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Note

A sharp global stability result for a discrete population model

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Abstract

We get a sharp global stability result for a first order difference equation modelling the growth of bobwhite quail populations. The corresponding higher-dimensional model is also discussed, and our stability conditions improve other recent results for the same equation.

Keywords: Discrete population model; Difference equation; Global stability

1. Introduction

Recently [1–3], we have developed a new method for proving global stability in a family of discrete single-species modeled by a difference equation

$$x_{n+1} = x_n F(x_{n-k}), \quad n = 0, 1, \ldots, \tag{1.1}$$

where $F : (0, \infty) \to (0, \infty)$ is continuous. Assuming that there is a unique positive equilibrium $\bar{x}$ of (1.1), we say that $\bar{x}$ is globally stable if it is stable and all solutions of (1.1) converge to $\bar{x}$ as n tends to ∞. Due to biological reasons, by a solution of (1.1) we mean a sequence $\{x_n\}$ defined by the recurrence (1.1) from a set of initial conditions $x_{-k}, \ldots, x_{-1}, x_0 \in (0, \infty)$. Notice that $x_n > 0$ for all $n \geqslant 0$.

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For $k = 0$ in (1.1), the above mentioned method gives us the best possible global stability conditions, since we prove that $\bar{x}$ is globally stable whenever it is asymptotically stable (see [3] and Proposition 3).

For $k > 0$, we believe that local stability continues to imply global stability under the same assumptions on F. Although this is still an open problem, a result in the good direction may be derived from [1,2], and it is stated below as Proposition 4.

This note is devoted to apply our results to the model

$$x_{n+1} = \alpha x_n + \frac{\beta x_n}{1 + x_{n-k}^r}, \quad n = 0, 1, \ldots, \tag{1.2}$$

where $0 \leqslant \alpha < 1 < \alpha + \beta$, and $r > 0$. For $k = 0$, Eq. (1.2) was proposed by Milton and Bélair [4] to describe the growth of bobwhite quail populations, and the global stability of the equilibrium was investigated later by Graef et al. [5,6], including also the case $k > 0$.

For $k = 0$, Eq. (1.2) is of the form $x_{n+1} = f(x_n)$, where f is in general a *hump-with-tail* map, that is, with two extremum points. We notice that many of the known methods for proving global stability in discrete equations only work for monotone or unimodal (one-humped) maps. A typical example is precisely Eq. (1.2) with $\alpha = 0$ and $k = 0$, which becomes the well-known unimodal population model proposed by Maynard Smith in 1974 (see, e.g., [7]). However, the presence or two extrema is not a problem when applying our approach, since the important point is that Eq. (1.2) can be written in the form (1.1) with F strictly decreasing in $(0, \infty)$.

In Section 2, we first recall some previous stability results for (1.2), and then we apply our techniques to prove our main theorem, showing how it improves the previous ones. Moreover, we emphasize that for the one-dimensional model ($k = 0$ in (1.2)), our result is the best possible, and we propose an open problem for $k > 0$.

2. Main result

As it was proved by Levin and May [8], if F is differentiable, then the boundary of the region for which the positive equilibrium $\bar{x}$ of (1.1) is asymptotically stable is given by the relation

$$\bar{x}\left|F'(\bar{x})\right| = 2\cos\left(\frac{k\pi}{2k+1}\right).$$

In the particular case of Eq. (1.2), the equilibrium $\bar{x} = ((\alpha + \beta - 1)/(1 - \alpha))^{1/r}$ is asymptotically stable if

$$r < \frac{2\beta}{(\alpha + \beta - 1)(1 - \alpha)} \cos\left(\frac{k\pi}{2k+1}\right). \tag{2.1}$$

The global stability of $\bar{x}$ was addressed by Graef et al. In their first paper [5], they use Lyapunov functions to get several sufficient conditions. However, as it is pointed out in [6], these conditions are complicated and hard to verify in some cases. In this latest paper, the authors propose an alternative approach to improve their previous estimations. Their main result is the following one:

Theorem 1. [6, Theorem 3.1] *The equilibrium $\bar{x}$ of Eq.* (1.2) *is globally stable if*

$$r < \frac{2\alpha + \beta + 2\sqrt{\alpha^2 + \alpha\beta}}{\beta} \frac{3k + 4}{2(k+1)^2}. \tag{2.2}$$

For $k=0$, (2.2) reads

$$r<2\frac{2\alpha+\beta+2\sqrt{\alpha^2+\alpha\beta}}{\beta},$$

which is in general far from the local stability condition

$$r<\frac{2\beta}{(\alpha+\beta-1)(1-\alpha)}.$$

We are in a position to state our main result.

Theorem 2. *The equilibrium $\bar{x}$ of Eq.* (1.2) *is globally stable if*

$$r<\frac{\beta}{(\alpha+\beta-1)(1-\alpha)}\frac{3k+4}{2(k+1)^2}\quad \text{and}\quad k\geqslant 1, \tag{2.3}$$

or

$$r\leqslant\frac{2\beta}{(\alpha+\beta-1)(1-\alpha)}\quad \text{and}\quad k=0. \tag{2.4}$$

In order to demonstrate Theorem 2, we will use two recent results which make use of the Schwarzian derivative. We recall that the Schwarzian derivative of a C^3-map f is defined by

$$(Sf)(x)=\frac{f'''(x)}{f'(x)}-\frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2,$$

whenever $f'(x)\neq 0$.

Proposition 3. [3, Corollary 2.7] *Assume that $F:[0,\infty)\to(0,\infty)$ is differentiable, $F'(x)<0$ for all $x>0$, $F(0)>1$, and function $g(x)=-\ln(F(\bar{x}e^{-x}))$ has negative Schwarzian derivative everywhere, where $F(\bar{x})=1$.*

Then, the unique positive equilibrium $\bar{x}$ of

$$x_{n+1}=x_nF(x_n),\quad n=0,1,\ldots,$$

is globally stable if and only if

$$\bar{x}\left|F'(\bar{x})\right|\leqslant 2.$$

A similar result can be proved for the higher order difference equation (1.1). The following proposition is a consequence of Corollary 4.3 in [1] and Theorem 1.3 in [2].

Proposition 4. *Let the assumptions of Proposition* 3 *hold. Then, the unique positive equilibrium of* (1.1) *is globally stable if $k\geqslant 1$ and*

$$\bar{x}\left|F'(\bar{x})\right|<\frac{3k+4}{2(k+1)^2}.$$

To prove Theorem 2, there only remains to check that all conditions of Proposition 3 are fulfilled. Indeed, Eq. (1.2) can be rewritten as (1.1) with

$$F(x)=\alpha+\frac{\beta}{1+x^r},$$

which satisfies $F(0)=\alpha+\beta>1$, and $F'(x)=-\beta r x^{r-1}(1+x^r)^{-2}<0$ for all $x>0$. Regarding the Schwarzian derivative of $g(x)=-\ln(F(\bar{x}e^{-x}))$, direct computations show that

$$(Sg)(x)=\frac{-r^2}{2}\frac{2\alpha\beta(1+h(x))^3+\alpha^2(1+h(x))^4+\beta^2(1+2h(x))}{(1+h(x))^2(\alpha+\beta+\alpha h(x))^2},$$

where

$$h(x)=\frac{\alpha+\beta-1}{1-\alpha}e^{-rx}>0, \quad \forall x\in\mathbb{R}.$$

Therefore, $(Sg)(x)<0$ for all $x\in\mathbb{R}$, and we are done.

Final remarks. Notice that, for $k=0$, Theorem 2 gives us the best possible result on the global stability of the equilibrium $\bar{x}$, showing that it is globally stable whenever it is asymptotically stable. For $k>0$, we conjecture that the same is true, that is, the equilibrium $\bar{x}$ of Eq. (1.2) is globally stable if $k>1$ and (2.1) holds. This conjecture is a particular case of Conjecture 4.8 in [1], which was motivated by a remark made by Levin and May in [8]. In any case, it is clear that Theorem 2 also improves Theorem 1 for $k>0$. In fact, it is a substantial improvement; notice, for example, that condition (2.3) is always true for any fixed r, k, β when α is sufficiently close to 1. This fact cannot be derived from Theorem 1.

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