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Sufficient conditions for the global stability of nonautonomous higher order difference equations

LEONID BEREZANSKY†§, ELENA BRAVERMAN‡‖ and EDUARDO LIZ¶#*

†Department of Mathematics, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel
‡Department of Mathematics and Statistics, University of Calgary, 2500 University Drive N.W., Calgary, AB, Canada T2N 1N4
¶Departamento de Matemática Aplicada II, E.T.S.I. Telecomunicación, Campus Marcosende, Universidad de Vigo 36280, Vigo, Spain

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We present some explicit sufficient conditions for the global stability of the zero solution in nonautonomous higher order difference equations. The linear case is discussed in detail. We illustrate our main results with some examples. In particular, the stability properties of the equilibrium in a nonlinear model in macroeconomics is addressed.

Keywords: Difference equations; Exponential stability; Asymptotic stability; Nonlinear model

Mathematics Subject Classifications (2000): 39A10; 39A11

1. Introduction

The stability analysis of higher order difference equations is of great importance in many areas where discrete processes occur, especially in economics and population dynamics.

As it is well-known, the linear equation with real constant coefficients

$$x_{n+1} - x_n + \sum_{k=0}^{N} a_k x_{n-k} = 0, \quad n \geq 0, \tag{1.1}$$

is exponentially stable, if and only if all roots of its characteristic equation

$$\lambda^{N+1} - \lambda^N + \sum_{k=0}^{N} a_k \lambda^{N-k} = 0$$

lie inside the unit circle $D = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. Although necessary and sufficient conditions to ensure this fact can be obtained from the Schur criterion (see, e.g. [4,6,9]), as it is noticed in [6], the stability conditions become increasingly complicated as the order of the

*Corresponding author. Tel.: +34-986-812127. Fax: +34-986-812116. E-mail: eliz@dma.uvigo.es
§Partially supported by Israeli Ministry of Absorption. E-mail: brznsky@cs.bgu.ac.il
‖Partially supported by the NSERC Research Grant and the AIF Research Grant. E-mail: maelena@math.ucalgary.ca
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equation increases, and as a result, their interpretation in the considered model is more and more difficult. This discussion has motivated the investigation of explicit sufficient conditions, which can be easily employed in the applications. In the literature, many results can be found in this direction (see, e.g. [6,8,9] and references therein). Explicit necessary and sufficient conditions were found only for very particular cases of equation (1.1) (see, e.g. [3,12,13,18]). The situation is even more complicated when the considered model is nonautonomous, and the corresponding linear equation has a more general form

$$x_{n+1} - x_n + \sum_{k=0}^{p} a_k(n) x_{g(k,n)} = 0, \quad n \geq 0, \tag{1.2}$$

where $g(k,n) \leq n$ for all $k = 0, 1, \ldots, p$ and all $n \geq 0$.

Recently, some new approaches led to different stability results for linear and nonlinear difference equations. We mention the use of weak contraction arguments [20–22], discrete Halanay-type inequalities [14–16], monotonicity arguments [10,11,17], delay perturbation methods [7] and Bohl-Perron type theorems [1,2].

In this paper, we further develop some ideas from [2,14,17] to obtain new explicit conditions for the global stability of a nonlinear difference equation, which in particular, allows us to generalize some previous stability results for the linear equations (1.1) and (1.2). We notice that, for the particular case of equation (1.1), our conditions can be seen as new explicit conditions to ensure that the moduli of all the roots of a given polynomial are less than one.

We give some examples to illustrate the applicability of our results. Moreover, Section 4 is devoted to show how the stability properties of the equilibrium in a nonlinear version of the well-known Samuelson's multiplier–accelerator model [6] are preserved for some values of the involved parameters.

2. Halanay-type results

First of all, we establish a simple result, which can be obtained from [15]. However, due to the simplicity of its proof, we include it here for the sake of completeness. We recall that the zero solution of a difference equation

$$x_{n+1} = f(n, x_n, \ldots, x_{n-T}), \quad n \geq 0, \tag{2.1}$$

is *globally exponentially stable* if there exist constants $M > 0$, $\lambda \in [0, 1)$ such that, for every solution $\{x_n\}_{n \geq -T}$ of equation (2.1), the inequality

$$|x_n| \leq M\lambda^n \left(\max_{-T \leq i \leq 0} \{|x_i|\} \right) \tag{2.2}$$

holds for all $n \geq 0$. We say that the zero solution of equation (2.1) is *globally asymptotically stable*, if it is stable and for every solution $\{x_n\}$ of the equation, we have $\lim_{n \to \infty} x_n = 0$.

THEOREM 2.1 *Assume that* $f : \mathbb{N} \times \mathbb{R}^{T+1} \to \mathbb{R}$ *satisfies*

$$|f(n, u_0, \ldots, u_T)| \leq b \max\{|u_0|, \ldots, |u_T|\} \tag{2.3}$$

for some constant $b < 1$, and for all $(n, u_0, \ldots, u_T) \in \mathbb{N} \times \mathbb{R}^{T+1}$. Then

$$|x_n| \le b^{n/(T+1)} M_0, \quad n \ge 0,$$

for every solution $\{x_n\}$ of equation (2.1)*, where $M_0 = \max_{-T \le i \le 0}\{|x_i|\}$. In particular, the zero solution of equation* (2.1) *is globally exponentially stable* (*take $\lambda = b^{1/(T+1)} < 1$ in equation* (2.2)).

Proof. We first show that $|x_n| \le bM_0$ for all $n \ge 1$. Indeed, by (2.3),

$$|x_1| = |f(0, x_0, \ldots, x_{-T})| \le bM_0.$$

Fix $j > 1$ and assume that $|x_n| \le bM_0$ is true for $n = 1, 2, \ldots, j$. Thus,

$$|x_{j+1}| = |f(j, x_j, \ldots, x_{j-T})| \le b \max\{|x_j|, \ldots, |x_{j-T}|\} \le b \max\{M_0, bM_0\} = bM_0.$$

In particular, $|x_n| \le bM_0 \le b^{n/(T+1)} M_0$, for all $n = 1, \ldots, T+1$.
Assume now that $|x_n| \le b^{n/(T+1)} M_0$, for all $n = 1, \ldots, p (p \ge T+1)$. Thus,

$$|x_{p+1}| \le b \max\{|x_p|, \ldots, |x_{p-T}|\} \le bM_0 \max\{b^{p/(T+1)}, \ldots, b^{(p-T)/(T+1)}\} = b^{(p+1)/(T+1)} M_0.$$

An application of the induction principle completes the proof. □

Remark 2.2 For the autonomous version of equation (2.1), i.e.

$$x_{n+1} = f(x_n, \ldots, x_{n-T}), \quad n \ge 0, \tag{2.4}$$

an analogous result to Theorem 2.1 was obtained by Sedaghat in [20, Corollary 2]. See also Section 4.3 in the recent monograph [21] of the same author, where other related interesting results and discussion on their significance in the stability theory of difference equations can be found.

For further generalizations and applications of Theorem 2.1, see [14,16,17,22].

Let us consider equation (1.2) with bounded delays, i.e. we assume the existence of an integer $N > 0$ such that $n - N \le g(k, n) \le n$, for all $k = 0, 1, \ldots, p$ and all $n \ge 0$. Notice that the general form of a linear nonautonomous equation of an order not exceeding N is

$$x_{n+1} - x_n = -\sum_{k=0}^{N} a_k(n) x_{n-k}. \tag{2.5}$$

A direct application of Theorem 2.1 shows that equation (2.5) is exponentially stable, if $a_0(n) \in (0, 2)$ and

$$|1 - a_0(n)| + \sum_{k=1}^{N} |a_k(n)| \le b < 1,$$

for some constant b and sufficiently large n. (Compare with [5].)

However, Theorem 2.1 does not work directly to obtain exponential stability results for equation (2.5) with $a_0(n) \le 0$. In particular, the case $a_0(n) = 0$ has attracted much attention (see some related references in [2]). To overcome this difficulty, we use some ideas from [2] and [14]. Our main result in this section improves Theorem 6 in [2] and Theorem 2.4 in [14].

For convenience in the presentation of our results, we consider the following nonlinear difference equation:

$$x_{n+1} - x_n = -\sum_{k=0}^{N} a_k(n)x_{n-k} + f(n, x_n, \ldots, x_{n-T}). \tag{2.6}$$

(Without loss of generality, we can suppose that $N \leq T$.) We assume that there exist constants $b_n \geq 0$ such that

$$|f(n, u_0, \ldots, u_T)| \leq b_n \max\{|u_0|, \ldots, |u_T|\}, \tag{2.7}$$

for all $n \geq 0$ and $(u_0, \ldots, u_T) \in \mathbb{R}^{T+1}$.

THEOREM 2.3 *Assume that, for large n, equation* (2.7) *holds and there exists a constant* $\gamma > 1$ *such that*

$$c_n := \left|1 - \sum_{k=0}^{N} a_k(n)\right| + \sum_{k=0}^{N} |a_k(n)| \sum_{m=n-k}^{n-1} \left(b_m + \sum_{k=0}^{N} |a_k(m)|\right) + b_n \leq \gamma. \tag{2.8}$$

Then the zero solution of equation (2.6) *is globally exponentially stable. Moreover, if inequalities* (2.7) *and* (2.8) *hold for* $n \geq 0$, *then*

$$|x_n| \leq \gamma^{n/(N+T+1)} \max\{|x_N|, \ldots, |x - T|\}, \quad n \geq N,$$

for every $\{x_n\}$ *solution of equation* (2.6).

Proof. In order to apply Theorem 2.1, for $n \geq N$, we rewrite equation (2.6) in the form

$$\begin{aligned}
x_{n+1} &= (1 - a_0(n))x_n - \sum_{k=1}^{N} a_k(n)x_{n-k} + f(n, x_n, \ldots, x_{n-T}) \\
&= \left(1 - \sum_{k=0}^{N} a_k(n)\right)x_n + \sum_{k=1}^{N} a_k(n)(x_n - x_{n-k}) + f(n, x_n, \ldots, x_{n-T}) \\
&= \left(1 - \sum_{k=0}^{N} a_k(n)\right)x_n + \sum_{k=1}^{N} a_k(n) \sum_{m=n-k}^{n-1} (x_{m+1} - x_m) + f(n, x_n, \ldots, x_{n-T}) \\
&= \left(1 - \sum_{k=0}^{N} a_k(n)\right)x_n + \sum_{k=1}^{N} a_k(n) \sum_{m=n-k}^{n-1} \left[-\sum_{k=0}^{N} a_k(m)x_{m-k} + f(m, x_m, \ldots, x_{m-T})\right] \\
&\quad + f(n, x_n, \ldots, x_{n-T}) := h(n, x_n, \ldots, x_{n-l}), \quad l = N + T.
\end{aligned}$$

Using inequalities (2.7) and (2.8), we have

$$|h(n, u_0, \ldots, u_l)| \leq c_n \max\{|u_0|, \ldots, |u_l|\} \leq \gamma \max\{|u_0|, \ldots, |u_l|\},$$

for large n and all $(u_0, \ldots, u_l) \in \mathbb{R}^{l+1}$. The result follows from Theorem 2.1. □

Next, we establish a corollary for linear equations.

Let $I \subset \{1, 2, \ldots, N\}$ be a set of indices and $J = \{1, 2, \ldots, N\} \setminus I$. We can rewrite equation (2.5) as

$$x_{n+1} - x_n = -a_0(n)x_n - \sum_{k \in I} a_k(n)x_{n-k} - \sum_{k \in J} a_k(n)x_{n-k}, \quad n \geq 0. \tag{2.9}$$

Thus, observing that

$$\left| \sum_{k \in J} a_k(n)x_{n-k} \right| \leq \left(\sum_{k \in J} |a_k(n)| \right) \max\{|x_n|, \ldots, |x_{n-N}|\},$$

we have the following consequence of Theorem 2.3:

COROLLARY 2.4 *Assume there exist a set of indices $I \subset \{1, 2, \ldots, N\}$ and a constant $\gamma < 1$ such that, for large n,*

$$\left| 1 - \sum_{k \in I_0} a_k(n) \right| + \sum_{k \in I} |a_k(n)| \sum_{m=n-k}^{n-1} \sum_{k=0}^{N} |a_k(m)| + \sum_{k \in J} |a_k(n)| \leq \gamma, \tag{2.10}$$

where $I_0 = \{0\} \cup I$. Then equation (2.5) *is exponentially stable.*

Remark 2.5 It is easy to check that Theorem 6 in [2] can be obtained from Corollary 2.4. As it was noticed there, we can get 2^N explicit conditions for the exponential stability of equation (2.5) by choosing different partitions (I, J) of $\{1, 2, \ldots, N\}$.

Next, taking $I = \{k : a_k > 0\} \subset \{1, 2, \ldots, N\}$ and $a_0 = 0$ in equation (1.1), Corollary 2.4 gives the stability condition

$$\sum_{k=1}^{N} |a_k| \sum_{k=1}^{N} k a_k^+ < \sum_{k=1}^{N} a_k,$$

for equation

$$x_{n+1} - x_n + \sum_{k=1}^{N} a_k x_{n-k} = 0, \quad n \geq 0. \tag{2.11}$$

Here, $a_k^+ = \max\{a_k, 0\}$. This is equivalent to Corollary 2.5 in [14].

For the nonlinear equation (2.1), Theorem 2.3 gives the following

COROLLARY 2.6 *Assume that inequality* (2.7) *holds and*

$$\limsup_{n \to \infty} b_n = b < 1, \tag{2.12}$$

then the zero solution of equation (2.1) *is globally exponentially stable.*

Remark 2.7 For the autonomous equation (2.4), the global asymptotic stability of the equilibrium is ensured in [20] under the weaker assumption

$$|f(u_0, \ldots, u_T)| < \max\{|u_0|, \ldots, |u_T|\} \tag{2.13}$$

for all $(u_0, \dots, u_T) \in \mathbb{R}^{T+1} \setminus \{(0, \dots, 0)\}$, instead of equation (2.3). Clearly, such a result does not apply to the nonautonomous case. For example, equation

$$x_{n+1} = f(n, x_n) := \frac{1 - e^{-n-1}}{1 - e^{-n-2}} x_n, \quad n = 0, 1, \dots$$

satisfies $|f(n,u)| < |u|$ for all $u \neq 0$. However, the solution starting at x_0 converges to $x_0(1 - e^{-1})$, so that the zero solution is not asymptotically stable.

Our next result is a nonautonomous version of the mentioned *weak contraction* result of Sedaghat. It ensures the global asymptotic stability of the equilibrium in some cases where inequality (2.12) fails.

THEOREM 2.8 *Assume that* $f : \mathbb{N} \times \mathbb{R}^{T+1} \to \mathbb{R}$ *satisfies inequality* (2.7) *for some constants* $b_n \leq 1$ *and for all* $(n, u_0, \dots, u_T) \in \mathbb{N} \times \mathbb{R}^{T+1}$. *Let*

$$B_k = \max\{b_{(T+1)k}, b_{(T+1)k+1}, \dots, b_{(T+1)k+T}\}. \tag{2.14}$$

If $\{x_n\}$ *is a solution of equation* (2.1)*, then*

$$|x_n| \leq M_0 \prod_{i=0}^{k} B_i, \tag{2.15}$$

for any integer n *such that* $(T+1)k + 1 \leq n \leq (T+1)(k+1)$, *where* $M_0 = \max_{-T \leq i \leq 0}\{|x_i|\}$.

Proof. Let us first prove that $|x_i| \leq B_0 M_0$, $i = 1, 2, \dots, T+1$. By inequality (2.7)

$$|x_1| = |f(0, x_0, \dots, x_{-T})| \leq b_0 M_0 \leq B_0 M_0 \leq M_0,$$

$$|x_2| = |f(1, x_1, \dots, x_{-T+1})| \leq b_1 \max_{-T+1 \leq j \leq 1} |x_j| \leq b_1 M_0 \leq B_0 M_0 \leq M_0.$$

Similarly, $|x_i| \leq b_{i-1} M_0 \leq B_0 M_0$, $i = 1, 2, \dots, T+1$, since the previous $T+1$ elements of the sequence do not exceed M_0.

Like in Theorem 2.1, let us prove inequality (2.15) by induction. As shown above, inequality (2.15) is satisfied for $k = 0$. Further, let us assume

$$|x_n| \leq M_0 \prod_{i=0}^{k} B_i, \quad (T+1)k + 1 \leq n \leq (T+1)(k+1).$$

Then

$$\begin{aligned} x_{(T+1)(k+1)+1} &\leq b_{(T+1)(k+1)} \max\{|x_{(T+1)(k+1)}|, \dots, |x_{(T+1)k+1}|\} \\ &\leq b_{(T+1)(k+1)} M_0 \prod_{i=0}^{k} B_i \leq M_0 \prod_{i=0}^{k} B_i. \end{aligned}$$

Similarly, for $(T+1)(k+1)+1 \le j \le (T+1)(k+2)$,

$$|x_j| \le b_{j-1} M_0 \prod_{i=0}^{k} B_i.$$

Thus

$$|x_j| \le b_{j-1} M_0 \prod_{i=0}^{k} B_i \le B_{k+1} M_0 \prod_{i=0}^{k} B_i = M_0 \prod_{i=0}^{k+1} B_i,$$

for $(T+1)(k+1)+1 \le j \le (T+1)(k+2)$. The reference to the induction principle completes the proof. □

Corollary 2.9 *Assume that inequality* (2.7) *holds and* $\lim_{k\to\infty} \prod_{i=0}^{k} B_i = 0$, *where* B_i *were defined in equation* (2.14). *Then the zero solution of equation* (2.1) *is globally asymptotically stable.*

Example 2.10 Let us consider the following nonautonomous linear equation

$$x_{n+1} = \frac{k(n)+1}{k(n)+2} \sum_{i=0}^{n-k(n)} \alpha_i(n) x_{n-i}, \quad \text{where} \quad k(n) = \left[\frac{n}{l}\right] l, \tag{2.16}$$

$l \ge 1$ is a fixed positive integer, $[y]$ is the greatest integer not exceeding y, and $\alpha_i(n) \ge 0$, $i = 0, \ldots, n-k(n)$, $\sum_{i=0}^{n-k(n)} \alpha_i(n) = 1$. Equation (2.16) can be written in the form (2.1) with $T = l - 1$, and

$$f(n, u_0, \ldots, u_{l-1}) = \frac{k(n)+1}{k(n)+2} \sum_{i=0}^{n-k(n)} \alpha_i(n) u_i, \quad k(n) = \left[\frac{n}{l}\right] l.$$

Thus inequality (2.7) holds with $b_n = (k(n)+1)/(k(n)+2)$, so that $\limsup_{n\to\infty} b_n = 1$ and Corollary 2.6 does not apply. Next, $B_i = (li+1)/(li+2)$, and $lim_{k\to\infty} \prod_{i=0}^{k} B_i = 0$. Hence, Corollary 2.9 ensures that equation (2.16) is asymptotically stable.

We notice that, in general, equation (2.16) is not exponentially stable. Indeed, take $\alpha_{n-k(n)} = 1$, $\alpha_i = 0$, $i \ne n - k(n)$. Then equation (2.16) reads

$$x_{n+1} = \frac{k(n)+1}{k(n)+2} x_{k(n)}, \quad k(n) = \left[\frac{n}{l}\right] l. \tag{2.17}$$

It is not difficult to check that, for an arbitrary $x_0 \in \mathbb{R}$, the solution of equation (2.17) is given by

$$x_n = \left(\prod_{i=0}^{k} B_i\right) x_0, \quad \text{for } lk+1 \le n \le (l+1)(k+1).$$

Since $\prod_{i=0}^{k} B_i \ge 1/(k+2) \ge 1/(n+2)$, it is clear that equation (2.17) cannot be exponentially stable.

While Corollary 2.6 is given in terms of the coefficients b_n in equation (2.7), Corollary 2.9 requires the computation of the new coefficients B_k given in equation (2.14). Thus, a natural question arises: *Does the conclusions of Corollary* 2.9 *remain valid if we replace B_i by b_i?* The answer to this question is negative, as the following example shows:

Example 2.11 Consider the following equation

$$x_{n+1} = \frac{3+(-1)^n}{4} x_{n-1}, \quad n = 0, 1, \ldots \tag{2.18}$$

It is clear that the zero solution of equation (2.18) is not asymptotically stable, since, for every initial data (x_{-1}, x_0), we have $x_{2k+1} = x_{-1}$ for all $k \geq 0$. Notice that inequality (2.7) holds with $b_n = (3+(-1)^n)/4$ and hence,

$$\lim_{k\to\infty} \prod_{i=0}^{k} b_i = \lim_{k\to\infty} 2^{-[(k+1)/2]} = 0.$$

On the other hand, $B_k = 1$ for all $k \geq 0$, and therefore $\lim_{k\to\infty} \prod_{i=0}^{k} B_i = 1 \neq 0$.

3. Monotonicity arguments

As it was noticed above, one of the interesting features of Theorem 2.3 is that it gives us the possibility to choose the most appropriate set of indices for each particular linear difference equation in order to ensure the exponential stability (see, e.g. [2, Example 3]). In this section, we show that the results from [17,19], based on monotonicity arguments, provide another way to obtain different stability conditions. We recall Theorem 1.4 in [17]

THEOREM 3.1 [17, *Theorem* 1.4] *Suppose that there exist $b > 0$ and $\mu \in (0,1)$ such that equation* (2.3) *holds and*

$$\mu + \sum_{k=1}^{N} a_k^{+} \mu^{-k} \leq 1 - a_0, \tag{3.1}$$

$$b < \sum_{k=0}^{N} a_k. \tag{3.2}$$

Then, the zero solution of equation

$$x_{n+1} - x_n = -\sum_{k=0}^{N} a_k x_{n-k} + f(n, x_n, \ldots, x_{n-T}) \tag{3.3}$$

is globally exponentially stable.

Remark 3.2 A version of Theorem 3.1 valid to obtain the global asymptotic stability of equation (2.6) (with variable coefficients in the linear part) can be found in [19]. (Notice that for a linear equation with constant coefficients the asymptotic stability is equivalent to the exponential stability).

Let us consider the linear equation with constant coefficients (1.1). Arguing as in Corollary 2.4, we have the following consequence of Theorem 3.1

COROLLARY 3.3 *Assume there exist a set of indices $I \subset \{1,2,\ldots,N\}$ and a constant $\mu \in (0,1)$ such that*

$$\mu + \sum_{k\in I} a_k^+ \mu^{-k} \leq 1 - a_0, \tag{3.4}$$

and

$$0 < \sum_{k\in J} |a_k| < \sum_{k\in I_0} a_k, \tag{3.5}$$

where $J = \{1,2,\ldots,N\} \setminus I$ and $I_0 = \{0\} \cup I$. Then equation (1.1) *is exponentially stable.*

Remark 3.4 If $\sum_{k\in J} |a_k| = 0$, that is, when $I = \{1,2,\ldots,N\}$, the conclusion of Corollary 3.3 still holds if $\sum_{k\in I_0} a_k > 0$ and $a_i < 0$ for some $i \in I$.

Corollary 3.3 complements Corollary 10 in [2], providing a new set of stability conditions for equation (1.1). We illustrate this fact by considering the equation with two delays

$$x_{n+1} - x_n = -a_0 x_n - a_1 x_{n-p} - a_2 x_{n-q}, \tag{3.6}$$

with $p > 0$, $q > 0$.

In order to simplify condition (3.1), we will use the following result, which follows easily from Proposition 3 in [11].

PROPOSITION 3.5 *Condition* (3.1) *holds for equation*

$$x_{n+1} - x_n = -a_0 x_n - a_k x_{n-k},$$

where $k \geq 1$ is an integer, if

$$\frac{-1}{k} \leq a_0 < 1 \quad \text{and} \quad 0 < a_k \leq (1 - a_0)^{k+1} \frac{k^k}{(k+1)^{k+1}}.$$

Using Proposition 3.5, and considering the different possibilities for the set I in Corollary 3.3, we have the following result.

COROLLARY 3.6 *Assume that $a_0 < 1$ and at least one of the following conditions holds:*

(a) $|a_1| + |a_2| < a_0$;
(b) $|a_1| < a_0 + a_2$, $qa_0 \geq -1$ *and* $0 < a_2 \leq (1 - a_0)^{q+1} q^q (q+1)^{-(q+1)}$;
(c) $|a_2| < a_0 + a_1$, $pa_0 \geq -1$ *and* $0 < a_1 \leq (1 - a_0)^{p+1} p^p (p+1)^{-(p+1)}$;
(d) $a_0 + a_1 + a_2 > 0$ *and there exists* $\mu \in (0,1)$ *such that*

$$\mu + a_1^+ \mu^{-p} + a_2^+ \mu^{-q} \leq 1 - a_0, \tag{3.7}$$

with $a_1 > 0$ or $a_2 > 0$.

Then equation (3.6) *is exponentially stable.*

Example 3.7 *Consider equation* (3.6) *with* $a_0 = a_1 = 1/2$, i.e.

$$x_{n+1} = \frac{x_n - x_{n-p}}{2} - a_2 x_{n-q}. \tag{3.8}$$

If $a_2 > 0$, Corollary 13 in [2] does not apply. However, an application of Corollary 3.6 (b) gives the exponential stability for small enough a_2, namely, for $a_2 < q^q(2+2q)^{-q-1}$.

Now, for the nonautonomous equation (2.6), we have the following result.

COROLLARY 3.8 *Suppose there exist constants* b, c_0, $c_1,\ldots,c_N$ *and* $\mu \in (0,1)$, *such that inequality* (2.3) *holds and*

$$\mu + \sum_{k=1}^{N} c_k^+ \mu^{-k} \le 1 - c_0, \tag{3.9}$$

$$\limsup_{n\to\infty} \sum_{k=0}^{N} |c_k - a_k(n)| + b < \sum_{k=0}^{N} c_k. \tag{3.10}$$

Then, the zero solution of equation (2.6) *is globally exponentially stable.*

Proof. Without loss of generality we may assume

$$\sum_{k=0}^{N} |c_k - a_k(n)| + b < \sum_{k=0}^{N} c_k, \quad n \ge 0$$

(otherwise, we shift the initial point). Equation (2.6) can be rewritten as

$$x_{n+1} - x_n = -\sum_{k=0}^{N} c_k x_{n-k} + \sum_{k=0}^{N} [c_k - a_k(n)] x_{n-k} + f(n, x_n, \ldots, x_{n-T}).$$

Applying Theorem 3.1, with f and b replaced, respectively, by

$$g(n, u_0, \ldots, u_T) = \sum_{k=0}^{N} [c_k - a_k(n)] u_k + f(n, u_0, \ldots, u_T),$$

$$B = \sup_{n\ge 0} \sum_{k=0}^{N} |c_k - a_k(n)| + b,$$

completes the proof. □

Corollaries 3.3 and 3.8 imply the following result for the linear equation (2.5):

COROLLARY 3.9 *Suppose there exist a set of indices* $I \subset \{1,2,\ldots,N\}$, *constants* $c_0,\ldots,c_N$, *and* $\mu \in (0,1)$, *such that*

$$\mu + \sum_{k\in I} c_k^+ \mu^{-k} \le 1 - c_0, \tag{3.11}$$

$$\limsup_{n\to\infty} \sum_{k=0}^{N} |c_k - a_k(n)| + \sum_{k\in J} |c_k| < \sum_{k\in I_0} c_k, \tag{3.12}$$

where $J = \{1,2,\ldots,N\} \setminus I$, *and* $I_0 = \{0\} \cup I$.

Then the zero solution of equation (2.5) *is globally exponentially stable*

.

Let us apply this result to the equation with two delays and nonconstant coefficients

$$x_{n+1} - x_n = -a_0 x_n - a_1(n) x_{n-p} - a_2(n) x_{n-q}. \tag{3.13}$$

Denote $c_i = (1/2)[\sup_n a_i(n) + \inf_n a_i(n)]$, $d_i = \sup_n |a_i(n) - c_i|$, $i = 1, 2$.

COROLLARY 3.10 *Assume that* $a_0 < 1$ *and at least one of the following conditions holds:*

(a) $|c_1| + |c_2| + |d_1| + |d_2| < a_0$;
(b) $|c_1| + |d_1| + |d_2| < a_0 + c_2$, $qa_0 \geq -1$ *and* $0 < c_2 \leq (1 - a_0)^{q+1} q^q (q + 1)^{-(q+1)}$;
(c) $|c_2| + |d_1| + |d_2| < a_0 + c_1$, $pa_0 \geq -1$ *and* $0 < c_1 \leq (1 - a_0)^{p+1} p^p (p + 1)^{-(p+1)}$;
(d) $|d_1| + |d_2| < a_0 + c_1 + c_2$ *and there exists* $\mu \in (0, 1)$ *such that*

$$\mu + c_1^+ \mu^{-p} + c_2^+ \mu^{-q} \leq 1 - a_0, \tag{3.14}$$

with $c_1 > 0$ or $c_2 > 0$.
Then equation (3.13) *is exponentially stable.*

Example 3.11 Consider the equation

$$x_{n+1} - x_n = \frac{1}{2} x_n - \left(\frac{1}{2} + b_1 \cos n\right) x_{n-p} - (a_2 + b_2 \sin n) x_{n-q}. \tag{3.15}$$

Let us apply Corollary 3.10(b). Here $c_1 = (1/2)$, $c_2 = a_2$, $d_1 = |b_1|$, $d_2 = |b_2|$. Hence, if

$$|b_1| + |b_2| < a_2 < \frac{q^q}{(2 + 2q)^{q+1}},$$

then equation (3.15) is exponentially stable. This result also cannot be obtained by the methods in [2,17].

4. An example in macroeconomics

In this section, we consider a generalization of the multiplier–accelerator model for the national income determination proposed by Samuelson [6, section 5.1]. Assuming that the autonomous investment is a constant G, the model is given by the second order difference equation

$$y_{n+1} - \beta(1 + k) y_n + \beta k y_{n-1} = G, \tag{4.1}$$

where $\beta \in (0, 1)$ is the propensity to consume, and k is the acceleration coefficient.

For a general discussion, historical remarks, and recent results on model (4.1) and other related multiplier–accelerator models, we refer the reader to [21, Section 5.1].

Equation (4.1) is a linear (nonhomogeneous) equation of the second order. It is well known that the unique equilibrium $y* = G/(1 - \beta)$ is asymptotically stable if and only if $\beta k < 1$.

In order to have a more realistic model, it is natural to assume that the investment depends on the national income in the previous years. Thus, we have a nonlinear equation

$$y_{n+1} - \beta(1+k)y_n + \beta k y_{n-1} = G(y_n, y_{n-1}). \tag{4.2}$$

We assume some hypotheses on function G defined in $\mathbb{R}^+ \times \mathbb{R}^+$. First, we suppose that there exists a unique equilibrium $y*$ of equation (4.2), which is the solution of the nonlinear equation $y* = G(y*, y*)/(1-\beta)$. We will also assume that G is differentiable and

$$R = \sup\left\{ \left|\frac{\partial G}{\partial x}(x,y)\right| + \left|\frac{\partial G}{\partial y}(x,y)\right| : x \geq 0, y \geq 0 \right\} < +\infty.$$

For example, if $G(x,y) = r\ln(p + \lambda(x+y))$ for some constants $r > 0$, $\lambda > 0$ and $p > 1$, then we have $R = 2r\lambda/p$.

The change of variables $x_n = y_n - y*$ transforms equation (4.2) into

$$x_{n+1} - \beta(1+k)x_n + \beta k x_{n-1} = F(x_n, x_{n-1}), \tag{4.3}$$

where $F(x,y) = G(x + y*, y + y*) - G(y*, y*)$. Obviously, the exponential stability of the equilibrium $y*$ of equation (4.2) is equivalent to the exponential stability of the zero solution for equation (4.3). Next, the mean value theorem gives the relation

$$|F(x,y)| \leq R \max\{|x|, |y|\},$$

whenever $x + y* \geq 0$, $y + y* \geq 0$. Thus, we can apply our results to get sufficient conditions to ensure that all solutions of equation (4.2) converge exponentially to the positive equilibrium. We have the following consequence of Theorems 2.1 and 3.1:

PROPOSITION 4.1 *Under the assumptions made for function G, the equilibrium $y*$ of equation* (4.2) *is exponentially stable if at least one of the following conditions holds:*

(A1) $0 \leq R < 1 - (2k+1)\beta, \quad \beta < 1/(2k+1)$.
(A2) $0 \leq R < 1 - \beta$ *and* $\beta \geq 4k/(1+k)^2$.

Proof. Assume that (A1) holds. Equation (4.3) can be written in the form (2.1) with $f(n, u_0, u_1) = \beta(1+k)u_0 - \beta k u_1 + F(u_0, u_1)$. Thus, condition (2.3) is fulfilled with $b = (2k+1)\beta + R < 1$. The result follows from Theorem 2.1. Next, assume that (A2) is satisfied. In this case, we write equation (4.3) as equation (3.3) with $a_0 = 1 - \beta(1+k)$, $a_1 = \beta k$ and $f(n, u_0, u_1) = F(u_0, u_1)$. By Proposition 3.5, condition (3.1) holds for some $\mu \in (0,1)$ if $0 < 1 - a_0 \leq 2$ and $(1-a_0)^2 \geq 4a_1$, i.e. for $1 > \beta \geq 4k/(1+k)^2$. Finally, condition (3.2) holds for $R < a_0 + a_1 = 1 - \beta$, and therefore the result follows from Theorem 3.1. □

Hence, we can conclude that the equilibrium is globally exponentially stable for sufficiently small R in the region of the plane of parameters (k,β) defined by the relations $1 > \beta \geq 4k/(1+k)^2$ and $0 < \beta < 1/(2k+1) < 1$. In figure 1, we represent such a region. The dashed line delimits the corresponding region for which the linear model (4.1) is exponentially stable.

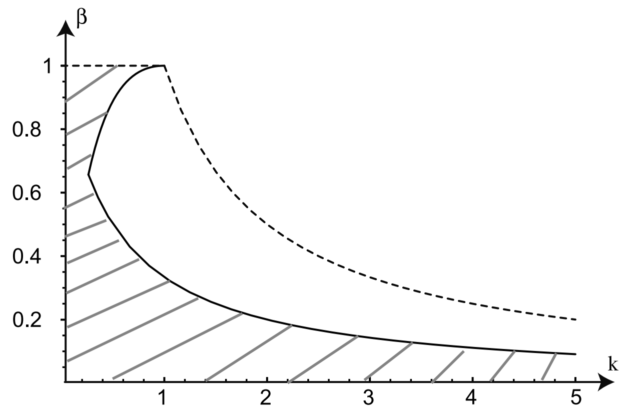

Figure 1. Domains of global exponential stability for equations (4.1) and (4.2) in coordinates (k,β).

Remark 4.2 Proposition 4.1 shows that the theoretical methods presented in Sections 2 and 3 are complementary. Indeed, the stability region for the nonlinear model (4.2) consists of two parts; for small β, the Halanay-type result Theorem 2.1 ensures the global stability for small R, whereas for β close to 1, such a result does not apply and Theorem 3.1, based on monotonicity arguments, is needed.

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References

[1] Berezansky, L. and Braverman, E., 2004, On Bohl-Perron type theorems for linear difference equations, *Functional Differential Equations*, **11**, 19–29.
[2] Berezansky, L. and Braverman, E., 2005, On exponential dichotomy, Bohl-Perron type theorems and stability of difference equations, *Journal of Mathematical Analysis and Applications*, **304**, 511–530.
[3] Dannan, F.M., 2004, The asymptotic stability of $x(n+k)+ax(n)+bx(n-l)=0$, *Journal of Difference Equations and Applications*, **10**, 589–599.
[4] Elaydi, S., 1996, *An Introduction to Difference Equations* (New York: Springer-Verlag).
[5] Elaydi, S., 1994, Periodicity and stability of linear Volterra difference systems, *Journal of Mathematical Analysis and Applications*, **181**, 483–492.
[6] Gandolfo, G., 1980, *Economic Dynamics: Methods and Models* (Amsterdam: North Holland).
[7] Győri, I. and Hartung, F., 2001, Stability in delay perturbed differential and difference equations, *Fields Institute Communication*, **20**, 181–194.
[8] Győri, I. and Ladas, G., 1991, *Oscillation Theory of Delay Differential Equations* (Oxford University Press).
[9] Kocić, V.L. and Ladas, G., 1993, Global behavior of nonlinear difference equations of higher order with applications. *Mathematics and its Applications* (Dordrecht: Kluwer Academic Publishers Group), Vol. 256.
[10] Krause, U., 2001, The asymptotic behavior of monotone difference equations of higher order, *Computers and Mathematics with Applications*, **42**, 647–654.
[11] Krause, U. and Pituk, M., 2004, Boundedness and stability for higher order difference equations, *Journal of Difference Equations and Applications*, **10**, 343–356.
[12] Kuruklis, S.A., 1994, The asymptotic stability of $x_{n+1}-ax_n+bx_{n-k}=0$, *Journal of Mathematical Analysis and Applications*, **188**, 719–731.
[13] Levin, S.A. and May, R., 1976, A note on difference delay equations, *Theoretical Population Biology*, **9**, 178–187.
[14] Liz, E., 2005, On explicit conditions for the asymptotic stability of linear higher order difference equations, *Journal of Mathematical Analysis and Applications*, **303**, 492–498.

[15] Liz, E. and Ferreiro, J.B., 2002, A note on the global stability of generalized difference equations, *Applied Mathematics Letters*, **15**, 655–659.

[16] Liz, E., Ivanov, A. and Ferreiro, J.B., 2003, Discrete Halanay-type inequalities and applications, *Nonlinear Analysis*, **55**, 669–678.

[17] Liz, E. and Pituk, M., 2005, Asymptotic estimates and exponential stability for higher order monotone difference equations, *Advances in Difference Equations*, 41–55.

[18] Ogita, R., Matsunaga, H. and Hara, T., 2000, Asymptotic stability condition for a class of linear delay difference equations of higher order, *Journal of Mathematical Analysis and Applications*, **248**, 83–96.

[19] Pituk, M., 2003, Global asymptotic stability in a perturbed higher-order linear difference equation, *Computers and Mathematics with Applications*, **45**, 1195–1202.

[20] Sedaghat, H., 1998, Geometric stability conditions for higher order difference equations, *Journal of Mathematical Analysis and Applications*, **224**, 255–272.

[21] Sedaghat, H., 2003, Nonlinear difference equations. Theory with applications to social science models. *Mathematical Modelling: Theory and Applications* (Dordrecht: Kluwer Academic Publishers), Vol. 15.

[22] Xiao, H. and Yang, X., 2003, Existence and stability of equilibrium points in higher order discrete systems, *Far East Journal of Dynamical Systems*, **5**, 141–147.