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ATTRACTIVITY PROPERTIES OF INFINITE DELAY MACKEY-GLASS TYPE EQUATIONS

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Abstract. In this paper, several sufficient conditions are established for the global stability of the positive steady state of a scalar functional differential equation $x' = -Lx_t + f(x_t),\ x \geq 0$ (1). The basic idea of the paper is to reduce an infinite dimensional system generated by (1) in some "friendly" spaces to the study of associated one-dimensional maps. In this way, we improve earlier results concerning not only the scalar Lasota-Wazewska and Mackey-Glass equations with infinite distributed delay but also the multidimensional Goodwin oscillator with infinite delay.

1. Introduction

In this paper, we study the attractivity properties of a unique positive equilibrium of functional differential equations like

$$x'(t) = -\delta x(t) + f(x_t),\ x \geq 0, \tag{1.1}$$

where f is a nonlinear functional defined on some functional space C_I^+ and $\delta > 0$. Taking $f(x_t) = f(x(t-h))$, we can obtain from (1.1), correspondingly, Lasota-Wazewska ($f(x) = p\exp(-\alpha x),\ p, \alpha > 0$), Nicholson's ($f(x) =$

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$px\exp(-\alpha x),\ p,\alpha > 0)$ and Mackey-Glass $(f(x) = \alpha x^m(1+x^n)^{-1},\ m = 0,1, n > 0, \alpha > 0)$ equations. Precisely these particular systems have motivated our study. The positivity of the variable x in (1.1) could be explained by multiple applications that these equations have in biological modeling (see also Section 4). The global attractivity in Eq. (1.1) with finite delay was widely studied in various works (see, e.g. [5, 7, 8, 15, 22] for references). Also, the last years were marked by important advances in studies of Lotka-Volterra systems with infinite delay which are dynamically similar to (1.1) (see [16]). However, the case of continuous delay distributed over an infinite interval in the mentioned Mackey-Glass type equations was considered only in a few works. In this paper, we propose to reduce the infinite dimensional system generated by Eq. (1.1) in some "friendly" space C_I^+ (allowing consideration of infinite delay) to the study of some associated one-dimensional maps. Moreover, our methods can be applied to some multidimensional differential equations, including the Goodwin oscillator [14, 22, 23]. As an application, we improve earlier results from [6, 14, 17, 21, 23].

2. Preliminaries

In the paper, we consider the Banach space $(C([-\tau,0]),\ |\bullet|_0)$ endowed with the supremum norm. It is customary to write C instead of $C([-\tau,0])$, and we will follow this tradition.

2.1. **Functional spaces.** In the case of the finite delay in (1.1), we will write $I = [-\tau, 0],\ C_I = C = C([-\tau,0])$ and

$$C_I^+ = C^+([-\tau,0]) = \{\phi \in C([-\tau,0]) \,:\, \phi \geq 0\}.$$

Otherwise, when delay is infinite, we set $I = (-\infty, 0]$ and $C_I = UC_g$, where UC_g stands for the Banach space $(UC_g,\ |\bullet|_g)$ of fading memory type (see the definition below and [2, 10, 13, 15]). The corresponding cone of nonnegative functions will be denoted as $C_I^+ = UC_g^+$.

Definition 2.1. Let

- **(g1)** $g : (-\infty,0] \to [1,\infty),\ g(0) = 1$ be a continuous nonincreasing function and
- **(g2)** $g(s+u)/g(s) \to 1$ uniformly on $(-\infty, 0]$ as $u \to 0^-$, and
- **(g3)** $g(s) \to +\infty$ as $s \to -\infty$.

Then UC_g denote the space of all continuous functions $\phi : (-\infty, 0] \to \mathbb{R}$ such that ϕ/g is bounded and uniformly continuous on $(-\infty, 0]$. This UC_g equipped with the norm $|\phi|_g = \sup_{s\leq 0}[|\phi(s)|/g(s)]$ is a Banach space.

It is well known (see [2, 10, 13, 15] and also [16, 23]) that standard uniqueness, continuation and continuous dependence theorems hold for Eq. (1.1) in the space UC_g. Moreover, the bounded solutions to Eq. (1.1) corresponding to initial values $\phi \in BC$ (that is, bounded and continuous) have precompact orbits in UC_g (e.g., see [15, Section 2.7]).

2.2. **Nonlinearity.** We will assume that the functional $f : C_I^+ \to \mathbb{R}^+$ is bounded continuous and locally Lipschitzian, so that the existence and uniqueness of solution $x(\phi)(t)$ for the initial value problem $x(s) = \phi(s),\ s \in I, \phi \in C_I^+$ associated to Eq. (1.1) is guaranteed. The relation $f^*(m) = f(m(t))$, where $m(t) \equiv m \in \mathbb{R}^+$, defines a continuous bounded map $f^* : \mathbb{R}^+ \to \mathbb{R}^+$. By abuse of notation, we shall use the same letter f for f^*. We will suppose that $f(+\infty) = 0$ and that the equation

$$x = \delta^{-1} f(x) \stackrel{def}{=} h(x)$$

has at most one positive solution x_2. This means that there exist at most two constant nonnegative solutions $x_1(t) \equiv 0$ and $x_2(t) \equiv x_2 > 0$ to (1.1). Moreover, we will consider only functionals f satisfying the following monotonicity property:

$$m = \inf_I \psi(t) \le \sup_I \psi(t) = M \text{ implies } \ f(\psi) \in f([m, M]). \tag{2.1}$$

Finally, we list several additional hypotheses on h which will be assumed only whenever this is explicitly indicated:

(H1) h is a strictly decreasing function.

(H2) $h^{-1}(0) = 0$, $\liminf_{x \to 0^+} \frac{h(x)}{x} > 1$ and $h : \mathbb{R}^+ \to \mathbb{R}^+$ has only one local extremum x^* (maximum).

(H3) $h(x) < x$ for every $x > 0$.

2.3. **Global attractivity.** Following [12, Section 3.4], we give the definition of global attractivity (which is stronger than used in the population dynamics, cf. with [21, 26]):

Definition 2.2. Let X be a metric space and $T = \mathbb{R}^+$ or $\mathbb{Z}^+$. The semi-dynamical system $\Phi^t x : T \times X \to X$ has a global attractor A if A is a maximal compact invariant set which attracts each bounded set $B \subset X$ (thus $\Phi^t x \to A$ as $t \to +\infty$ for every $x \in X$).

We will study the existence of a global attractor in Eq. (1.1) using some associated one-dimensional maps. The next proposition, which can be deduced from the Singer's results [3, 24], will be very important in the sequel.

Remember that the Schwarz derivative $(Sh)(x)$ of a real smooth function h at the point x is defined as

$$(Sh)(x) = h'''(x)(h'(x))^{-1} - (3/2)\left(h''(x)(h'(x))^{-1}\right)^2.$$

Proposition 2.3. *Assume that the function $h : [a,b] \to [a,b]$, $h \in C^3[a,b]$, is either strictly decreasing or it has only one critical point x^* (maximum) in $[a,b]$. If a unique fixed point $x_2 \in [a,b]$ of h is locally asymptotically stable and the Schwarzian derivative $(Sh)(x) < 0$ for all $x \neq x^*$, then x_2 is a global attractor of h.*

We shall use also the following statement (e.g., see [5] for the proof):

Proposition 2.4. *Let $h : [a,b] \to [a,b]$, $h \in C[a,b]$ be such that the equation $h^2(x) = x$ has a unique solution $x = x_2$. Then x_2 is a global attractor of the discrete dynamical system $x_{n+1} = h(x_n)$.*

As an immediate consequence of the previous proposition, we have the following

Corollary 2.5. *Let $h : [a,b] \to [a,b]$ be a continuous function such that $h^n(x) \to x_2$ as $n \to \infty$ for every $x \in [a,b]$. Then $h^n[a,b] \to x_2$ as $n \to \infty$.*

3. Attractivity.

In this section, for Eq. (1.1), we establish global stability and uniform persistence results. Here and subsequently, we will use the symbol $x(\phi)(t)$ to denote the solution of Eq. (1.1) satisfying the initial condition $x(\phi)(s) = \phi(s),\ s \in I,\ \phi \in C_I^+$. By definition, $(x_t(\psi))(s) = x(\psi)(t+s),\ s \in I$. Next, for simplicity of notation and up to Section 5, we will write $v(t)$ instead of $\exp(-\delta t)$. We begin with the following statement:

Lemma 3.1. *The map $F : \mathbb{R}^+ \times C_I^+ \to C_I^+,\ F^t\psi = x_t(\psi)$ defines a continuous semiflow. Next, for every bounded and continuous $\phi \in C_I^+$, the $\omega-$limit set $\omega(\phi)$ is nonempty, connected, invariant and compact. If $x(t) : \mathbb{R} \to \mathbb{R}^+$ is a bounded solution of* (1.1), *then*

$$x(t) = \int_{-\infty}^{t} v(t-s) f(x_s) ds. \tag{3.1}$$

Proof. Consider $I = (-\infty, 0]$, the case $I = [-\tau, 0]$ being analogous and simpler. As we have mentioned, under assumptions imposed on right-hand side of (1.1) and functional spaces, $F^t : C_I^+ \to C_I^+$ defines a local continuous semiflow. Let $x(\phi)(t),\ \phi \in C_I^+$ be a solution of (1.1) defined on the maximal interval of existence $[0,\kappa)$. By the variation of constants formula,

$$x(\phi)(t) = v(t)\phi(0) + \int_0^t v(t-s) f(x_s(\phi))ds, \ \ t \in [0,\kappa). \tag{3.2}$$

We have

$$0 \le \int_0^t v(t-s) f(x_s(\phi))ds \le \delta^{-1} \sup_{C_I^+} f,$$

and therefore $0 \le x(\phi)(t) \le \phi(0) + \delta^{-1}\sup f$ for all $t \in [0,\kappa)$. Now, since UC_g is an admissible space (see e.g. [9, Definition 2.1]), there exists $K > 0$ such that

$$|x_t(\phi)|_g \le K\Big(\sup_{s\in[0,\kappa)} |x(\phi)(s)| + |\phi|_g\Big), \, t \in [0,\kappa)\,.$$

Hence, since $|x_t(\phi)|_g$ is bounded over $[0,\kappa)$ and since the functional $-\delta\phi(0)+f(\phi)$ takes closed bounded sets of UC_g^+ into bounded sets, we can apply Continuation Theorem 2.4 from [13] to conclude that $\kappa = +\infty$. This means that $F^t\psi$ maps $\mathbb{R}^+ \times C_I^+$ into C_I^+. Moreover, each bounded orbit $\{F^t\phi,\ t \ge 0\}$, $\phi \in BC^+$ is precompact in C_I^+ (see Lemma 7.1, p. 47 from [15]). Therefore the $\omega-$limit set $\omega(\phi)$ of $\phi \in BC^+$ is nonempty, compact, connected and invariant (there is a complete orbit $\Gamma(\xi) \in \omega(\phi)$ through every $\xi \in \omega(\phi)$). Additionally, $\omega(\phi)$ attracts ϕ ([10, Theorem 3.1]).

Finally, the equality (3.1) is immediate since $\delta > 0$ and the scalar continuous function $f(x_t)$, $t \in \mathbb{R}$, is bounded. □

Remark 3.2. Suppose that x is a complete (that is defined over $\mathbb{R}$) bounded nonnegative solution to Eq. (1.1) and $x(s) = 0$ at some point $s \in \mathbb{R}$. Then the positivity of $v(t)$ and (3.1) imply that $f(x_u) = 0$ for all $u < s$. Hence, again by (3.1), $x(t) = 0$ for all $t < s$ that gives $f(0) = 0$ and, finally, $x(t) \equiv 0$ by the uniqueness theorem.

Using Lemma 3.1 we can easily prove the following key result:

Lemma 3.3. *If $x : \mathbb{R} \to \mathbb{R}^+,\ x \not\equiv 0$, is a complete bounded solution of Eq.* (1.1) *with $m = \inf_{t\in\mathbb{R}} x(t)$, $M = \sup_{t\in\mathbb{R}} x(t)$, then*

$$[m,M] \subset h([m,M]) \subset h^2([m,M]) \subset \cdots \subset h^j([m,M]) \subset \cdots \tag{3.3}$$

Proof. Since $m \le x(t) \le M$ for all $t \in \mathbb{R}$, it follows from (2.1) and (3.1) that

$$m = \inf_{t\in\mathbb{R}} x(t) \ge \int_{-\infty}^t v(t-s) \ (\min_{y\in[m,M]} f(y)) \ ds = \min_{y\in[m,M]} h(y);$$

$$M = \sup_{t\in\mathbb{R}} x(t) \le \int_{-\infty}^{t} v(t-s)\ (\max_{y\in[m,M]} f(y))\ ds = \max_{y\in[m,M]} h(y). \tag{3.4}$$

Hence, $[m, M] \subset h([m, M])$, that implies immediately (3.3). $\square$

Corollary 3.4. *Let (H3) hold. Then $\omega(\phi) = 0$ for every bounded and continuous $\phi \in C_I^+$.*

Proof. Let $M = \sup_{t\in\mathbb{R}} x(\beta)(t) \ne 0$, where $\beta \in \omega(\phi)$. Then (3.3) and (H3) imply that $M = 0$, a contradiction. $\square$

Let us remember the concept of uniform persistence:

Definition 3.5. Equation (1.1) is said to be uniformly persistent if there exists a positive number m such that

$$\liminf_{t\to\infty} x(\phi)(t) \ge m \tag{3.5}$$

for every solution $x(\phi)(t),\ \phi \in BC^+$, to (1.1) such that $x(\phi)(t) \not\equiv 0$ over $\mathbb{R}^+$.

It is immediate to check that Eq. (1.1) is uniformly persistent under hypothesis (H1). The following result shows that this is also true when (H2) holds.

Theorem 3.6.

a) *If (H2) holds and $I = [-\tau, 0]$, then Eq. (1.1) is uniformly persistent.*
b) *Let $I = (-\infty, 0]$ and let $\pi : UC_g^+ \to C^+([-\tilde{\tau}, 0])$ denote the restriction operator. If there are $\tilde{\tau} > 0$ and a continuous functional $\tilde{f} : C^+([-\tilde{\tau}, 0]) \to \mathbb{R}^+$ such that $f(\phi) \ge \tilde{f}(\pi(\phi))$ for every $\phi \in UC_g^+$, then* (3.5) *holds for every $x(\phi)(t) \not\equiv 0,\ \phi \in BC^+$ once the induced function $\tilde{f}^* : \mathbb{R}^+ \to \mathbb{R}^+,\ \tilde{f}^*(m) = \tilde{f}(m),\ m(t) \equiv m \in \mathbb{R}^+$ satisfies the assumption (H2).*

Proof. a) By Lemma 3.1, there exists $M > 0$ such that $M > \limsup_{t\to\infty} x(\phi)(t)$ for every initial function $\phi \in C_I^+$. Let $d, m > 0$ be such that

$$\eta = \inf_{z\in(0,m]} \frac{f(z)}{z} \int_0^d v(u)du > 1, \quad \min_{t\in[m,M]} f(t) = f(m).$$

We are going to show that, with this m, the inequality (3.5) is satisfied. Actually, since $x(\phi)(t) \not\equiv 0$ over positive semiaxis, then there is $t_0 > 0$ such that $x(\phi)(t) > 0$ for all $t \geq t_0$. Now, take $s > t_0$ and $m_1 = m_1(s) \leq m$ such that $0 < m_1(s) \leq \min_{s \leq t \leq s+d+\tau} x(\phi)(t)$. Then

$$x(\phi)(s+d+\tau) = v(d)x(\phi)(s+\tau) + \int_{s+\tau}^{s+d+\tau} v(s+d+\tau-u)f(x_u(\phi))du$$

$$\geq f(m_1)\int_{s+\tau}^{s+\tau+d} v(s+\tau+d-u)du \geq m_1\frac{f(m_1)}{m_1}\int_0^d v(u)du \geq \eta m_1(s) > m_1(s).$$

This implies that $x(\phi)(t) > m_1(s) > 0$ for all $t > s+d+\tau$. Indeed, otherwise there exists a point $s_1 > s+d+\tau$ such that $x(\phi)(s_1) = m_1(s)$ and hence there is an interval $[s_1-\tau-d, s_1]$ such that $x(\phi)$ reaches its minimum in that interval at s_1. This leads us to a contradiction using the previous argument. Moreover, repeating the precedent arguments, we find that $x(\phi)(t) \geq \eta m_1(s) > 0$ for all $t > s+d+\tau$.

Now, let $L = \liminf_{t\to\infty} x(\phi)(t)$. By the above discussion $L > 0$ and we can indicate $r > T$ such that $x(\phi)(t) \geq L(\Delta+1)(2\Delta)^{-1}$ for all $t > r$. The above inequalities imply immediately that, in the case $L < m$, we have the contradiction $x(\phi)(t) \geq L(\Delta+1)(2\Delta)^{-1}\Delta > L$ for all $t > r+d+\tau$.

b) When delay is infinite, we can also use the variation of parameters formula and employ the arguments used in the proof of (a). We need only to replace τ with $\tilde{\tau}$ and use the properties of $\tilde{f}^*$. More precisely, the unique difference lies in the chain of inequalities proving that $x(\phi)(s+d+\tau) > m_1(s)$, which should read:

$$\begin{aligned} x(\phi)(s+d+\tilde{\tau}) &= v(d)x(\phi)(s+\tilde{\tau}) + \int_{s+\tilde{\tau}}^{s+d+\tilde{\tau}} v(s+d+\tilde{\tau}-u)f(x_u(\phi))du \\ &\geq \int_{s+\tilde{\tau}}^{s+d+\tilde{\tau}} v(s+d+\tilde{\tau}-u)\tilde{f}(\pi(x_u(\phi)))du \\ &\geq \tilde{f}^*(m_1)\int_{s+\tilde{\tau}}^{s+\tilde{\tau}+d} v(s+\tilde{\tau}+d-u)du \geq m_1\frac{\tilde{f}^*(m_1)}{m_1}\int_0^d v(u)du \\ &\geq \eta m_1(s) > m_1(s). \end{aligned}$$

□

As an easy consequence of Lemmas 3.1 and 3.3 and the previous theorem, we get

Corollary 3.7. *Let (H2) hold, $I = [-\tau, 0]$ and $K = [\alpha_1, \beta_1] \ni x_2$ be an invariant globally attracting set of the map $h : [\mathbf{a}, \mathbf{b}] \to [\mathbf{a}, \mathbf{b}]$ defined for an arbitrary small $\mathbf{a} > 0$ and $\mathbf{b} = \max h(x)$. Then we can set $m = \alpha_1$ in* (3.5).

As a direct application of Corollary 2.5 and Lemma 3.3, we get

Theorem 3.8. *Let Eq.* (1.1) *be uniformly persistent. We suppose that the map*

$$h : [\mathbf{a}, \mathbf{b}] \to [\mathbf{a}, \mathbf{b}] \ni x_2, \quad h(x_2) = x_2, \tag{3.6}$$

is well defined for an arbitrary small $\mathbf{a} > 0$, $\mathbf{b} = \max h(x)$. *If* $h^n(x) \to x_2$ *as* $n \to \infty$ *for all* $x \in [\mathbf{a}, \mathbf{b}]$, *then* $\lim_{t\to+\infty} x(\phi)(t) = x_2$ *for any bounded and continuous* $\phi \in C_I^+$. *Furthermore, in the case of finite delay, the solution* $x(t) \equiv x_2$ *of Eq.* (1.1) *is globally attracting over the phase space* C^+.

Remark 3.9. The part of this theorem assuming uniform persistence and well-posedness of (3.6) is satisfied if the assumption (H1) (or (H2) when delay is finite) holds.

Combining Theorem 3.8 and Proposition 2.3, we obtain the following:

Corollary 3.10. *Let either (H1) or (H2) hold. If* $2f'(x)f'''(x) < 3(f''(x))^2$ *for all* $x > 0$, $x \neq x^*$ *and* $f'(x_2) > -\delta$, *then the equilibrium* x_2 *of* (1.1) *is a global attractor if delay is finite. In the general case,* $\lim_{t\to+\infty} x(\phi)(t) = x_2$ *for any* $\phi \in BC^+$ *such that* $x(\phi)(t) \not\equiv 0$ *over* $[0, +\infty)$.

In the case when delay is finite, we can find even sharper conditions for the global attractivity. We note again that, if either condition (H1) or (H2) is satisfied, then Eq. (1.1) is uniformly persistent. Moreover, in this case, for some $\mathbf{a}, \mathbf{b} > \mathbf{0} : \mathbf{a} \leq x_2 \leq \mathbf{b}$, the map

$$\zeta \overset{def}{=} \exp(-\delta\tau)x_2 + \frac{(1-\exp(-\delta\tau))}{\delta} f : [\mathbf{a}, \mathbf{b}] \to [\mathbf{a}, \mathbf{b}], \tag{3.7}$$

is well defined and also satisfies (H1) or it has only one critical point at which the global maximum is reached. Obviously, $\zeta(x_2) = x_2$.

Theorem 3.11. *Consider Eq.* (1.1) *with finite delay* τ. *Assume that either hypothesis* $(H1)$ *or* $(H2)$ *holds and the map* (3.7) *is well defined for arbitrary small* $\mathbf{a} > 0$ *and* $\mathbf{b} = \max \zeta(x)$. *If* $\zeta^n(x) \to x_2$ *as* $n \to \infty$ *for every* $x \in [\mathbf{a}, \mathbf{b}]$, *then the equilibrium* x_2 *of Eq.* (1.1) *is globally attracting.*

Proof. Let $x(t) = x(\phi)(t)$ be a nonzero trajectory of the uniformly persistent equation (1.1). The limit set $\omega(\phi)$ is a nonempty invariant and compact set. Hence, the values $m = \min\limits_{\beta\in\omega(\phi)} \inf\limits_{t\in\mathbb{R}} x(\beta)(t)$ and $M = \max\limits_{\beta\in\omega(\phi)} \sup\limits_{t\in\mathbb{R}} x(\beta)(t)$ are well defined.

We now establish the conditions guaranteeing the equality $m = M$. It is clear that these conditions are sufficient for the global attractivity of the steady state $x(t) \equiv x_2$.

Since $\omega(\phi)$ is a compact invariant set, there exist solutions $y(t)$, $z(t)$ to Eq. (1.1) such that $y(0) = M$ and $z(0) = m$.

Next we prove the following statement:

Claim: *If $m < M$, then there exist $s, s' \in [-\tau, 0]$ such that $y(s) = z(s') = x_2$.*

Proof. First, let us assume that (H1) holds and suppose the case $z(0) = m$. It follows easily from (3.3) that $m \leq x_2$. If $z(s) < x_2$ for all $s \in [-\tau, 0]$ then

$$\delta x_2 > \delta z(0) = f(z_0) \geq f(x_2) = \delta x_2,$$

a contradiction. The case $y(0) = M \geq x_2$ is analogous.

Now let us assume that (H2) holds. If $x_2 \leq x^*$, being x^* the unique local maximum of f, then it is easy to check that $h^n(x) \to x_2$ as $n \to \infty$, for all $x > 0$ and Theorem 3.8 ensures that x_2 is the global attractor of Eq. (1.1). Hence $m = M$ and therefore we have to consider only the case $x_2 > x^*$. If $y(0) > x_2$ or $z(0) > x_2$, we can argue as it was done under assumption (H1) since f is decreasing in the interval $[x_2, \infty)$. Let us suppose that $z(0) = m < x_2$ and $z(s) \in [m, x_2]$ for all $s \in [-\tau, 0]$. Using condition (2.1) we have that $m = h(z_0) \in h([m, x_2])$. A simple graphical analysis of h shows that this is impossible unless $m = x_2$. Finally, if $y(0) = M < x_2$, we can use Lemma 3.3 to obtain again that $m \in h([m, M]) \subset h([m, x_2])$. This concludes the proof of the claim. Now,

$$M = y(0) = \exp(\delta s)x_2 + \int_s^0 \exp(\delta u) f(y_u)du\,.$$

Taking into account that the function $w_c : [-\tau, 0] \to \mathbb{R}$ defined by $w_c(x) = e^{\delta x}x_2 + (1 - e^{\delta x})c$ is nonincreasing for $c \geq x_2$ and $x_2 = h(x_2) \leq \max_{x \in [m,M]} h(x)$, we have that

$$\begin{aligned} M &\leq \exp(\delta s)x_2 + (1 - \exp(\delta s)) \max_{x \in [m,M]} h(x) \\ &\leq \exp(-\delta\tau)x_2 + (1 - \exp(-\delta\tau)) \max_{x \in [m,M]} h(x)\,. \end{aligned} \tag{3.8}$$

Analogously,

$$m = z(0) = \exp(\delta s')x_2 + \int_{s'}^0 \exp(\delta u) f(z_u)du,$$

and we have (now we use that $w_c(x)$ is nondecreasing for $c \leq x_2$):

$$\begin{aligned} m &\geq \exp(\delta s')x_2 + (1 - \exp(\delta s')) \min_{x \in [m,M]} h(x) \\ &\geq \exp(-\delta\tau)x_2 + (1 - \exp(-\delta\tau)) \min_{x \in [m,M]} h(x). \end{aligned} \tag{3.9}$$

Finally, from the estimates (3.8), (3.9) we obtain that

$$[m, M] \subset \zeta([m, M]) \subset \zeta^2([m, M]) \subset \cdots \subset \zeta^j([m, M]) \subset \cdots$$

The statement of Theorem 3.11 is now an immediate consequence of these relations and Corollary 2.5 $\square$

Finally, we have the following application of Proposition 2.3 and Theorem 3.11:

Corollary 3.12. *Let (H1) or (H2) be satisfied and the interval* $I = [-\tau, 0]$ *be finite. If* $(1 - \exp(-\delta\tau))f'(x_2) > -\delta$ *and* $2f'(x)f'''(x) < 3(f''(x))^2$ *for* $x > 0$, $x \neq x^*$, *then the equilibrium* x_2 *of Eq.* (1.1) *is a global attractor.*

4. Applications

In this section we apply our results to study several physiological and biochemical models. In particular we consider the model for the survival of red blood cells in an animal used by Lasota and Wazewska [27], the model of haematopoiesis (blood cell production) proposed by Mackey and Glass [19, 20] and the model for biochemical reaction sequences with end product inhibition known as Goodwin oscillator [18]. We emphasize that in each case we improve some previous results.

4.1. **Lasota-Wazewska model with infinite delay.** First, in the cone UC_g^+ we consider the system with infinite delay:

$$x'(t) = -\delta x(t) + \int_0^\infty e^{-\gamma(s)x(t-s)} dq(s), \tag{4.1}$$

where $q : [0, +\infty) \to (0, +\infty)$ is a nonconstant nondecreasing function, $\int_0^\infty dq(s) = Q > 0$ and $\gamma : [0, \infty) \to (0, \infty)$ is a bounded continuous function with $\gamma_0 = \sup_{t \in \mathbb{R}^+} \gamma(t)$. We suppose that $g : (-\infty, 0] \to [1, +\infty)$ satisfies the hypotheses (g1-g3) from Definition 2.1 as well as the inequality $\int_0^\infty g(-s)\gamma(s)dq(s) < \infty$. Such g exists according to [2, Theorem 3.1]. It is easy to verify that the functional

$$f(\phi) = \int_0^\infty e^{-\gamma(s)\phi(-s)} dq(s)$$

is well defined and Lipschitzian over such UC_g^+ and satisfies there (H1) together with (2.1). Finally, applying the results of the previous section, we get the following propositions:

Theorem 4.1. *Suppose that*

$$\int_0^\infty \gamma(s)e^{-\gamma(s)x_2}dq(s) < \delta \tag{4.2}$$

where

$$\delta x_2 = \int_0^\infty e^{-\gamma(s)x_2}dq(s)\,. \tag{4.3}$$

If, for all $x \in (0, \delta^{-1}\int_0^\infty dq(s)]$, *we have*

$$\int_0^\infty \gamma^3(s)e^{-\gamma(s)x}dq(s)\int_0^\infty \gamma(s)e^{-\gamma(s)x}dq(s) < \frac{3}{2}\Big(\int_0^\infty \gamma^2(s)e^{-\gamma(s)x}dq(s)\Big)^2, \tag{4.4}$$

then $x(\phi)(t) \to x_2$ *for every solution of* (4.1) *with initial value* $\phi \in BC^+$.

Proof. By Lemma 3.1, (4.1) defines a continuous semigroup on the phase space UC_g^+ and every trajectory with initial value $\phi \in BC^+$ has a compact invariant ω-limit set. Moreover, in this case the strictly decreasing function

$$h(x) = \frac{1}{\delta}\int_0^\infty e^{-\gamma(s)x}dq(s) : [0, +\infty) \to (0, +\infty)$$

is well defined and has only one nonnegative fixed point x_2. Thus the conditions of Corollary 3.10 are satisfied if the inequalities (4.2) and (4.4) are verified. □

We can replace (4.2) by a condition which does not depend on x_2.

Corollary 4.2. *The conclusion of Theorem* 4.1 *remains valid if we replace* (4.2) *by the inequality*

$$\int_0^\infty \gamma(s)e^{-\gamma(s)/\gamma_0}dq(s) < \delta. \tag{4.5}$$

Proof. Let us prove that (4.5) implies (4.2). Indeed, for each $\delta > 0$ let us denote by $x_2(\delta)$ the unique positive solution of Eq. (4.3) and by $z(\delta)$ the unique positive root of the equation

$$\delta z = \int_0^\infty \frac{\gamma(s)}{\gamma_0}e^{-\gamma(s)z}dq(s)\,.$$

It is immediate to check that $z(\delta) \le x_2(\delta)$ for all $\delta > 0$, z is a decreasing function of δ and $z(\delta^*) = \gamma_0^{-1}$ for $\delta^* = \int_0^\infty \gamma(s)e^{-\gamma(s)/\gamma_0}dq(s)$.

Hence, for all $\delta > \delta^*$, we have

$$\int_0^\infty \gamma(s)e^{-\gamma(s)x_2(\delta)}dq(s) \le \int_0^\infty \gamma(s)e^{-\gamma(\tau)z(\delta)}dq(s) = \delta\gamma_0 z(\delta) < \delta.$$

Remark 4.3. We also note that if $\gamma(s)$ is close to some positive constant so that $\sup_{s\geq 0}\gamma(s) < \frac{3}{2}\inf_{s\geq 0}\gamma(s)$, then (4.4) holds in an evident way.

From Theorem 3.8 we deduce immediately the following result

Theorem 4.4. *The conclusion of Theorem* 4.1 *is valid when* $h^2(x) = x$ *if and only if* $x = x_2$.

Now, for all positive x, setting $H(x) = h^2(x)$, we have

$$\begin{aligned} H'(x) &= \Big(\frac{1}{\delta}\int_0^\infty \gamma(s)e^{-\gamma(s)h(x)}dq(s)\Big)\Big(\frac{1}{\delta}\int_0^\infty \gamma(s)e^{-\gamma(s)x}dq(s)\Big) \\ &\leq \frac{\gamma_0}{\delta}\int_0^\infty \gamma(s)h(x)e^{-\gamma(s)h(x)}dq(s) \leq \frac{\gamma_0}{e\delta}\int_0^\infty dq(s). \end{aligned}$$

Actually, it is not difficult to prove that the strict inequality $H'(x) < \frac{\gamma_0}{e\delta}\int_0^\infty dq(s)$ holds excepting at most one point. Thus, since $H = h^2$ is increasing, we obtain the following result as a consequence of Theorem 4.4:

Corollary 4.5. *Let*

$$\delta e \geq \gamma_0 \int_0^\infty dq(s). \tag{4.6}$$

Then $x(\phi)(t) \to x_2$ *for every solution of* (4.1) *with initial value* $\phi \in BC^+$.

Remark 4.6. It should be noted that (4.6) implies (4.5). However, in Theorem 4.1 we have the additional restriction (4.4). In the case $\gamma_0 \equiv \gamma(s)$ theorems 4.1 and 4.4 give about the same result.

4.2. **Lasota-Wazewska model with finite delay.** Considering the equation

$$x'(t) = -\delta x(t) + \int_0^\tau e^{-\gamma(s)x(t-s)}dq(s),\ x \geq 0, \tag{4.7}$$

with finite $\tau > 0$, we can obtain new results applying Theorem 3.11. (Here, γ and q are functions defined on $[0,\tau]$ which have the properties indicated in the previous subsection). The unique positive equilibrium x_2 of this equation is determined from the equation

$$x_2 = \frac{1}{\delta}\int_0^\tau e^{-\gamma(s)x_2}dq(s) = h(x_2).$$

Consider the function

$$\zeta(x) = e^{-\delta\tau}x_2 + \Big(1 - e^{-\delta\tau}\Big)\frac{1}{\delta}\int_0^\tau e^{-\gamma(s)x}dq(s),$$

mapping the interval $D = [0, e^{-\delta\tau}x_2 + \frac{(1-e^{-\delta\tau})}{\delta}\int_0^\tau dq(s)]$ into itself. Since (H1) holds, we have the following easy consequence of Theorem 3.11:

Theorem 4.7. *Let $x = x_2$ be the global attractor of the map $\zeta : D \to D$. Then the equilibrium x_2 of Eq.* (4.7) *is global attractor. The same is true when ζ^2 has the unique fixed point $x = x_2$.*

Corollary 4.8. *Let $\gamma(s) \equiv \gamma > 0$. Then $\gamma(1 - e^{-\delta\tau})x_2 < 1$ is sufficient to guarantee that x_2 is the global attractor for Eq.* (4.7).

Proof. Indeed, since ζ has negative Schwarzian derivative, we obtain from Corollary 3.12 that the inequality

$$\gamma(1 - \exp(-\delta\tau)) \int_0^\tau dq(s) \exp(-\gamma x_2) = \delta\gamma(1 - e^{-\delta\tau})x_2 < \delta$$

is sufficient for the global attractivity of x_2. □

Remark 4.9. This condition is sharper than $\gamma(1-e^{-\delta\tau})x_2 < \ln 2$ of Kulenovic, Ladas and Sficas (see [17, Theorem 1]).

4.3. **Mackey-Glass model.** Let us consider the following generalization of the Mackey-Glass equation proposed in [6]:

$$x'(t) = -\delta x(t) + \alpha \int_0^\infty \frac{dq(s)}{1 + x^n(t-s)},\ t > 0,\ x \geq 0, \tag{4.8}$$

with $\alpha, \delta > 0$ and $n \in (0, +\infty)$.

Again we can choose $g : (-\infty, 0] \to [1, +\infty)$ which satisfies the hypotheses (g1-g3) indicated in Definition 2.1 as well as the inequality $\int_0^\infty g(-s)dq(s) < \infty$. Then the functional

$$f(\phi) = \int_0^\infty \frac{dq(s)}{1 + \phi^n(-s)}$$

is Lipschitzian over the corresponding UC_g^+. Moreover, assuming $\int_0^\infty dq(s) = 1$, we obtain that $h(x) = \alpha(\delta(1 + x^n))^{-1}$ has a unique positive fixed point which we will denote as x_2. When $n > 1$, it is easy to check [5, Lemma 1] that h has negative Schwarzian derivative.

Theorem 4.10. *Suppose that either $n \in (0, 1]$ or $\alpha(n-1)^{(n+1)/n} < \delta n$ holds. Then $x(\phi)(t) \to x_2$ for every solution of* (4.8) *with initial value $\phi \in BC^+$.*

Proof. The case $n \in (0, 1]$ is a straightforward consequence of Theorem 3.8 and [5, Lemma 2]. Next, assume that $n > 1$. Since h has negative Schwarzian derivative, we need only to check that $h'(x_2) > -1$ and then to apply Corollary 3.10.

Let us observe that $h(x) = x$ if and only if $x^{n+1} + x = \alpha/\delta$. Then it is easy to prove that the solution $x_2 > 0$ of $h(x) = x$ satisfies $h'(x_2) > -1$ if and only if

$$x_2 > \frac{\alpha(n-1)}{\delta n} \stackrel{def}{=} x_1 \,. \tag{4.9}$$

Now, let us write $F(x) = x^{n+1} + x - \alpha/\delta$. We have that $F(0) < 0$ and x_2 is the unique positive solution of $F(x) = 0$. Thus, if $F(x_1) < 0$ it follows immediately (4.9) and the proof is done.

Finally, $F(x_1) < 0$ if and only if $\alpha(n-1)^{(n+1)/n} < \delta n$. □

Remark 4.11. In [6], the asymptotical stability in (4.8) was proved (by means of Lyapunov functionals) under considerably more restrictive conditions: $\alpha < \delta n^{-1}$, $n > 1$ and $\int_0^\infty s dq(s) < \infty$. Moreover, notice the assumptions of Theorem 4.10 are the best possible delay independent global stability conditions for Eq. (4.8). Indeed, for a given $\tau > 0$, take $q(s) = 0$ for $s \le \tau$ and $q(s) = 1$ when $s > \tau$. For that $q(s)$, Eq. (4.8) linearized along x_2 has the form $x'(t) = -\delta x(t) + \delta h'(x_2)x(t-\tau)$. Now, it is a well known fact that the exact delay independent asymptotic stability condition for this equation with negative $h'(x_2)$ has the form $h'(x_2) \ge -1$. In the other words, $h'(x_2) < -1$ implies local instability of the steady state x_2 for some values of τ. Finally, a slight modification of the proof of Theorem 4.10 shows that its statement is still valid if we replace $< \delta n$ with $\le \delta n$.

4.4. **Goodwin oscillator.** In this section we shall give new conditions sufficient for the asymptotical stability in the following Goodwin oscillator with infinite delay (see, for example, [18] and [22, Chapter 6]):

$$\begin{aligned} x_1'(t) + a_1 x_1(t) &= b_1\Big[1 + \Big(\int_0^\infty k(s)x_n(t-s)ds\Big)^\mu\Big]^{-1} \\ x_i'(t) + a_i x_i(t) &= b_i \int_0^\infty k(s)x_{i-1}(t-s)ds \,,\ i = 2,3,\dots n, \end{aligned} \tag{4.10}$$

where $a_i, b_i > 0$, $x_i \ge 0$ for $i = 1,2,\dots,n$, $k \in L^1(\mathbb{R}^+) \cap C(\mathbb{R}^+)$, $k \ge 0$ and $\mu \in \mathbb{N}$. We will also write $k_1 = \|k\|_1$.

In particular, we improve some results of [14, 23] investigating the asymptotic behavior of solutions to the initial value problem $x_i(s) = \phi_i(s)$, $s \in (-\infty, 0)$, $\phi_i \in BC^+ \subset UC_g^+$, $i = 1,\dots,n$, where $\int_0^\infty g(-s)k(s)ds < \infty$.

It is easy to see that the unique positive equilibrium $(z_1, z_2, \dots, z_n)$ of (4.10) is determined by

$$h(z_n) = z_n \quad ; \quad z_{i-1} = k_1^{-1}\frac{a_i}{b_i} z_i \,,\ i = 2,3,\dots n,$$

being

$$h(x) = k_1^{n-1}(1 + (k_1 x)^\mu)^{-1} \prod_{i=1}^{n} \frac{b_i}{a_i} \tag{4.11}$$

Let us observe that the functional defined by the right hand side of (4.10), satisfies a global Lipschitz condition on $UC_{g,n}^+ = UC_g^+(\mathbb{R}^n)$ and Eq. (4.10) defines a continuous semiflow in this phase space. Moreover, the first equation in (4.10) has properties similar to Eq. (4.8) and hence it is not difficult to prove that the component $x_1(t)$ of each solution $x(t) = (x_1(t), \dots, x_n(t))$ with the initial condition $x_0 \in BC^+(\mathbb{R}^n)$ is bounded over $\mathbb{R}$. Then we can proceed recursively and prove that the remainder components $x_2(t), \dots,$ $x_n(t)$ are also bounded on $\mathbb{R}$. Thus we can conclude that the $\omega-$limit set $\omega(\phi)$ for every $\phi \in BC^+(\mathbb{R}^n)$ is nonempty and compact (see [15]). Moreover, if $x(t) = (x_1(t), \dots, x_n(t)) : \mathbb{R} \to \mathbb{R}_+^n$ is a complete bounded solution, we can get the following integral representations (see Lemma 3.1):

$$x_1(t) = b_1 \int_{-\infty}^{t} e^{-a_1(t-s)} f(x_{ns}) ds \tag{4.12}$$

$$x_i(t) = b_i \int_{-\infty}^{t} e^{-a_i(t-s)} (\int_0^\infty k(u) x_{i-1}(s-u) du) ds \, , \; i = 2, \dots n, \tag{4.13}$$

where f is a nonlinear scalar functional defined by (4.10). As before, it determines a strictly decreasing function $f = f^* : \mathbb{R}^+ \to \mathbb{R}^+$ in the way indicated in the introductory section.

From the integral equation (4.12) it follows that

$$\sup_{t\in\mathbb{R}} x_1(t) \le \frac{b_1}{a_1} f(0) = M_1 \quad ; \quad \inf_{t\in\mathbb{R}} x_1(t) \ge \frac{b_1}{a_1} f(M_1) = m_1$$

Next, using equations (4.13), we obtain for each $i = 2, \dots, n$:

$$\sup_{t\in\mathbb{R}} x_i(t) \le k_1 \frac{b_i}{a_i} \sup_{t\in\mathbb{R}} x_{i-1}(t) = M_i \quad ; \quad \inf_{t\in\mathbb{R}} x_i(t) \ge k_1 \frac{b_i}{a_i} \inf_{t\in\mathbb{R}} x_{i-1}(t) = m_i$$

Hence system (4.10) is uniformly persistent.

Using again (4.12) and (4.13), we can easily prove that

$$[m_1, M_1] \subset \frac{b_1}{a_1} f([m_n, M_n]), \quad [m_i, M_i] \subset k_1 \frac{b_i}{a_i} [m_{i-1}, M_{i-1}] \, , \; i = 2, \dots, n.$$

Therefore,

$$[m_n, M_n] \subset k_1^{n-1} \frac{b_1 \dots b_n}{a_1 \dots a_n} f([m_n, M_n]) = h([m_n, M_n])$$

and we can conclude that

$$[m_n, M_n] \subset h([m_n, M_n]) \subset h^2([m_n, M_n]) \subset \cdots$$

As a consequence, if the unique fixed point z_n of h is a global attractor, then $m_n = M_n$, from where $m_i = M_i$ for each $i = 1, \dots, n-1$ and system (4.10) has an asymptotically stable equilibrium $(z_1, \dots, z_n)$.

Next, applying Corollary 3.10 as it was done in Theorem 4.10 we obtain the following:

Theorem 4.12. *Suppose that*

$$\prod_{i=1}^{n} \frac{a_i}{b_i} > k_1^n \frac{\mu - 1}{\mu} (\mu - 1)^{1/\mu} \,. \tag{4.14}$$

Then $x_i(\phi)(t) \to z_i,\ i = 1, ..., n$ *for every solution of* (4.10) *with initial values* $\phi_i \in BC^+,\ i = 1, ..., n.$

Now we compare our estimation (4.14) with those obtained in [14, 23].

First, in [14, Section 5, pag. 229] the authors prove, using linearization, the (local) exponential stability of the equilibrium of system (4.10) with finite delay under exactly the same condition (4.14). Thus we improve their result since we consider infinite delay (also proving the global attractivity in the case of finite delay).

On the other hand, Ruess and Summers consider system (4.10) and give the following condition for the asymptotical stability of the equilibrium in the case $n = 2$ (see [23, Example 4.3, pp. 1248 − 1249]): $M < \alpha$ where $\alpha = \min\{a_1, a_2\}$ and $M = k_1((b_1 c(\mu))^2 + b_2^2)^{1/2}$, being $c(1) = 1$ and

$$c(\mu) = \frac{(\mu+1)^2}{4\mu} \Big(\frac{\mu - 1}{\mu + 1} \Big)^{(\mu-1)/\mu}, \ \mu > 1.$$

Next we prove that our condition is better than the previous one for all $\mu \geq 1$. It is clear that $\alpha^2 \leq a_1 a_2$ and then estimation $M^2 < a_1 a_2$ is better than $M < \alpha$ in [23]. But $M^2 < a_1 a_2$ if and only if

$$k_1^2((b_1 c(\mu))^2 + b_2^2) < a_1 a_2.$$

Therefore, our condition is sharper than Ruess-Summers condition if

$$b_1^2 c^2(\mu) + b_2^2 > b_1 b_2 \frac{\mu - 1}{\mu} (\mu - 1)^{1/\mu},$$

or, equivalently,

$$c^2(\mu) - \frac{1}{4} \Big[\frac{\mu - 1}{\mu} (\mu - 1)^{1/\mu} \Big]^2 > 0.$$

Finally, it is easy to check that this inequality holds whenever $\mu \geq 1$.

Remark 4.13. Finally we note that, regardless of the fact that all considered models (Mackey-Glass, Lasota-Wazewska, Goodwin) led us to the study of one-dimensional maps h satisfying (H1), the application of our main results is similar when h is an unimodal map (see hypothesis (H2)). It should be noted that it is not the case for delay-differential equations with variable (e.g. almost periodic) coefficients [4].

5. A generalization

In the last section of the paper, we will indicate how our results about the convergence of all solutions to some *positive* equilibrium can be generalized to the functional differential equations

$$x'(t) = -Lx_t + f(x_t),\ x \in \mathbb{R}, \tag{5.1}$$

where $Lx_t = \int_0^\tau x(t-s)dp(s)$ for some nonconstant nondecreasing function p with $p(0) = 0$. In this case, the cone C_I^+ is not semiflow invariant, so that the nonlinearity f should be considered on the whole space C_I. We will suppose that f satisfies there all general properties indicated in Section 2.2. Moreover, since we will be interested in eventually positive solutions to (5.1), we will suppose that the restriction of f on C_I^+ also satisfies the assumptions formulated in Section 2.2. In particular, this means that $h(\mathbb{R}^+) \subset \mathbb{R}^+$. To simplify the exposition, we will suppose that the hypothesis **(H1)** is verified.

To have the eventual positivity of all solutions, we need some special positivity properties of the linear part to Eq. (5.1). We discuss briefly this aspect of the problem in the following subsection of paper.

5.1. **Linear equation.** Throughout the section, we will suppose that the equation

$$x'(t) = -Lx_t \tag{5.2}$$

has a nonnegative fundamental solution $v : [-\tau, \infty) \to \mathbb{R}^+$, $v(t) = 0$, $t \in [-\tau, 0)$, $v(0) = 1$ and $v \in L^1([0, +\infty))$.

Lemma 5.1. *Let $\delta = p(\tau) > 0$. Then $\delta \int_0^{+\infty} v(t)dt = 1$ and $v(t) > 0$ for all $t \geq 0$.*

Proof. Since $Lv_t = \int_0^\tau v(t-s)dp(s)$ for nondecreasing $p(s)$, we have

$$v(a) - 1 = -\int_0^a Lv_t dt = -\int_0^a \int_0^\tau v(t-s)dp(s)dt$$

$$= -\int_0^\tau \int_0^a v(t-s)dtdp(s) = -\int_0^\tau \int_0^{a-s} v(t)dtdp(s)$$
$$= -\delta \int_0^a v(t)dt + \int_0^\tau \int_{a-s}^a v(t)dtdp(s). \tag{5.3}$$

Now, since $\lim_{a\to+\infty} v(a) = 0$, we have

$$0 \le \left|\int_0^\tau \int_{a-s}^a v(t)dtdp(s)\right| \le p(\tau)\int_{a-\tau}^a v(t)dt \to 0$$

as $a \to +\infty$. Finally, $-1 = -\delta \int_0^\infty v(t)dt$ and the first part of Lemma 5.1 is proved.

Next, if $v(a) = 0$ at some point $a > 0$, we have that $v(t) = 0$ for $t \ge a$ and $\delta \int_0^a v(t)dt = 1$. Hence (5.3) implies that

$$\int_0^\tau \int_{a-s}^a v(t)dtdp(s) = 0\,. \tag{5.4}$$

Since $\int_{a-s}^a v(t)dt > 0$ for all $s \in (0, a]$, by the mean value theorem, (5.4) holds only if $Lx_t = \delta x(t)$. Therefore $v(a) = \exp(-\delta a) > 0$, a contradiction. □

Notice that the characteristic equation corresponding to (5.2) is given by

$$\Delta(\lambda) = \lambda + \int_0^\tau \exp(-\lambda s)dp(s) = 0. \tag{5.5}$$

Obviously, the characteristic function $\Delta(\lambda)$ does not have nonnegative real roots, and it is an easy exercise to generalize the proof of the above lemma to get the equality

$$\int_0^{+\infty} \exp(-\lambda t)v(t)dt = 1/\Delta(\lambda),\ \lambda \ge 0, \tag{5.6}$$

establishing the relation between the Laplace transform $\tilde{v}(\lambda)$ of the fundamental solution and the characteristic function to (5.2) (compare with [11, p.19]).

On the other hand, Eq. (5.2) has at least one real eigenvalue as a consequence of the positivity property of $v(t)$, we will prove this fact following an idea from [1]:

Lemma 5.2. $\Delta(\lambda) > 0$ *for all* $\lambda \ge 0$ *and* $\Delta(\mu) = 0$ *for some* $\mu < 0$.

Sketch of proof. (see [1] for more details) Lemma is trivial when $Lx_t = \delta x(t)$, therefore we can assume that $\tau > 0$ and that $p(s)$ is not constant in some subinterval $[n, m] \subset (0, \tau]$. Suppose now, contrary to our claim, that $\Delta(\lambda) > 0$ for all $\lambda \in \mathbb{R}$. Then, since $\Delta(\lambda)$ is complex analytic on the whole real axis, the Laplace transform $\tilde{v}(\lambda)$ (analytic over the semi-plane $\{\Re\lambda > 0\}$) has an analytic continuation on $\mathbb{R} \subset \mathbb{C}$. This implies, via the multiple applications of Beppo-Levi's theorem (see [1, p.102]), that the formula (5.6) also holds for all real λ. Moreover, for $\lambda < 0$, we have

$$\Delta(\lambda) \geq \lambda + \int_n^m \exp(-\lambda s) dp(s) \geq \lambda + \exp(-\lambda n)(p(m) - p(n)),$$

and therefore $0 < \tilde{v}(0) \leq \lim_{\lambda \to -\infty} \tilde{v}(\lambda) = 1/\lim_{\lambda \to -\infty} \Delta(\lambda) = 0$, a contradiction. $\square$

Take now the biggest $\mu < 0$ such that $\Delta(\mu) = 0$ and consider the closed cone

$$K_\mu = \{\phi \in C : \phi \geq 0 \text{ and } \phi(s)\exp(-\mu s) \text{ is nondecreasing on } [-\tau, 0]\}. \tag{5.7}$$

Corollary 5.3. *The linear semigroup $T(t) : C \to C$ generated by Eq.* (5.2) *is asymptotically stable and the cone K_μ is $T(t)$- invariant.*

Proof. The invariance of K_μ follows from [25, Proposition 1.4, p. 105 and Theorem 1.1, p. 102]. Next, by [25, Theorem 4.1, pp. 110 and 117], the stability modulus $s_L = \sup\{\Re\lambda : \Delta(\lambda) = 0\}$ of Eq. (5.2) also satisfies the characteristic equation. This means that $s_L < 0$ and therefore there exist constants $k_1, k_2 > 0$ such that $|T(t)\phi|_0 \leq k_1 \exp(-k_2 t)|\phi|_0$ for all $t \geq 0$ and $\phi \in C$ [11, Corollary 6.1, p. 215]. $\square$

5.2. **Attractivity properties of Eq. (5.1).** Notice that the conclusion of Lemma 3.1 will still hold if we replace the space C_I^+ in its formulation by C_I. Indeed, in this case, the variation of parameters formula (3.2) takes the following form

$$x(\phi)(t) = (T(t)\hat{\phi})(0) + \int_0^t v(t-s) f(x_s(\phi)) ds, \;\; t \in [0, \kappa), \tag{5.8}$$

where we write $\hat{\phi}(s) = \phi(s), \; s \in [-\tau, 0]$ for every $\phi \in C_I$. Now, the second term in the right-hand side of (5.8) is positive in virtue of **(H1)** and the positivity of v and f, while the first term is exponentially decreasing by Corollary 5.3. Therefore the arguments of Lemma 3.1 continue to work in the new situation. In particular, (3.1) implies immediately the eventual positivity of all solutions. Moreover, since the trajectory $x_t(\phi)$ is attracted by its ω-limit set $\omega(\phi)$ consisting from bounded and positive solutions only, we obtain that

$$\liminf_{t\to+\infty} x(\phi)(t) \geq \min_{\psi\in\omega(\phi)} \inf_{t\in\mathbb{R}} \int_{-\infty}^{t} v(t-s)f(x_s(\psi))ds$$
$$\geq \int_{-\infty}^{t} v(t-s)f(h(0))ds \geq h^2(0).$$

In this way, the eventual uniformly persistence of Eq. (5.1) is established and thus we can apply all attractivity results given in the third section of the work. We will summarize them in the following statement:

Theorem 5.4. *Let $\phi \in C_I$ be bounded and continuous. Then the solution $x(\phi)(t)$ to Eq.* (5.1) *exists for all $t \geq 0$ and $\liminf_{t\to+\infty} x(\phi)(t) \geq h(0) > 0$. Furthermore, $x(\phi)(t) > 0$ for all $t > 0$ when $\hat{\phi} \in K_\mu$. Finally, $\lim_{t\to+\infty} x(\phi)(t) = x_2$, when the unique fixed point x_2 of the strictly decreasing map $h : [0, h(0)] \to [0, h(0)]$ is globally attracting.*

5.3. **Example.** In the whole Banach space UC_g, we consider now the following generalization of the Lasota-Wazewska equation from Section 4.1:

$$x'(t) = -\delta x(t-\tau) + \int_0^\infty e^{-\gamma(s)|x(t-s)|}dq(s),\ \delta\tau \in (0, \frac{1}{e}], \qquad (5.9)$$

where $q : [0, +\infty) \to (0, +\infty)$ and $\gamma : [0, \infty) \to (0, \infty)$ satisfy all hypotheses of Section 4.1. Consider also g with the properties mentioned there. Again, it is easy to verify that the functional

$$f(\phi) = \int_0^\infty e^{-\gamma(s)|\phi(-s)|}dq(s)$$

is well defined and Lipschitzian over such UC_g and satisfies there (H1) together with (2.1). Moreover, notice that the condition $0 < \delta\tau \leq 1/e$ implies the positivity of the fundamental solution $v(t)$ in the case $Lx_t = \delta x(t-\tau)$ (see [7, Theorem 3.3.1]). By Lemma 5.2, this gives the existence of a negative real root μ to the characteristic equation $\mu + \delta\exp(-\tau\mu) = 0$.

Now, applying Theorem 5.4, we see that all results proved in Section 4.1 still hold for Eq. (5.9) if we replace there the cone BC^+ of initial functions by the whole space BC. Moreover, all solutions $x(\phi)(t)$ to (5.9) with $\hat{\phi} \in K_\mu$ are positive. Hence, we can see that small delays ($\tau \leq (\delta e)^{-1}$) in the linear part of (5.9) are harmless for the global stability of x_2, improving in this sense the results in [21].

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