gamma^* = c/f(c)$. We notice that, for this parameter value, the equilibrium of the controlled system is superstable, that is, $g'_{\gamma^*}(K_{\gamma^*}) = 0$.
- Any population size S between 0 and $f(c)$ can be reached by PF control. Indeed, we only have to solve equation $f(\gamma S) = S$ for $\gamma \in (\gamma_1, \gamma_2)$.

Before giving the proof, we present an example to illustrate how the theorem works.

Since for the quadratic map everything is easily computable by solving algebraic equations of order at most two, we consider the exponential map $f(x) = x\exp[r(1-x)]$. Following our previous remarks, we determine γ_1 and γ_2. First, $\gamma_1 = 1/f'(0) = \exp(-r)$; to get γ_2, we solve the system $f(\gamma_2 x) = x$, $\gamma_2 f'(\gamma_2 x) = -1$. This leads to

$$\gamma_2 \exp\bigl[r(1-\gamma_2 x)\bigr] = 1, \qquad \gamma_2 \exp\bigl[r(1-\gamma_2 x)\bigr](1 - r\gamma_2 x) = -1.$$

By substitution of the first equation into the second one, we easily get $\gamma_2 x = 2/r$. Therefore,

$$1 = \gamma_2 \exp\bigl[r(1-\gamma_2 x)\bigr] = \gamma_2 \exp\bigl[r(1-2/r)\bigr] = \gamma_2 \exp(r-2),$$

which gives $\gamma_2 = \exp(2-r)$. Thus, for $\gamma \in (e^{-r}, e^{2-r})$, the controlled system $x_{n+1} = \gamma x_n \exp[r(1-\gamma x_n)]$ has a globally attracting positive fixed point. The critical value of f is $c = 1/r$. Hence, we can choose any value between 0 and $f(c) = \exp(r-1)/r$, and find the value of γ to stabilize the system into it. In particular, $f(c)$ is attained for $\gamma^* = c/f(c) = \exp(1-r)$. We give a numerical example with $r = 2.8$, that is,

$$x_{n+1} = \gamma x_n \exp\bigl[2.8(1-\gamma x_n)\bigr]. \tag{1}$$

The uncontrolled system ($\gamma = 1$) is chaotic. A global attractor is achieved for $\gamma \in (\gamma_1, \gamma_2) = (0.0608, 0.4493)$. The maximum value

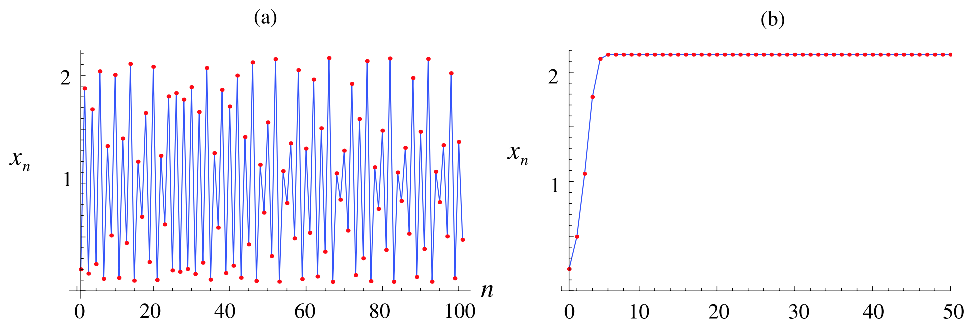

Fig. 1. (Color online.) Time series for (a) the uncontrolled system (1) with $\gamma = 1$, and (b) the superstable equilibrium at the maximum value for $\gamma = 0.1653$.

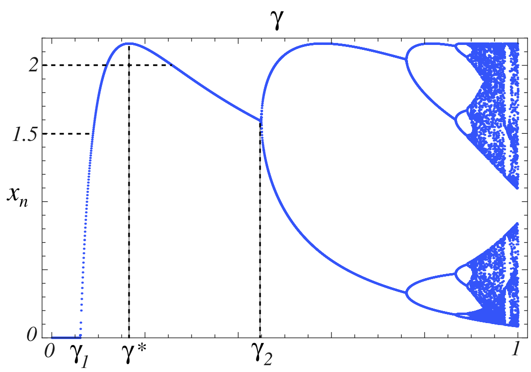

Fig. 2. (Color online.) Bifurcation diagram for Eq. (1) with the bifurcation parameter $\gamma \in (0, 1)$. See the text for an explanation of the horizontal and vertical lines.

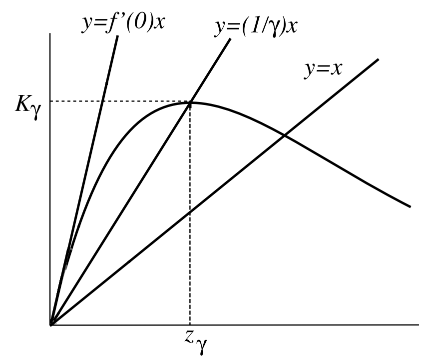

Fig. 3. For each γ between γ_1 and 1, K_γ is found as the projection on the vertical axis of the intersection point between the graph of f and the line $y = (1/\gamma)x$.

$f(c) = 2.16059$ is attained for $\gamma^* = 0.1653$. In Fig. 1(a), we plot a time series of the chaotic system, while the solution of the controlled system with $\gamma = \gamma^*$ is displayed in Fig. 1(b).

In Fig. 2, we plot the bifurcation diagram of Eq. (1) with the parameter γ ranging between $\gamma = 0$ and $\gamma = 1$ (uncontrolled system). The values of γ_1, γ_2 and γ^* are represented there. Notice that the equilibrium is globally stable for $\gamma \in (\gamma_1, \gamma_2)$. The enrichment effect is easily observable here; for example, a reduction of 83% of the population modeled by (1) leads to stabilization of the population around its maximum value.

If we want to reach a globally stable population size S between 0 and $f(c)$, we solve numerically equation $f(\gamma S) = S$. For example, for $S = 1.5$, there is a unique solution $\gamma = 0.088$ of this equation in the interval (γ_1, γ_2). For $S = 2$, there are two possible values: 0.1172 and 0.2582. These values are represented in Fig. 2 as the intersection between the horizontal dashed lines and the part of the bifurcation diagram between γ_1 and γ_2.

Proof of Theorem 1. For each $\gamma \in (0, 1]$, let us define the function $g_\gamma : I \to I$ by $g_\gamma(x) = f(\gamma x)$. A positive fixed point K_γ of g_γ is defined by equation $f(\gamma x) = x$, which is equivalent to $f(z) = z/\gamma$, with $z = \gamma x$. The solution $z = z_\gamma$ has a clear geometric interpretation: it is the abscissa of the intersection between the graph of f and the line $y = (1/\gamma)x$, whereas $K_\gamma = z_\gamma/\gamma$ is the ordinate (see Fig. 3).

It is clear from the properties of f that there exists a unique solution K_γ for each γ between $\gamma_1 = 1/f'(0)$ and $\gamma = 1$. Thus we can define a continuous map $P : (\gamma_1, 1] \to (0, f(c)]$ by $P(\gamma) = K_\gamma$ (recall that $f(c)$ is the maximum of f). Define $\gamma^* = c/f(c)$. The equality $f(\gamma^* f(c)) = f(c)$ implies that $K_{\gamma^*} = P(\gamma^*) = f(c)$. Since f is increasing in $(0, c)$ and decreasing in (c, b), it follows that P is increasing in (γ_1, γ^*) and decreasing in $(\gamma^*, 1)$. Moreover, since $\lim_{\gamma \to \gamma_1} P(\gamma) = 0$, P takes all values between 0 and $f(c)$. Next, K_γ is asymptotically stable while $|g'_\gamma(K_\gamma)| < 1$. Notice that $g'_\gamma(K_\gamma) = \gamma f'(\gamma K_\gamma) = \gamma f'(z_\gamma)$ and therefore $g'_\gamma(K_\gamma)$ is positive for $\gamma < \gamma^*$ and negative for $\gamma > \gamma^*$.

Since $g'_{\gamma^*}(K_{\gamma^*}) = 0$ and, by hypothesis, $g'_1(K_1) = f'(K) < -1$, there exists at least one value of $\gamma \in (\gamma^*, 1)$ such that $g'_\gamma(K_\gamma) = \gamma f'(\gamma K_\gamma) = -1$. Denoting by γ_2 the minimum of such values, it follows that K_γ is asymptotically stable for all $\gamma \in (\gamma_1, \gamma_2)$.

To complete the proof, it only remains to notice that, for $\gamma \in (\gamma_1, \gamma_2)$, g_γ has negative Schwarzian derivative because it is the composition of f with a linear map (see [19]), and thus Singer's results [20] apply to conclude that K_γ is a global attractor. □

Remark. We notice that the one-parameter family of maps $\{g_\gamma\}$ defined in the proof undergoes a period-doubling bifurcation at $\gamma = \gamma_2$, and a transcritical bifurcation at $\gamma = \gamma_1$. For $\gamma < \gamma_1$, $x = 0$ is the unique fixed point, and it is globally attracting. This can be observed in Fig. 2 for Eq. (1).

3. Discussion

We proved that the proportional feedback control is an effective tool both for suppressing chaos, and, not less important, to control the population size. The possibility to choose a strategy of control based on removing a constant percentage of the system variable at each iteration to stabilize the population into any desired value between zero and the maximum possible size has a great potential for applications. For example, it can be used to design policies of fishing for sustainable development, control of plagues, and in many other disciplines as genetics, economics or social sciences [6,7]. Here, we proved analytically that this method works well in one-dimensional discrete dynamical systems as the quadratic and the exponential maps.

In order to emphasize the good properties of the PF method stated in Theorem 1, we give a comparison with other related control methods considering the chaotic quadratic equation

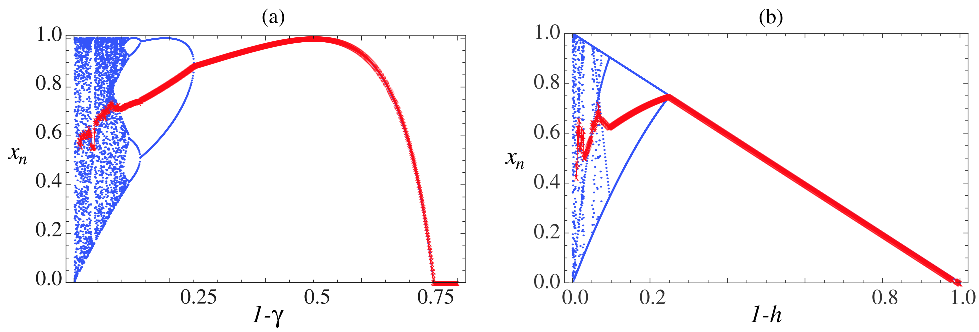

Fig. 4. (a) Bifurcation diagram for Eq. (4); (b) Bifurcation diagram for Eq. (5). The thick line indicates the average of population size.

$$x_{n+1} = 4x_n(1 - x_n), \tag{2}$$

largely used as a model in population dynamics. Function $f(x) = 4x(1 - x)$ maps the interval $[0, 1]$ into itself, and reaches its maximum value at $x = 1/2$, with $f(1/2) = 1$.

Since the most clear biological interpretation of the PF method is a strategy of harvesting, we first compare this technique with the constant feedback control scheme. For the quadratic map (2), it writes

$$x_{n+1} = 4x_n(1 - x_n) - c, \tag{3}$$

where $c > 0$ means that a constant harvesting or migration takes place every period of time. This example was studied in [15]. Although it is possible to stabilize the chaotic equation (2) into a stable equilibrium for some positive values of c using the control (3), this method has two clear disadvantages with respect to the PF method, which in this case reads

$$x_{n+1} = 4\gamma x_n(1 - \gamma x_n), \tag{4}$$

with $\gamma \in [0, 1]$.

First, applying Theorem 1, we easily prove that the positive equilibrium K_γ of (4) is globally attracting for $\gamma \in (0.25, 0.75)$. Moreover, K_γ takes all possible values between 0 and 1. In contrast, the range of attainable values of a stable positive equilibrium using (3) reduces to the interval $J = (0.5, 0.5625)$. Moreover, for $c \in J$, the region of attraction of the equilibrium K_c is (see [15]) $I_c = (x_L(c), x_R(c))$, where $x_L(c) = [3 - (9 - 16c)^{1/2}]/8$, $x_R(c) = [3 + (9 - 16c)^{1/2}]/8$. For example, for $c = 0.55$, the basin of attraction is $I_c = (0.3191, 0.4309)$. Moreover, most of initial population sizes out of I_c are driven to extinction by (3) due to the Allee effect (see [18]). Thus, a strategy of captures based on PF control prevents the risk of extinction observed for the CF method, and it is more robust against noise. We also notice that the enrichment effect is not observed for the CF control method.

Next we consider the limiter control method, for which the paradoxical effect of enrichment was reported in [7] using the quadratic map. This strategy of control is based in preventing the population from exceeding a threshold level h. Thus, this scheme applied to (2) reads

$$x_{n+1} = \min\{4x_n(1 - x_n), h\}, \tag{5}$$

where h ranges from $h = 0$ to $h = 1$ (uncontrolled system).

In Fig. 4, we plot the bifurcation diagrams both for (4) and (5). To show the stabilizing effect as the control increases, we used $1 - \gamma$ and $1 - h$, respectively, as the bifurcation parameters. The thick line represents the mean of the asymptotic state variables. Although the enrichment property is observed both for the PF and the LC methods, the maximum mean value attained with the LC technique is $x = 0.75$, while the maximum population size $x = 1$ is reached with the PF control method.

In addition, we recall that, as reported in [7], unimodal maps approaching zero for large state variables (like the Ricker map) do not show the paradoxical effect of enrichment when limiter control is applied. Thus, the PF method is more versatile, since it can also be used to enhance the population size in models governed by this class of maps.

The control methods discussed in this section assume that in each generation a portion of the population is harvested. In Eqs. (3) and (5), the harvesting takes place after the density dependent mechanism of population dynamics; in contrast, in Eq. (4) harvest is prior to the reproductive season.

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