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WRIGHT TYPE DELAY DIFFERENTIAL EQUATIONS WITH NEGATIVE SCHWARZIAN

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Abstract. We prove that the well-known 3/2 stability condition established for the Wright equation (WE) still holds if the nonlinearity $p(\exp(-x)-1)$ in WE is replaced by a decreasing or unimodal smooth function f with $f'(0) < 0$ satisfying the standard negative feedback and below boundedness conditions and having everywhere negative Schwarz derivative.

1. **Introduction.**

In this paper we study the global stability properties of the scalar delay-differential equation

$$x'(t) = f(x(t-1)), \tag{1.1}$$

where $f \in C^3(\mathbb{R}, \mathbb{R})$ satisfies the following additional conditions **(H)**:

(H1) $xf(x) < 0$ for $x \neq 0$ and $f'(0) < 0$.

(H2) f is bounded below and has at most one critical point $x^* \in \mathbb{R}$ which is a local extremum.

(H3) $(Sf)(x) < 0$ for all $x \neq x^*$, where $Sf = f'''(f')^{-1} - (3/2)(f'')^2(f')^{-2}$ is the Schwarz derivative of f.

The negative feedback condition **(H1)** and boundedness condition **(H2)** are very typical in the theory of (1.1); the first one causes solutions to tend to oscillate about zero, while both of them guarantee the existence of the global compact attractor to Eq. (1.1) (e.g. see [9]). On the other hand, the Schwarzian negativity condition **(H3)** is rather common in the theory of one-dimensional dynamical systems (see [12, 16]) while it is not very usual for the studies of delay-differential equations. Here, we introduce **(H3)** in hope of obtaining an analogue of the Singer global stability result for an one-dimensional map $f : I \to I$; this result (see its complete formulation below) states that the local stability of a unique fixed point $e \in I$

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plus an appropriate (monotone or unimodal) form of the map f imply the global attractivity of the equilibrium $e \in I$ [12]. The negative Schwarzian condition is not artificial at all, it appears naturally also in some other contexts of the theory of delay differential equations, see e.g. [10, Sections 6–9], where it is explicitly used, and [3, Theorem 7.2, p. 388], where the condition $Sf < 0$ is implicitly required; moreover, many nonlinear delay differential equations used in mathematical modelling in biology (e.g. Mackey-Glass, Lasota-Wazewska, Nicholson, Goodwin equations) have their right-hand sides satisfying the hypothesis **(H3)**. Take also, for example, the celebrated Wright equation which was used to describe the distribution of prime numbers or to model population dynamics of a single species:

$$x'(t) = -px(t-1)[1+x(t)], \quad p > 0. \tag{1.2}$$

For $x(t) > -1$, Eq. (1.2) is reduced to (1.1) after applying the transformation $y = -\ln(1+x)$. In this case $f(x) = (\exp(-x) - 1)$ and, by abuse of notation, we will again refer to the transformed system

$$x'(t) = p(\exp(-x(t-1)) - 1), \; p > 0, \tag{1.3}$$

as the Wright equation. In this case, f is strictly decreasing and has no inflexion points; both these facts simplify considerably the investigation of (1.3). Below, we will present two other important examples, with nonlinearities which may have an inflexion point (some “food-limitation” models) or even a local extremum (population model exhibiting the Allee effect).

Eq. (1.1) has been considered before by several authors but only assuming conditions **(H1)** and **(H2)**, see [9, 11, 22, 23] and references therein. In particular, the Morse decomposition of its compact attractor has been described in detail in [9]. Moreover, it was proved also that the Poincaré-Bendixson theorem holds for (1.1) with the decreasing nonlinearity f so that the asymptotic periodicity is the “most complicated” type of behavior in (1.3) [9, 11, 23]. It should be noted also that the above information has essentially a “qualitative” character. So that adding **(H3)**, we can hope to obtain some additional information of analytic nature about possible bifurcations in parametrized families of (1.1). The following variational equation along its unique steady state $x = 0$ plays a very important role in the study of such bifurcations:

$$x'(t) = f'(0)x(t-1). \tag{1.4}$$

As is well-known, this equation is unstable when $-f'(0) > \pi/2$, and this instability implies the existence of slowly oscillating periodic solutions to (1.1) (see e.g. [22]). Surprisingly, the dynamically simpler case $-f'(0) < \pi/2 = 1.571...$ has not been studied thoroughly before, and, in particular, it seems that the following Wright conjecture has not been solved: the inequality $p < \pi/2$ is sufficient for the global stability in (1.3). On the other hand, the sufficiency of the stronger condition $p < 3/2$ for the global stability of Eq. (1.3) was proved in ‘a very difficult theorem of Wright [26]’ (see [2, page 64]), where also the sharper conditions $p < 37/24 = 1.5416...$ and $p < 1.567...$ were announced. It should be noted that proofs of the 3/2 stability condition for Eq. (1.3) have strongly used the specific exponential form of the nonlinearity $f(x) = p(\exp(-x) - 1)$ and, in particular, the monotonicity properties of such f. This fact explains why any analogue of this Wright result has not been proved for other, essentially different (nonexponential), right-hand sides in Eq. (1.1) (even for monotone f, the general situation being considerably more complicated).

An important step in solving the Wright conjecture was made in Theorem 3 from [22] which provides us with some examples of Eq. (1.1) which satisfy **(H1), (H2)** and have slowly oscillating periodic solutions, even when the corresponding linearized equation (1.4) is exponentially stable. This means that the local exponential stability of the steady state in (1.1) (or, what is the same, exponential stability in (1.4)) with f having only these standard and usual properties **(H1), (H2)** in general does not imply the global asymptotic stability in (1.1). Moreover, as a simple consequence of an elegant approach towards stable periodic orbits for scalar equations of the form $x'(t) = -\mu x(t) + f(x(t-1))$, $\mu \geq 0$ with Lipschitz nonlinearities proposed recently in [24] (see also [25]), we get the following

Theorem 1.1 ([24])**.** *For every $\alpha \geq 0$ there exists a smooth strictly decreasing function $f(x)$ satisfying* **(H1)**, **(H2)**, *$-f'(0) = \alpha$ and such that Eq. (1.1) has a nontrivial periodic solution which is hyperbolic, stable and exponentially attracting with asymptotic phase (so therefore (1.1) is not globally stable).*

This result is of special importance for us, since it shows clearly that the strong correlation between local (at zero) and global properties of Eq. (1.3) can not be explained only with the concepts presented in **(H1), (H2)**.

On the other hand, Walther's result from [22] cannot be applied to Eq. (1.3) so that the original Wright conjecture remains open. We explain here this particularity of Eq. (1.3) by its additional property of having negative Schwarz derivative Sf: in fact, no function from [22, Theorem 3] can have $Sf < 0$. Moreover, bearing in mind the following result of D. Singer for one-dimensional maps: *"Assume that the function $h \in C^3[a,b]$ is either strictly decreasing or has only one critical point x^* (local extremum) in $[a,b]$. If h has a unique fixed point $e \in [a,b]$ which is locally asymptotically stable and $(Sh)(x) < 0$ for all $x \neq x^*$, then e is the global attractor of the dynamical system $h : [a,b] \to [a,b]$"*, we propose to generalize Wright's conjecture in the form stated below.

Conjecture 1.2. *Let all conditions* **(H)** *be satisfied and $-f'(0) < \pi/2$. Then $\lim_{t\to+\infty} x(t) = 0$ for every solution $x(t)$ to Eq. (1.1).*

We remark that this conjecture is very close to the Hal Smith conjecture [17] to the effect that the local and global asymptotic stabilities for Nicholson's blowflies equation

$$x'(t) = -\delta x(t) + px(t-1)\exp(-a(x(t-1))), \quad x, \delta, p, a > 0,$$

are equivalent. Observe that this equation also has a unique positive steady state and nonlinearity satisfying the negative feedback and the negative Schwarzian conditions (see [1, 8, 18] for further discussions).

Furthermore, due to recent results of Krisztin [5], now we can indicate some class of symmetric and monotone nonlinearities (e.g. $f(x) = -p\tanh x, f(x) = -p\arctan x$, $p > 0$) for which the above conjecture is true. Although for both the mentioned functions condition $Sf < 0$ holds, in general the additional convexity condition imposed on f in [5] (see also [6]) is different from **(H3)**: evidently, the requirement of the negative Schwarzian is not the unique way to approach the problem (the same situation that we have in the theory of one-dimensional maps). In fact, to prove our main result (Theorem 1.3), we only need some geometric consequences of the inequality $Sf < 0$ for the graph of f. For instance, if $f''(0) = 0$ then this geometric consequence is given by $(f(x) - f'(0)x)x > 0$ for $x \neq 0$

(that necessarily holds also under the above mentioned convexity, symmetry and monotonicity assumptions from [5, 6]). This geometric approach was developed further in [8], where a generalization of the Yorke condition [7, Section 4.5] was proposed instead of **(H3)**.

In this paper we carry out the first step towards Conjecture 1.2, showing how all the conditions (**H**) come together in proving

Theorem 1.3. *If, in addition to* **(H)**, *we have* $-f'(0) \leq 1.5$, *then the steady state solution* $x(t) = 0$ *of Eq. (1.1) is globally attracting.*

To prove Theorem 1.3, we will essentially use an idea from [4], which allows us to construct some one-dimensional map inheriting some properties of Eq. (1.1). Roughly speaking, we consider maps $F_k = F_k(z) : \mathbb{R} \to \mathbb{R},\ F_k(0) = 0$, which give the values of the k-th consecutive extremum of the oscillating solutions $x(t, z),\ z \neq 0$, satisfying $x(s, z) \equiv z,\ s \in [-1, 0]$. Then we investigate some relations existing between the global attractivity properties of F_k and (1.1), trying to deduce in this way the global asymptotical stability of (1.1) from the corresponding property of the discrete dynamical system generated by F_k. Since the computation becomes more and more complicated with the growth of k, we only consider the simplest case $k = 1$ here. Computer experiments show that, increasing k, we obtain better approximations to the condition $-f'(0) \leq \pi/2$ given in Conjecture 1.2 (for example, for $k = 2$ we get $-f'(0) \leq 37/24$ and so on). However, due to the technical complications, this way to approach the above conjecture could be used only for very special cases.

Curiously, the note [4] devoted to the study of the Yorke type functional differential equations with sublinear nonlinearity (see [27]) and, in particular, the Yorke 3/2 stability criterion, still can be extended to the class of nonlinear Wright's type delay differential equations. We consider the special nature of the number 3/2 (which was found almost simultaneously by A.D. Myshkis [14] and by E.M. Wright [26]) as an invariant of such a prolongation. Moreover, there exists an interesting interplay between both these types of functional differential equations if we consider a variable coefficient $p(t)$ instead of the constant p in Eq. (1.3) (see [19] for details). In any case, it should be noted that the Yorke and the Wright type equations have rather different structures (see [4, 19, 20] for more comments).

Completing our discussion, we consider briefly two other Wright type equations studied recently by several authors:

Example 1.4. The "food-limitation" model [19, p. 456] or, what is basically the same, the Michaelis-Menten single species growth equation with one delay (see [7, p. 132]):

$$x'(t) = -r(1 + x(t))\frac{x(t-h)}{1 + cr(1 + x(t-h))},\ x > -1,\ c \geq 0,\ r, h > 0\ . \tag{1.5}$$

Note that (1.5) is of the form $x'(t) = -r(1 + x(t))g(x(t-h))$ with $Sg = 0$. The change of variables $x = \exp(-y) - 1$ reduces (1.5) to $y'(t) = rf(y(t-h))$, where $f(y) = g(\exp(-y)-1)$ is strictly decreasing with $f(0) = 0,\ f'(0) = -(1+cr)^{-1}$, and $(Sf)(y) = -1/2 < 0$ for all y. By Theorem 1.3, the inequality $rh \leq (3/2)(1 + cr)$ implies the global stability of the zero solution to Eq. (1.5) (compare with [7] and [19]). We also point out that Eq. (1.5) with $c = 0$ coincides with the Wright equation (1.2), so that Wright's 3/2 stability theorem is a very special case of our result.

Example 1.5. Consider now a population model described by

$$x'(t) = x(t)[a + bx(t-h) - cx^2(t-h)],\ a, c \in (0, +\infty),\ b \in \mathbb{R}. \tag{1.6}$$

This equation has a unique positive equilibrium e^* and the change of variables $x = e^* \exp(-y)$ transforms (1.6) into (1.1) with $f(x) = -(a + be^* \exp(-x) - c(e^* \exp(-x))^2)$. Note that f has negative S-derivative as a composition of a quadratic polynomial and the real exponential function. If $b \le 0$ then f is strictly decreasing, and if $b > 0$, then f has exactly one critical point (minimum). In the latter case, the population model exhibits the so-called Allee effect [13]. Applying Theorem 1.3, we see that e^* attracts all positive solutions of (1.6) once $(2ce^* - b)he^* \le 1.5$ (compare e.g. with [7, pp. 143-146], where also other references can be found).

The paper is organized as follows. In Section 2 we define several auxiliary scalar functions and study their properties as well as relations connecting them. Finally, in the last section, we use these functions to prove Theorem 1.3 (notice that, in contrast with [1], we may not use the above formulated Singer's result for this purpose).

2. **Auxiliary functions.** To prove our main result, we will proceed in analogy to [4], so that the construction of one-dimensional maps inheriting attractivity properties of the dynamical system generated by Eq. (1.1) is the main tool here. In this section, we introduce several such scalar maps and study their properties as well as the relations existing among them.

First we note that we can only have eight different possibilities for the maps satisfying hypotheses **(H)**, depending on the situation of the eventual critical point and the inflexion points. A graphic representation of all these cases is given in Fig. 1 below, where x^* denotes the critical point and c_1, c_2 are the inflexion points. We recall that a real function has at most one inflexion point in any interval in which the Schwarz derivative is well defined and is negative (see [16]).

In the sequel, up to the proof of Theorem 1.3 and with the unique exception made for Corollary 2.2, we will always assume that f satisfies **(H)** and $f''(0) > 0$. This situation corresponds to the pictures (a)-(e) in Fig. 1.

Next, for $a < 0$, $b > 0$, we introduce the set

$$\begin{aligned} K_{a,b} \quad = \quad & \{y \in C^3(\mathbb{R}) : y \text{ satisfies } \textbf{(H1), (H2)},\ y'(0) = a,\ y''(0) = 2b, \\ & (Sy)(x) \le 0 \text{ for all } x \text{ such that } y'(x) \neq 0\}. \end{aligned}$$

Also, for every $a < 0$ and every $b > 0$, consider the rational function $r(x, a, b) = a^2x/(a - bx)$ defined over $(ab^{-1}, +\infty)$. Let $K^+_{a,b}$ (respectively, $K^-_{a,b}$) be the set of restrictions of elements of $K_{a,b}$ to $[0, +\infty)$ (respectively, to $(ab^{-1}, 0]$). We will denote by r^+ and r^- the restrictions of $r(\cdot, a, b)$ to the intervals $[0, +\infty)$ and $(ab^{-1}, 0]$ respectively. The following properties are elementary:

i) $r^+ \in K^+_{a,b}$, $r^- \in K^-_{a,b}$, $Sr \equiv 0$ and $r(x, a, b) \to -a^2b^{-1}$ as $x \to +\infty$;

ii) the inverse ρ of r is given by $\rho(x, a, b) = ax/(a^2 + bx)$, and $\rho'''(x) < 0$ for all $x > -a^2b^{-1}$;

iii) the equation $r(x, a, b) = -x$ has exactly two solutions: $x_1 = 0$ and $x_2 = (a + a^2)/b$.

Furthermore, it can be proved that r^+ and r^- are respectively the minimal element of $K^+_{a,b}$ and the maximal element of $K^-_{a,b}$ with respect to the usual order. The following slightly different result will play a key role in the sequel:

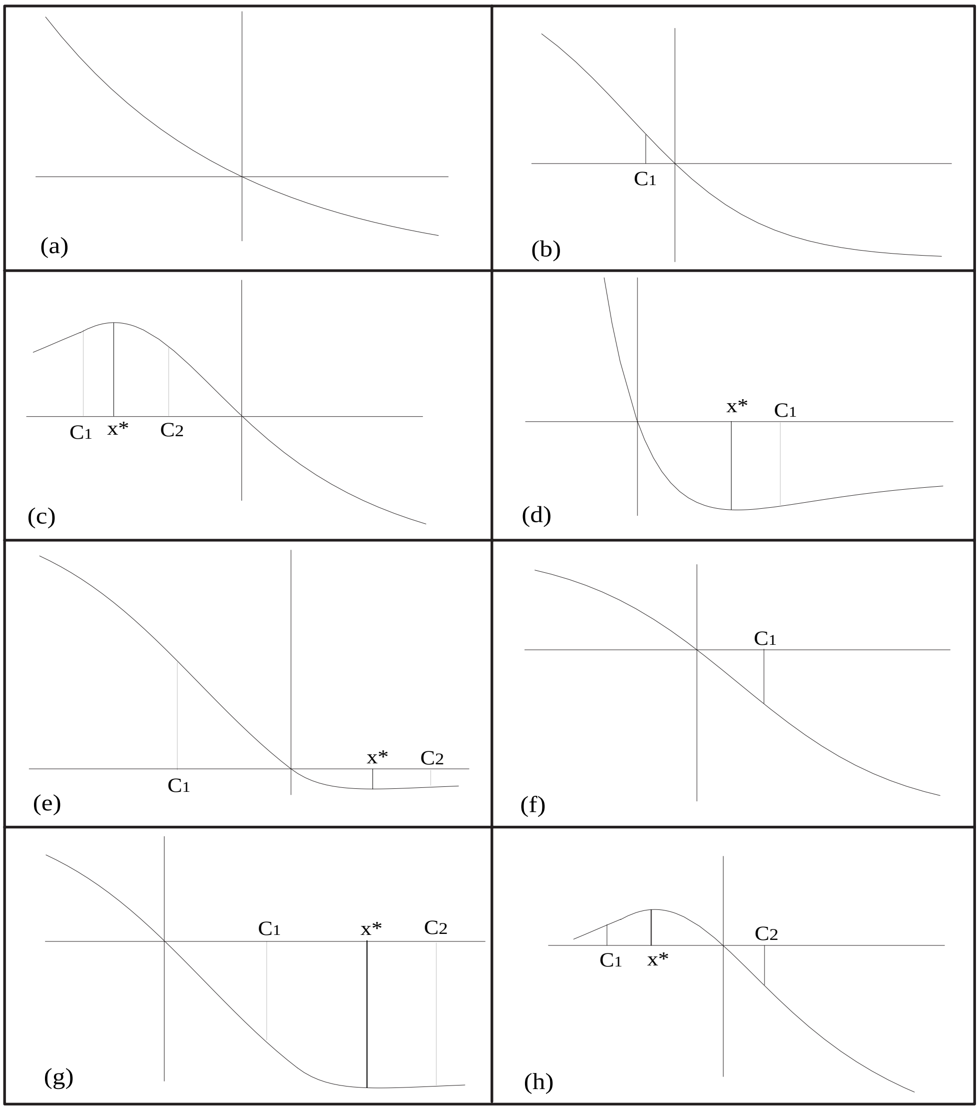

FIGURE 1. Different maps satisfying **(H)**

Lemma 2.1. *For all* $y \in K_{a,b}$ *with* $(Sy)(x) < 0$ *for* $x \notin \{w : y'(w) = 0\}$*, we have* $r(x, a, b) < y(x)$ *for all* $x > 0$ *and also* $r(x, a, b) > y(x)$ *for all* $x \in (ab^{-1}, 0)$*.*

Proof. Take $g \in C^3(\mathbb{R})$ and define $G(x) = g''(x)/g'(x)$ for all $x \in D_g = \{x : g'(x) \neq 0\}$. Then we have $(Sg)(x) = G'(x) - (1/2)G^2(x),\ x \in D_g$. Therefore, for every function $g \in C^3(\mathbb{R})$ with negative Schwarzian, the associated function $G(x)$ satisfies the differential Riccati inequality $G'(x) - (1/2)G^2(x) < 0$ for all $x \in D_g$. Now, the lemma follows from standard comparison results (see, e.g. [21, Theorem 5.III]) if we observe that $R = r''/r'$ and $Y = y''/y'$ satisfy $(Sr)(x) = R'(x) - (1/2)R^2(x) = 0$, $(Sy)(x) = Y'(x) - (1/2)Y^2(x) < 0$, for all $x \in (ab^{-1}, +\infty) \cap D_y$, and $Y(0) = R(0)$. Indeed, the above relations imply in cases (a)-(c) that $R(x) > Y(x)$ for all $x > 0$. Now, integrating R and Y over the interval $(0, x)$, we get $r'(x) < y'(x)$. Integrating

r' and y' from 0 to x again, we obtain $r(x) < y(x)$ for all $x > 0$. In cases (d)-(e) we obtain using the above arguments that $r(x) < y(x)$ for all $x \in (0, x^*)$. Now, since r is strictly decreasing and y reaches its minimum at x^*, it is obvious that the relation $r(x) < y(x)$ also holds for $x \geq x^*$.

Now the previous arguments allow us to prove that $r'(x) < y'(x)$ and $r(x) > y(x)$ for all $x \in (ab^{-1}, 0)$ in cases (a) and (d), where y has no negative inflexion points.

The proof for (b) and (e) is slightly different if the inflexion point c_1 of y belongs to the interval $(ab^{-1}, 0)$. In this case, we can use the same arguments only to show that $r'(x) < y'(x)$ and $r(x) > y(x)$ for $x \in [c_1, 0)$. Next, by convexity arguments,

$$r(x) > r(c_1) + r'(c_1)(x - c_1) > y(c_1) + y'(c_1)(x - c_1) > y(x)$$

for all $x \in (ab^{-1}, c_1)$.

Finally, case (c) can be studied analogously taking into account that r is strictly decreasing on $(ab^{-1}, 0)$, whereas y reaches its maximum at x^*. □

As a by-product of the proof of Lemma 2.1, we state the following corollary, which will be used in the proof of Theorem 1.3 when considering the case $f''(0) < 0$.

Corollary 2.2. *Suppose that f satisfies **(H)** and $f''(0) < 0$. Then f is bounded on $\mathbb{R}$.*

Proof. Since f satisfies **(H)**, the inequality $f''(0) < 0$ implies that either $f''(x) < 0$ for all $x \leq 0$ (cases (f) and (g) in Fig. 1) or f has a global maximum at $x^* < 0$ (see Fig. 1 (h)). Since in the latter case the statement of the corollary is evident, we can assume that $f''(x) < 0$ on $(-\infty, 0]$.

Next, the function g defined by $g(x) = -f(-x)$ satisfies $g'(0) = f'(0)$ and $g''(x) = -f''(-x) > 0$ for $x \geq 0$. Hence g has not inflexion points on $[0, +\infty)$ and we can use the property $(Sg)(x) = G'(x) - (1/2)G^2(x) < 0$ for $x \in [0, +\infty) \cap D_g$ as it was done in the first part of the proof of Lemma 2.1 to establish that $g(x) > r(x) = r(x, g'(0), g''(0)/2)$ for $x \in (0, +\infty)$. Since r is strictly decreasing and $r(+\infty) = 2(f'(0))^2/f''(0) \in \mathbb{R}$, we can conclude that g is bounded on $(0, +\infty)$. Thus $f(x) = -g(-x)$ is bounded on $(-\infty, 0)$. Finally, since f satisfies **(H)**, f is also bounded on $[0, +\infty)$. □

Now set

$$a = f'(0), 2b = f''(0), \mu = ab^{-1} = 2f'(0)/f''(0), r(x) = r(x, f'(0), f''(0)/2)$$

and define the continuous functions $A, B : (\mu, +\infty) \to \mathbb{R}$ and $D : \mathbb{R}_+ \to \mathbb{R}$ by

$$A(x) = x + r(x) + \frac{1}{r(x)} \int_x^0 r(t)dt, \quad B(x) = \frac{1}{r(x)} \int_{-r(x)}^0 r(s)ds \quad \text{for } x \neq 0,$$

$$A(0) = B(0) = 0, \quad D(x) = \begin{cases} A(x) & \text{if } r(x) < -x, \\ B(x) & \text{if } r(x) \geq -x. \end{cases}$$

For the case $f'(0) < -1$, we will also use the function $R(x) = r(x, A'(0), A''(0)/2)$ defined on the interval $(2A'(0)/A''(0), \infty) = (\nu, \infty)$. Note that $A'(0) = f'(0)+1/2 < 0$, $A''(0) = f''(0)(1 + (6f'(0))^{-1}) > 0$. It is easy to check that $f'(0) < -1/6$ implies $(\nu, +\infty) \subset (\mu, +\infty)$, and that $A(x_2) = B(x_2)$, where $r(x_2) = -x_2 < 0$. Also $B'(0) = -(r'(0))^2/2 = -(f'(0))^2/2$. In the following lemmas we state other properties of the functions A, B, D, R.

Lemma 2.3. *If $f'(0) < -1$ then $A'(x) < 0$ and $(SA)(x) < 0$ for all $x \in (\mu, x_2)$.*

Proof. Since $f'(0) < -1$, we have $x_2 > 0$ and $-xr(x) < r^2(x)$ for all $x \in (\mu, x_2) \setminus \{0\}$. Hence

$$\int_x^0 r(t)dt \le -xr(x) < r^2(x) \text{ and } A'(x) = r'(x)\Big(1 - \frac{\int_x^0 r(t)dt}{r^2(x)}\Big) < 0,\ x \neq 0.$$

We get also $A'(0) = r'(0) + 1/2 < -1/2 < 0$.

Next, integrating by parts, we obtain

$$\begin{aligned} A(x) &= r(x) + \frac{xr(x) + \int_x^0 r(t)dt}{r(x)} = r(x) + \frac{xr(x) + \int_{r(x)}^0 vd\rho(v)}{r(x)} \\ &= r(x) + \frac{1}{r(x)}\int_0^{r(x)} \rho(v)dv = G(r(x)), \end{aligned}$$

where $\rho(v) = r^{-1}(v)$ and $G(z) = z + \int_0^1 \rho(vz)dv$.

Then, by the formula for the Schwarzian derivative of the composition of two functions [12], we obtain

$$(SA)(x) = (SG)(r(x))(r'(x))^2 + (Sr)(x) = (SG)(r(x))(r'(x))^2.$$

Hence, in order to prove that $SA < 0$, we have to verify that $(SG)(r(x)) < 0$. On the other hand, $A'(x) < 0$ if and only if $G'(r(x)) > 0$, so it suffices to show that $(SG)(r(x)) < 0$ when $G'(r(x)) > 0$. Taking into account the fact that $\rho'''(z) < 0$ for $z = r(x)$, we have $G'''(z) = \int_0^1 v^3\rho'''(vz)dv < 0$ and therefore $(SG)(r(x)) < 0$. □

Lemma 2.4. *If $f'(0) < -1$ then $(A(x) - R(x))x > 0$ for $x \in (\nu, x_2) \setminus \{0\}$.*

Proof. Recall that $R(x) = r(x, A'(0), A''(0)/2)$ and apply Lemmas 2.1, 2.3. □

Next, we will compare the functions $B(x)$ and $R(x)$ over the interval $[x_2, +\infty)$. In order to do that, we need the following simple result:

Lemma 2.5. *If $(s, \zeta) \in \Pi = [-1, 0] \times [-1.5, -1.25]$, then*

$$L(\zeta, s) = s - \zeta - \ln(1 + s - \zeta) + \frac{2(\zeta + 1/2)^2(s - \zeta)^2}{(2\zeta + 1)\zeta^2 + (2/3)\zeta(s - \zeta)} < 0.$$

Proof. We have

$$\frac{\partial L(\zeta, s)}{\partial s} = \frac{(s - \zeta)(s - A_-(\zeta))(s - A_+(\zeta))(12\zeta^2 + 16\zeta + 3)}{\zeta(1 + s - \zeta)(6\zeta^2 + \zeta + 2s)^2},$$

where

$$\begin{aligned} A_\pm(\zeta) &= -\frac{72\zeta^4 + 108\zeta^3 + 46\zeta^2 + 15\zeta + 3}{2(12\zeta^2 + 16\zeta + 3)} \\ &\pm \frac{3(\zeta + 1)(2\zeta + 1)\sqrt{(2\zeta + 1)(72\zeta^3 - 12\zeta^2 - 6\zeta + 1)}}{2(12\zeta^2 + 16\zeta + 3)}. \end{aligned}$$

Note that A_+ and A_- are continuous on the interval $J = [-\frac{3}{2}, -\frac{2}{3} - \frac{\sqrt{7}}{6}) = [-1.5, -1.107..)$, and $A_-(\zeta) < A_+(\zeta)$ for all $\zeta \in J$. Now, it is straightforward to see (after several elementary transformations) that every root ζ of $A_+(\zeta) = -1$ satisfies $\zeta(\zeta+1)^2(12\zeta^2+16\zeta+3)(36\zeta^2+12\zeta-7) = 0$, and therefore belongs to the set $\{0, -1, -1/6 \pm \sqrt{2}/3, -2/3 \pm \sqrt{7}/6\} = \{0, -1, -0.638..., 0.304..., -1.107..., -0.225...\}$. Since $A_+(-1.5) = -1.29991... < -1$, we obtain that

$$A_-(\zeta) < A_+(\zeta) < -1 \le s \text{ for all } \zeta \in [-1.5, -1.107...),\ s \in [-1, 0].$$

This implies immediately that $\partial L(\zeta,s)/\partial s < 0$ for all $\zeta \in [-1.5,-1.25]$ and $s \in [-1,0]$. Thus

$$\max_{(s,\zeta)\in\Pi} L(\zeta,s) = \max_{\zeta\in[-1.5,-1.25]} L(\zeta,-1).$$

Finally, for $\zeta \in [-1.5,-1.25]$, we have

$$\partial L(\zeta,-1)/\partial\zeta = -(36\zeta^2+16\zeta-3)(\zeta+1)^3\zeta^{-2}(3\zeta+2)^{-2}(2\zeta-1)^{-2} > 0,$$

so that $\max_{\zeta\in[-1.5,-1.25]} L(\zeta,-1) = L(-1.25,-1) = -0.0006945... < 0.$ □

Lemma 2.6. *If $f'(0) \in [-1.5,-1.25]$ then $B(x) > R(x)$ for all $x \geq x_2$.*

Proof. Note that $B(x) = \tilde{B}(-r(x))$, and $R(x) = \tilde{R}(-r(x))$, where

$$\tilde{B}(u) = \int_0^1 r(zu)dz = \frac{\zeta}{u\theta}\big(u - \frac{1}{\theta}\ln(1+\theta u)\big),$$

and

$$\tilde{R}(u) = R(\rho(-u)) = -\frac{2(\zeta+1/2)^2 u}{(2\zeta+1)\zeta+(2/3)\theta u},$$

with $\zeta = f'(0)$, $\theta = -f''(0)/(2f'(0)) > 0$ and $\rho(x) = r^{-1}(x)$. Therefore $B(x) - R(x) > 0$ for $x \in [x_2,+\infty)$ if and only if $\tilde{B}(u) - \tilde{R}(u) > 0$ for all $u = -r(x) \in [x_2,-r(+\infty)) = \theta^{-1}[-\zeta-1,-\zeta)$. Finally, by Lemma 2.5, we have that

$$\tilde{B}(u) - \tilde{R}(u) = \frac{\zeta}{u\theta^2}L(\zeta,\theta u+\zeta) > 0$$

for all $\zeta \in [-1.5,-1.25]$ and $u \in \theta^{-1}[-\zeta-1,-\zeta)$, since in this case $s = \theta u + \zeta \in [-1,0]$. □

Lemmas 2.4 and 2.6 together yield the

Corollary 2.7. *If $f'(0) \in [-1.5,-1.25]$, then $D(x) > R(x)$ for $x > 0$.*

3. **Proof of Theorem 1.3.** In this section, we prove Theorem 1.3. Thus we assume that f satisfies all conditions **(H)**.

Denote by C the space of all continuous real functions γ on the interval $[-1,0]$, with the norm given by $\|\gamma\| = \max_{-1\leq t\leq 0}|\gamma(t)|$. Since f is continuous, each $\gamma \in C$ determines a unique solution $x = x(\cdot,\gamma)$ to Eq. (1.1) on $[-1,+\infty)$ such that $x(t) = \gamma(t)$, $t \in [-1,0]$. It is well-known that the application $F^t\gamma : \mathbb{R}_+ \times C \to C$ given by $F^t\gamma(s) = x_t(s) = x(t+s,\gamma)$, $s \in [-1,0]$, defines a continuous semiflow on C. Moreover, this semiflow is point dissipative:

Lemma 3.1. *Assume that f is bounded below and satisfies **(H1)**. Then there exists $K_0 > 0$ such that for any $\gamma \in C$ we have $\limsup_{t\to+\infty}\|F^t\gamma\| < K_0$.*

Proof. See for example [9, Proposition 2.1]. □

Now, fix an arbitrary $\gamma \in C$. It follows from Lemma 3.1 that the ω-limit set $\omega(\gamma)$ of the trajectory $\{F^t\gamma : t \in \mathbb{R}_+\} \subset C$ is an invariant and compact set. Write $m = m_\gamma = \min_{\alpha\in\omega(\gamma)}\alpha(0)$ and $M = M_\gamma = \max_{\alpha\in\omega(\gamma)}\alpha(0)$.

Evidently, Theorem 1.3 is proved if we demonstrate that $M_\gamma = m_\gamma = 0$ once $-f'(0) \in [0,1.5]$. We only need to consider the case $m < 0 < M$. Indeed, otherwise (i.e. $m \geq 0$ or $M \leq 0$) every solution $x(t,\alpha)$, $\alpha \in \omega(\gamma)$, is bounded and monotone (due to (**H1**)), and the limit values $x_\pm = x(\pm\infty,\alpha) \in \mathbb{R}$ are steady states of (1.1). Now, since (1.1) possesses a unique equilibrium $x(t) \equiv 0$, we conclude that either $0 = x_- \leq x(t,\alpha) \leq x_+ = 0$ or $0 = x_- \geq x(t,\alpha) \geq x_+ = 0$ for all $t \in \mathbb{R}$, so that

$\omega(\gamma) = \{0\}$. Therefore, $m = M = 0$ if $m \geq 0$ or $M \leq 0$, and in the sequel we will always assume that $m < 0 < M$.

In the following lemmas we establish some relations between m and M which are needed in the proof of Theorem 1.3.

Lemma 3.2. *We have $m > D(M)$ and $m > r(-r(M)/2)$.*

Proof. First we assume that $r(M) \leq -M$. Then $t_1 = M/r(M) \in [-1, 0]$. Next, $x(t) = r(M)t,\ t \in [t_1, t_1 + 1]$ is the solution of the initial value problem $x(s) = M,\ s \in [t_1 - 1, t_1]$ for

$$x'(t) = r(x(t-1)). \tag{3.1}$$

Since $m = \beta(0) = \min_{\alpha \in \omega(\gamma)} \alpha(0)$ for some $\beta \in \omega(\gamma)$, and since $\omega(\gamma)$ is an invariant set, we obtain the existence of a solution $\tilde{z}(t) : \mathbb{R} \to \mathbb{R}$ to (1.1) such that $\tilde{z}(s) = \beta(s),\ s \in [-1, 0]$ and $\tilde{z}_t \in \omega(\gamma)$ for every $t \in \mathbb{R}$. Obviously, $z(t) = \tilde{z}(t-1)$ also satisfies (1.1) for all $t \in \mathbb{R}$ and is such that $z(t) \geq z(1) = m$ for all $t \in \mathbb{R}$. This implies $0 = z'(1) = f(z(0))$. Finally, by hypothesis **(H1)**, we get $z(0) = 0$.

Let fix now this solution $z = z(t)$. Clearly $M = x(t) \geq z(t)$ for all $t \in [t_1 - 1, t_1]$. Moreover, we will prove that $x(t) \geq z(t)$ for all $t \in [t_1, 0]$.

Indeed, if this is not the case we can find $t_* \in [t_1, 0)$ such that $x(t_*) = z(t_*)$ and $x(t) \geq z(t)$ for all $t \in [t_1 - 1, t_*]$. We claim that

$$z'(t) > x'(t) \ \text{ for all } \ t \in [t_*, 0]. \tag{3.2}$$

We distinguish two cases: if $z(t-1) > 0$ then, using Lemma 2.1, we obtain $x'(t) = r(x(t-1)) \leq r(z(t-1)) < f(z(t-1)) = z'(t)$. Next if $z(t-1) \leq 0$, then $z'(t) = f(z(t-1)) \geq 0 > r(x(t-1)) = x'(t)$.

After integration over $(t_*, 0)$, and using $x(0) = z(0) = 0$, it follows from (3.2) that $z(t_*) < x(t_*)$, which is a contradiction.

Thus $x(t) \geq z(t)$ for $t \in [t_1, 0)$ and, arguing as above, we obtain

$$m = \int_0^1 z'(s)ds = \int_0^1 f(z(s-1))ds > \int_0^1 r(x(s-1))ds =$$
$$\int_{-1}^{t_1} r(M)ds + \int_{t_1}^{0} r(x(u))du = M + r(M) + \int_{t_1}^{0} r(r(M)u)du = A(M).$$

Now in the general case (i.e. we do not assume that $r(M) \leq -M$), we will prove that

$$f(z(t)) > r(r(M)t)\,,\ \forall\ t \in (-1, 0). \tag{3.3}$$

To do this, we first show that $z'(t) > r(M)$ for all $t \in (-1, 0)$. Indeed, if $z(t-1) > 0$ then, by Lemma 2.1, $z'(t) = f(z(t-1)) > r(z(t-1) \geq r(M)$. On the other hand, if $z(t-1) \leq 0$ then $z'(t) = f(z(t-1)) \geq 0 > r(M)$. Hence,

$$z(t) = -\int_t^0 z'(s)ds < -\int_t^0 r(M)ds = r(M)t\,,\ t \in (-1, 0).$$

To obtain (3.3) we only have to note that $f(z(t)) > r(z(t)) > r(r(M)t))$ if $z(t) > 0$ and $f(z(t)) \geq 0 > r(r(M)t))$ if $z(t) \leq 0$.

Now, using (3.3), we obtain

$$m = z(1) = \int_0^1 f(z(s-1))ds > \int_0^1 r(r(M)(s-1))ds = B(M).$$

Finally, applying Jensen's integral inequality (see [15, p. 110]) , we have

$$m > B(M) = \frac{1}{r(M)} \int_{-r(M)}^{0} r(s)ds \geq r(-r(M)/2).$$

This completes the proof. □

As a consequence of Lemma 2.6 and Lemma 3.2, we obtain that $R(m), r(m)$ and $r(r(-r(M)/2))$ are well-defined and that $R(\nu, +\infty) \subset (\nu, +\infty)$ for suitable values of $f'(0)$:

Corollary 3.3. *We have $m > \mu, r(-r(M)/2) > \mu$ if $f'(0) \in [-1.5, 0)$ and $m > \nu$ if $f'(0) \in [-1.5, -1.25]$.*

Proof. Indeed, for $f'(0) \in (-2, 0)$ and $f''(0) > 0$, we have

$$m > r(-r(M)/2) > r(-\frac{r(+\infty)}{2}) = \frac{(f'(0))^3}{f''(0)(1 - f'(0))} \geq \frac{2f'(0)}{f''(0)} = \mu.$$

Next, $-A'(0) = -(f'(0) + 0.5) \leq 1$ for $f'(0) \geq -1.5$, and Lemmas 2.6, 3.2 lead to the estimate

$$m > D(+\infty) = B(+\infty) \geq R(+\infty) = -A'(0)\nu \geq \nu.$$

This proves the corollary. □

Lemma 3.4. *Let $f'(0) \in [-1.5, 0)$. We have $M < r(m)$. Moreover, if $f'(0) \in [-1.5, -1.25]$ then $M < R(m)$.*

Proof. We have that $r(m)$ is well defined and $[m, +\infty) \subset [\mu, +\infty)$ since $f'(0) \in [-1.5, 0)$ (see Corollary 3.3). Take now $\theta \in \omega(\gamma)$ such that $y(t) = y(t, \theta)$ satisfies $y(1) = M$ and, consequently, $y'(1) = y(0) = 0$. First we prove that $f(y(s)) < r(m)$ for $s \in [-1, 0]$ and $f'(0) \in [-1.5, 0)$. Indeed, we have $f(y(s)) < r(y(s)) \leq r(m)$ if $y(s) < 0$ and $f(y(s)) \leq 0 < r(m)$ if $y(s) \geq 0$. Thus

$$M = y(1) = \int_0^1 f(y(s-1))ds < \int_0^1 r(m)ds = r(m).$$

Now, if $f'(0) < -1$ then $r(m) > -m$, from which it follows that $t_2 = m(r(m))^{-1} \in (-1, 0]$. Next, $x(t) = r(m)t$, with $t \in [t_2, t_2 + 1]$, is the solution of the initial value problem $x(s) = m,\ s \in [t_2 - 1, t_2]$ for Eq. (3.1). Now we only have to argue as in the proof of Lemma 3.2 to obtain the inequality $M < A(m)$. Finally, by Lemma 2.4 and Corollary 3.3, we obtain $M < A(m) < R(m)$ when $f'(0) \in [-1.5, -1.25]$. □

Remark 3.5. Assume that $(f(x) - f'(0)x)x > 0$ for all $x \neq 0$. Replacing $r(x)$ with $r_1(x) = f'(0)x$ in the proof of Lemma 3.2, we can observe that it still works without any change if we set $A(M) = (f'(0) + 1/2)M$ and $B(M) = -(f'(0))^2 M/2$. The same observation is valid for the proof of Lemma 3.4 (up to the last sentence beginning from the word "Finally"). Therefore, under the above assumption, we have $m > A(M) = (f'(0) + 1/2)M$ and $M < A(m) = (f'(0) + 1/2)m$ once $f'(0) \leq -1$. Also $m > B(M) = -(f'(0))^2 M/2$ and $M < r_1(m) = f'(0)m$ if $f'(0) < 0$.

Proof of Theorem 1.3. We will reach a contradiction if we assume that $m < 0 < M$. Suppose first that $f''(0) > 0$. If $f'(0) \in (-1.5, 0)$, in view of Lemmas 3.2, 3.4 and Corollary 3.3 we obtain that $M < r(m) \leq r \circ r(-r(M)/2) = \lambda(M)$ with the rational function $y = \lambda(x)$. Now, $\lambda(M) < M$ for $M > 0$ if $\lambda'(0) = (1/2)|f'(0)|^3 < 1$. Therefore, if $f'(0) \in [-1.25, 0)$ we obtain the desired contradiction under the assumption $M > 0$.

Now let $f'(0) \in [-1.5, -1.25]$ and, consequently, $R'(0) = f'(0)+0.5 \in [-1, -0.75]$. In this case Corollary 2.7, Lemma 3.2 and Lemma 3.4 imply that $M < R(R(M))$. As $R \circ R(x) \leq x$ for all $x > 0$ whenever $(R \circ R)'(0) = (R'(0))^2 \leq 1$, we obtain a contradiction again. Therefore the solution $x(t) \equiv 0$ of Eq. (1.1) is globally attracting if $f''(0) > 0$ and $f'(0) \in [-1.5, 0)$.

Assume now that $f''(0) < 0$. The change of variables $y(t) = -x(t)$ transforms (1.1) into $y'(t) = g(y(t-1))$ with $g(x) = -f(-x)$. It is easily seen that $g''(0) > 0$ and that g satisfies all properties from **(H)** (note that by Corollary 2.2, g is bounded below). Applying now the first part of the proof to the modified equation $y'(t) = g(y(t-1))$, we reach the same contradictions if we assume that $m < 0 < M$. Hence our theorem is also proved when $f''(0) < 0$. Note that this change of variables transforms the cases (f), (g) and (h) from Fig. 1 into (b), (c) and (e) of the same figure respectively.

Finally, take $f''(0) = 0$. In this case, $x = 0$ is an inflexion point for f and $(f(x) - f'(0)x)x > 0$ if $x \neq 0$. Therefore it is natural to employ the linear function $r_1(x) = f'(0)x$ instead of the rational function $r(x)$ used in the case $f''(0) > 0$. Now, by Remark 3.5 we have that $m > A(M) = (f'(0) + 1/2)M$ and $M < A(m) = (f'(0) + 1/2)m$ if $f'(0) \in [-1.5, -1)$. Hence $M < (f'(0) + 1/2)^2 M \leq M$, a contradiction. Let now $f'(0) \in [-1, 0)$. By the same Remark 3.5 we obtain that $m > B(M) = -(1/2)(f'(0))^2 M > -(1/2)(f'(0))^3 m$. Thus $-(f'(0))^3/2 > 1$, a contradiction. □

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REFERENCES

[1] I. Györi and S. Trofimchuk, *Global attractivity in $x'(t) = -\delta x(t) + pf(x(t-\tau))$,* Dynamic Syst. Appl. **8** (1999), 197–210.

[2] J. K. Hale, "Asymptotic behavior of dissipative systems," Mathematical Surveys and Monographs 25, A.M.S., Providence, Rhode Island, 1988.

[3] J. K. Hale and S. M. Verduyn Lunel, "Introduction to functional differential equations", Applied Mathematical Sciences, Springer-Verlag, 1993.

[4] A. Ivanov, E. Liz and S. Trofimchuk, *Halanay inequality, Yorke 3/2 stability criterion, and differential equations with maxima,* Tohoku Math. J. **54** (2002), 277–295.

[5] T. Krisztin, *Periodic orbits and the global attractor for delayed monotone negative feedback,* E. J. Qualitative Theory of Diff. Equ., Proc. 6'th Coll. Qualitative Theory of Diff. Equ. **15** (2000), 1–12.

[6] T. Krisztin and H.-O. Walther, *Unique periodic orbits for delayed positive feedback and the global attractor,* J. Dynam. Differential Equations **13** (2001), 1–57.

[7] Y. Kuang, "Delay differential equations with applications in population dynamics," Academic Press, 1993.

[8] E. Liz, V. Tkachenko and S. Trofimchuk, *A global stability criterion for scalar functional differential equations*, submitted (available at `http://arXiv.org/abs/math.DS/0112047).`

[9] J. Mallet-Paret, *Morse decompositions for differential delay equations,* J. Differential Equations **72** (1988), 270–315.

[10] J. Mallet-Paret and R. D. Nussbaum, *A differential-delay equation arising in optics and physiology,* SIAM J. Math. Anal. **20** (1989), 249–292.

[11] J. Mallet-Paret and G. Sell, *The Poincaré-Bendixson theorem for monotone cyclic feedback systems with delay,* J. Differential Equations **125** (1996), 441–489.

[12] W. de Melo and S. van Strien, "One-dimensional dynamics", Springer, 1993.

[13] J. D. Murray, "Mathematical Biology", Springer, 1993.

[14] A. D. Myshkis, "Lineare Differentialgleichungen mit Nacheilendem Argument," Deutscher Verlag Wiss., Berlin, 1955.

[15] H. L. Royden, "Real Analysis," The Macmillan Company, 1969.

[16] A. N. Sharkovsky, S. F. Kolyada, A. G. Sivak and V. V. Fedorenko, "Dynamics of one-dimensional maps," Kluwer Academic Publishers, 1997.

[17] H. L. Smith, "Monotone Dynamical Systems. An Introduction to the Theory of Competitive and Cooperative systems," AMS, Providence, RI, 1995.

[18] J. W.-H. So and Y. Yang, *Dirichlet problem for the diffusive Nicholson's blowflies equation,* J. Differential Equations **150** (1998), 317–348.

[19] J. W.-H. So and J. S. Yu, *Global stability for a general population model with time delays,* in "Differential Equations with Applications to Biology" (Ruan et al. Eds.), Fields Institute Communications series **21**, A.M.S. (1999), pp. 447–457.

[20] S. Trofimchuk and A. Ivanov, *Weak hyperbolicity of delay differential equations and the 3/2-type stability conditions,* Funkcial. Ekvac. **43** (2000), 39–56.

[21] W. Walter, "Differential and integral inequalities", Springer-Verlag, Berlin, 1970.

[22] H.-O. Walther, *A theorem on the amplitudes of periodic solutions of differential delay equations with applications to bifurcation,* J. Differential Equations **29** (1978), 396–404.

[23] H.-O. Walther, "The 2-dimensional attractor of $x'(t) = -\mu x(t) + f(x(t-1))$," Memoirs of the Amer. Math. Soc., Vol. 544, Amer. Math. Soc., Providence, RI, 1995.

[24] H.-O. Walther, *Contracting return maps for some delay differential equations,* in "Functional Differential and Difference Equations" (T. Faria and P. Freitas Eds.), Fields Institute Communications series **29**, A.M.S. (2001), pp. 349–360.

[25] H.-O. Walther, *Contracting return maps for monotone delayed feedback,* Discrete Contin. Dynam. Systems **7** (2001), 259–274.

[26] E. M. Wright, *A nonlinear difference-differential equation,* J. Reine Angew. Math. **194** (1955), 66–87.

[27] J. A. Yorke, *Asymptotic stability for one dimensional differential-delay equations,* J. Differential Equations **7** (1970), 189–202.

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