Nonlinear Dyn (2012) 67:2467–2475
DOI 10.1007/s11071-011-0160-x

ORIGINAL PAPER

Global stabilization of periodic orbits using a proportional feedback control with pulses

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Received: 24 February 2011 / Accepted: 7 July 2011 / Published online: 11 August 2011

Abstract We investigate the stabilization of periodic orbits of one-dimensional discrete maps by using a proportional feedback method applied in the form of pulses. We determine a range of the parameter μ values representing the strength of the feedback for which all positive solutions of the controlled equation converge to a periodic orbit.

An important feature of our approach is that the required assumptions for which our results hold are met by a general class of maps, improving in this way some previous results. We discuss the applicability of our scheme to some models of population dynamics, and give numerical simulations to illustrate our analytical results.

Keywords Chaos control · Proportional feedback · Population model · Periodic orbit · Global attractor

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1 Introduction

One of the meaningful mechanisms of control of chaos is the proportional feedback method (PF), which, roughly speaking, consists in a periodic reduction of the state variable, proportional to the size of this variable. There are many situations where this control makes sense; for example, in models of exploited populations (by harvesting or fishing), this means that the harvesting effort is proportional to the population size (referred to as *constant effort harvesting* [6]). The same strategy is usually applied to control of pests; see, e.g., [14, 17] and references therein.

Although both discrete and continuous mathematical models involving constant effort harvesting have a long tradition [3, 6, 12, 16], the study of this mechanism as a control method (that is, to avoid undesired chaotic behavior) is relatively recent. Güémez and Matías [8] considered the one-dimensional discrete equation

$$x_{n+1} = f(x_n), \tag{1}$$

with the quadratic ($f(x) = rx(1-x)$) and the Ricker ($f(x) = xe^{r(1-x)}$) maps, and applied the PF method in the form

$$x_{n+1} = \begin{cases} f(x_n), & \text{if } n \neq mk, \\ f((1-\mu)x_n), & \text{if } n = mk,\ k \in \mathbb{Z}^+, \end{cases} \tag{2}$$

where m is a positive integer; that is, the control is applied in the form of pulses after m iterations of f.

The parameter μ represents the strength of the feedback. Reference [8] contains some numerical experiments that reveal that this method has the potential to drive a chaotic system (1) into a T-periodic regime after control (2), where T is a multiple of m. Solé et al. [15] provided more comments on the PF method, related to its applicability in the control of populations.

For $m = 1$, the controlled system (2) reduces to the difference equation

$$x_{n+1} = f\big((1-\mu)x_n\big). \tag{3}$$

When (1) is a model of population dynamics, function f is the so-called stock-recruitment function. Thus, (3) means that a portion μx_n of the population is removed each year prior to reproduction. This is a common hypothesis in fishery models and control of plagues [1, 17]. Recently, model (3) has attracted the attention of ecologists because it can produce such counterintuitive effects as a population increasing in response to an increase in its per capita mortality rate [1, 10, 14, 17]. Seno [14] identified the range of μ values for which this paradoxical effect occurs for some typical population models. A related result, valid for a general family of maps f commonly used in the formulation of model (1), was given in [10]. Moreover, in paper [10], it was rigorously proved that the control scheme (3) allows us to stabilize chaotic dynamics towards a globally stable equilibrium, providing the exact range of μ values for which this stabilization is achieved. This is an important item which had not been addressed in previous papers such as [8, 15].

However, we can notice some shortcomings of the approach in [10]:

1. It is assumed that the control function is applied at each step, which is sometimes not feasible.
2. Keeping the system in the strict vicinity of the only positive equilibrium does not reflect the most typical types of stable behavior in nature, when stable oscillations more frequently occur than convergence to the equilibrium.
3. There are significant restrictions on the smoothness and the shape of the map, for example, the negativity of the Schwarzian derivative.

The aim of this paper is twofold. On the one hand, we are able to generalize the results of [10] by loosening the requirements on the production function f which still makes it possible to determine the range of the controlling parameter valid for stabilization. On the other hand, we provide an analogous result for the general control scheme (2), in which the control is implemented once in several steps, in the form of pulses. We notice that, contrary to (3), (2) with $m > 1$ is nonautonomous, and this fact makes its study much more difficult.

We emphasize that, for $m > 1$, the result of PF control is an attracting cycle of period m, where m is the frequency of control intervention; this feature is in accordance with the numerical observations in [8]. Our main result not only provides an analytical proof of the fact that the control scheme (2) leads system (1) to a stable periodic orbit, but also guarantees its global stability. Hence, our results accomplish two important aims of the mechanisms of control and targeting, namely, suppression of any possible chaotic behavior and making the basin of attraction of the targeted trajectory as large as possible (for more discussions on the problem of targeting, see Bocaletti et al. [5, Sect. 4]).

We organize this note as follows: in the second section, we state and prove the main results; in Sect. 3 we discuss the main features of our approach, compare it with previous results for some usual population models, provide numerical simulations to illustrate our results, and state some open problems.

2 Global stabilization

In this section we state and prove our main results for the global stabilization of periodic solutions using the control PF scheme (2). First, we list the hypotheses we assume for map f (it will be explicitly stated which of them are imposed) and discuss their biological meaning.

(A1) $f : [0,\infty) \to [0,\infty)$ is continuous, $f(0) = 0$, and $f(x) > 0$ for all $x > 0$.

(A2) There exists a point $c > 0$ such that f is twice continuously differentiable on $[0, c]$, $f'(0) > 1$, $f'(x) > 0$ and $f''(x) < 0$ for all $x \in (0, c)$.

(A3) The following inequality holds:

$$\frac{f(x)}{x} < \frac{f(c)}{c}, \quad \forall x > c. \tag{4}$$

Condition (A1) is almost always required in discrete population models. Assumption (A2) assumes a twice differentiable curve which is increasing, concave down, and lies above $y = x$ for x small enough.

This is a typical feature of *compensation* models [6, Sect. 1.2], for which the production function f is written as $f(x) = xF(x)$, where $F(0) > 1$ and F is decreasing and nonnegative for small x. Thus, $f'(0) = F(0) > 1$, and $f''(0) = 2F'(0) < 0$. For unimodal functions, it is natural to choose a maximum point as c; however, any smaller positive point will also do.

Condition (A3) is a technical assumption meaning that for $x > c$ all points of the graph $y = f(x)$ lie under the line connecting the origin with $(c, f(c))$. This means that after some point the stock-recruitment ratio can still increase, but it does not exceed the value at this point.

Remark 1 Although the domain of f is assumed to be $[0,\infty)$ in hypothesis (A1) because it is the usual condition in biological models, it is easy to check that all the results given in this paper hold if we consider a continuous function $f : [0, b] \to [0, b]$, where $b \in \mathbb{R}$ and $0 < c \le b$.

Remark 2 We notice that assumption (A2) implies that function

$$h(x) = \frac{f(x)}{x}$$

is twice differentiable on $(0, c)$, and $h'(x) < 0$ for all $x \in (0, c)$. Hence, if (4) is satisfied for a certain $c > 0$ then it also holds for any $\tilde{c}$ such that $0 < \tilde{c} < c$, since

$$\frac{f(\tilde{c})}{\tilde{c}} = h(\tilde{c}) > h(x) = \frac{f(x)}{x}, \quad \forall x \in (\tilde{c}, c],$$

and

$$\frac{f(\tilde{c})}{\tilde{c}} > \frac{f(c)}{c} > \frac{f(x)}{x}, \quad \forall x > c.$$

We need the following simple auxiliary statement in the proof of our main results. As usual, for a map $g : I \to I$, where I is a real interval, we denote $g^2 = g \circ g$, and $g^n = g \circ g^{n-1}$ for all integer $n \ge 3$.

Lemma 1 *Let $g : [0,\infty) \to [0,\infty)$ be a continuous function such that $g(0) = 0$ and*

(H) *g has a unique fixed point K such that $x < g(x) < K$ for all $x \in (0, K)$, and $0 < g(x) < x$ for all $x > K$.*

Then K is globally attracting for all positive solutions of the equation

$$x_{n+1} = g(x_n); \tag{5}$$

that is, every solution $\{x_n\}$ of (5) with $x_0 > 0$ converges to K:

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} g^n(x_0) = K. \tag{6}$$

Proof If for some $j \in \mathbb{N}$ we have $0 < x_j \le K$ then by (H) all x_n, $n \ge j$, also satisfy $0 < x_n \le K$. Moreover, since $g(x) > x$ for $0 < x < K$, it follows that the sequence $\{x_n\}_{n=j}^{\infty}$ is nondecreasing and thus has a limit d, $0 < d \le K$. Taking limits in both sides of (5) we obtain $d = g(d)$, as g is continuous. By (H), K is the only positive equilibrium of g, so $d = K$.

Assume now that $x_n > K$ for any $n \in \mathbb{N}$. Then by (H) the sequence $\{x_n\}$ is nonincreasing; again, it has a limit $d \ge K$, which by continuity of g is a fixed point, and therefore (6) holds. □

Now we are in a position to prove our main results. We begin with the case $m = 1$, that is, the control scheme (3).

Theorem 1 *Assume that f satisfies* (A1), (A2) *and* (A3). *Then there exist μ_1, μ_2 such that $0 < \mu_1 < \mu_2 < 1$ and* (3) *has a globally attracting positive equilibrium for all $\mu \in [\mu_1, \mu_2)$.*

Proof Throughout the proof, we will denote $\gamma = 1 - \mu$, so (3) writes $x_{n+1} = f(\gamma x_n)$.

First, let us choose $\gamma = c/f(c)$ and define $g(x) = f(\gamma x)$. We claim that the following statements are true:

(i) $f(c)$ is a fixed point of g;
(ii) $g(x) < f(c)$ for $0 < x < f(c)$;
(iii) $g(x) > x$ for $x < f(c)$ and $g(x) < x$ for $x > f(c)$.

Then, we will be able to apply Lemma 1 to establish the global attraction of the unique positive equilibrium $f(c)$ of (3).

(i) Since

$$g(f(c)) = f\left(\frac{c}{f(c)} f(c)\right) = f(c),$$

$f(c)$ is a fixed point of g.

(ii) Since f is monotone increasing on $[0, c]$, then g is increasing on $[0, f(c)]$, which implies

$$g(x) < g(f(c)) = f(c) \quad \text{for any } x \in (0, f(c)).$$

(iii) By Remark 2, $h(x) = f(x)/x$ is decreasing on $(0, c)$; in particular,

$$f(x) > \frac{f(c)}{c}x \quad \text{for any } x \in (0, c).$$

If $x < f(c)$ then $cx/f(c) \in (0, c)$, and therefore

$$g(x) = f\left(\frac{cx}{f(c)}\right) > \frac{f(c)}{c}\frac{cx}{f(c)} = x.$$

Further, for $x > f(c)$ we have $cx/f(c) > c$, so by (4),

$$g(x) = f\left(\frac{cx}{f(c)}\right) < \frac{cx}{f(c)}\frac{f(c)}{c} = x.$$

By Lemma 1, the point $f(c)$ attracts all positive solutions of the controlled equation (3).

Next, in view of Remark 2, the result holds for any $\tilde{\gamma} = \tilde{c}/f(\tilde{c})$ with $0 < \tilde{c} < c$. Since

$$\lim_{x\to 0} \frac{f(x)}{x} = f'(0),$$

we can ensure that (3) has a globally attracting equilibrium for all $\gamma \in (1/f'(0), c/f(c)]$, that is to say, for any $\mu \in [\mu_1, \mu_2)$, where

$$\mu_1 := 1 - \frac{c}{f(c)}, \qquad \mu_2 := 1 - \frac{1}{f'(0)}. \tag{7}$$

□

Next we prove a global stability result for the general equation (3) with $m > 1$. We need the following key auxiliary result.

Lemma 2 *Assume that f satisfies* (A1), (A2) *and* (A3) *for a given $c = c_1 > 0$. Then, for every $m \geq 2$, f^m satisfies* (A1), (A2) *and* (A3) *for $c = c_m = f^{-m+1}(c_1)$.*

Proof First, we notice that, if (A2) holds, then f is monotone increasing on $[0, c]$ and we can define $f^{-1} : [0, f(c)] \to [0, c]$, and $f^{-k} = (f^{-1})^k$, $\forall k > 1$. Denoting $c_k = f^{-k+1}(c)$, we have the following inequalities:

$$c_m < c_{m-1} < \cdots < c_2 < c_1 = c.$$

It is obvious that f^m satisfies (A1) if (A1) holds for f. Next, the proof of (A2) for f^m follows easily from basic derivation rules. So, we omit it and proceed with (A3). We have to prove that, for all $k \geq 1$,

$$\frac{f^k(x)}{x} < \frac{f^k(c_k)}{c_k}, \quad \forall x > c_k = f^{-k+1}(c). \tag{8}$$

We use induction in k. Since (8) holds for $k = 1$, let us assume that it is true for $k = 1, 2, \ldots, m-1$, and we demonstrate that it also holds for $k = m$.

We claim that

$$\frac{f^{m-1}(x)}{x} < \frac{f^{m-1}(c_m)}{c_m}, \quad \forall x > c_m. \tag{9}$$

Indeed, if $c_m < x \leq c_{m-1}$, then (9) is a consequence of the fact that condition (A2) holds for f^{m-1} on $[0, c_{m-1}]$. Next, if $x > c_{m-1} = f^{-m+2}(c)$, we get from the induction hypothesis that

$$\frac{f^{m-1}(x)}{x} < \frac{f^{m-1}(c_{m-1})}{c_{m-1}} < \frac{f^{m-1}(c_m)}{c_m}.$$

Next we consider two cases:

Case 1. $f^{m-1}(x) \geq c$. Then, by (A2),

$$\frac{f^m(x)}{f^{m-1}(x)} \leq \frac{f(c)}{c}. \tag{10}$$

From (9) and (10), it follows that

$$\frac{f^m(x)}{x} = \frac{f^m(x)}{f^{m-1}(x)}\frac{f^{m-1}(x)}{x} < \frac{f(c)}{c}\frac{f^{m-1}(c_m)}{c_m} = \frac{f(c)}{c_m} = \frac{f^m(c_m)}{c_m}.$$

Case 2. $f^{m-1}(x) < c$. Then, by (A2), $f^m(x) < f(c)$, and therefore

$$\frac{f^m(x)}{x} < \frac{f(c)}{x} < \frac{f(c)}{c_m}, \quad \forall x > c_m.$$

This ends the proof. □

Now we state and prove an analogous result to Theorem 1 for the general scheme (2).

Theorem 2 *Assume that f satisfies* (A1), (A2), *and* (A3). *Then there exist $\mu_1(m), \mu_2(m)$ such that $0 < \mu_1(m) < \mu_2(m) < 1$ and* (2) *has a globally attracting m-cycle for all $\mu \in [\mu_1(m), \mu_2(m))$.*

Proof As before, we denote $\gamma = 1 - \mu$. We choose

$$\gamma = \frac{c_m}{f^m(c_m)} = \frac{f^{-m+1}(c)}{f(c)},$$

and define the continuous function $g(x) = f^m(\gamma x)$.

Notice that if $\{x_n\}$ is a solution of (2) and we define $g(x) = f^m(\gamma x)$, then

$$x_{m+1} = f(\gamma x_m),$$

$$x_{m+2} = f(x_{m+1}) = f^2(\gamma x_m),$$

$$\vdots$$

$$x_{2m} = f^m(\gamma x_m) = g(x_m),$$

and, in general,

$$x_{mk} = g^k(x_m),$$

for all integer $k \geq 1$.

Lemma 2 ensures that f^m satisfies (A1), (A2), and (A3) with $c_m = f^{-m+1}(c)$ instead of c. Thus, if $x_0 > 0$, Theorem 1 applies to prove that there exists

$$\lim_{k\to\infty} x_{mk} = \lim_{k\to\infty} g^k(x_m) = f^m(c_m) = f(c).$$

By the definition of (2) and the continuity of f, we have

$$\lim_{k\to\infty} x_{mk+1} = f\big(\gamma f(c)\big) = f(c_m) = f^{-m+2}(c),$$

and, for $p = 2, 3, \ldots, m-1$:

$$\lim_{k\to\infty} x_{mk+p} = f^p\big(f^{-m+2}(c)\big) = f^{p-m+2}(c).$$

Thus, $\{x_n\}$ converges to the m-cycle

$$\Gamma = \big\{f^{-m+2}(c), f^{-m+3}(c), \ldots, c, f(c)\big\}. \tag{11}$$

As in the proof of Theorem 1, it follows that the range of μ values for which the global stabilization is achieved is $[\mu_1(m), \mu_2(m))$, where

$$\mu_1(m) := 1 - \frac{c_m}{f^m(c_m)} = 1 - \frac{f^{-m+1}(c)}{f(c)};$$

$$\mu_2(m) := 1 - \frac{1}{(f^m)'(0)} = 1 - \frac{1}{(f'(0))^m}. \qquad \square$$

From all requirements of Theorems 1 and 2, probably (A3) can be the most difficult to verify. We finish this section with a result that provides simple sufficient conditions to ensure that (A3) holds.

Proposition 1 *Assume that f satisfies* (A1), (A2), *and* (A4) *f has a positive fixed point K such that $f(x) > x$ for $0 < x < K$ and $f(x) < x$ for $x > K$. Assume also that the following condition holds for the point c in* (A2):

$$c \in [0, K] \quad \text{and} \quad f(c) = \max\big\{f(x) : x \in [0, K]\big\}. \tag{12}$$

Then, f fulfills (A3).

Proof Let us notice that $f(c) \geq c$ and, therefore,

$$\frac{f(c)}{c} \geq 1.$$

We distinguish between two cases. If $x > K$ then, it follows from (A4) that $f(x) < x$, and therefore

$$\frac{f(x)}{x} < 1 \leq \frac{f(c)}{c}.$$

Assume now that $c < x \leq K$. Then, by assumption (12), $f(x) \leq f(c)$, and therefore

$$\frac{f(x)}{x} \leq \frac{f(c)}{x} < \frac{f(c)}{c}.$$

Thus, inequality $f(x)/x < f(c)/c$ holds for all $x > c$, as we wanted to prove. $\square$

We notice that Assumption (A4) is very common in discrete population models (see [7] and [16, Chap. 9]).

3 Discussion and numerical results

For $m = 1$, Theorem 1 establishes the global stability of the fixed point in model (3), complementing Theorem 1 in [10], where the assumptions of unimodality and negative Schwarzian derivative are required for f. Such a class of maps includes the Ricker function $f(x) = xe^{r(1-x)}$, with $r > 0$, and the generalized Beverton–Holt map $f(x) = ax/(1 + x^b)$, with $a > 1$, $b \geq 2$.

It is obvious that the conditions of Proposition 1 hold for the family of maps considered in [10], so Theorem 1 applies. In this case, the main result in [10] is sharper because it provides a larger range of μ values for which global stabilization is achieved using (3). Actually, Theorem 1 establishes the μ values for which convergence to a globally stable equilibrium with the PF control scheme (3) is eventually monotone, that is, every positive solution $x = \{x_n\}$ of (3) converges to the equilibrium, and this convergence is monotone for all $n \geq k$, for an integer $k = k(x) \geq 0$ (see the proof of Lemma 1).

However, we notice that the scope of Theorem 1 is much more general, since many other maps fall within this setting; actually, we even assume neither existence nor uniqueness of fixed points of the production function f. For instance, the generalization $f(x) = xe^{q(x)}$ of the Ricker model, where $q(x)$ is a function satisfying $q(0) > 0$ and $q'(0) < 0$, can meet the assumptions of Theorem 1 even if q does not have positive zeros (so f does not have a positive equilibrium), or it has more than one zero (so f has several positive equilibria). For example, choose $q(x) = e^{-x}$; then (1) with $f(x) = xe^{q(x)}$ does not have any positive equilibria. However, by Theorem 1, the control scheme (3) has a globally stable equilibrium for all $\mu \in (0.169, 0.632)$.

Other known stock-recruitment functions f used in models of population dynamics for which the result in [10] does not apply are:

1. The family of bimodal maps $f(x) = \alpha x + \beta x e^{-x}$ (where $0 < \alpha < 1, \beta > 0$) suggested in [6, 16], which is a further generalization of the Ricker model allowing a probability α of adult survivorship after reproduction;
2. The generalization of the logistic map proposed by Beddington and May [2]:

   $$f(x) = x\left[1 + q\left(1 - x^z\right)\right]_+,$$

   where $q > 0$, $z > 0$, $[x]_+ = \max\{x, 0\}$. This map is unimodal but does not have a negative Schwarzian derivative for $z < 1$.

It is easy to check that these maps satisfy the conditions of Proposition 1, so both Theorems 1 and 2 apply.

It is worth emphasizing that if f meets the conditions of Proposition 1, then the PF scheme (3) has some additional interesting features. On the one hand, from the point of view of targeting, it gives the possibility to stabilize the size of the population into any desired positive number below the maximum possible value of the production function on $[0, K]$; in this direction, this method is better than other techniques of control of chaos (see [10] and references therein). On the other hand, when the attained equilibrium is a local maximum value $f(c)$, the convergence of the method is superstable, since $g'(f(c)) = \gamma f'(c) = 0$.

But the main contribution of this note is the statement and rigorous proof of a result (Theorem 2) which ensures that when the proportional feedback method is applied in the form of pulses (as it was introduced in [8]), it is still possible to get a global stabilization of all positive solutions. If the frequency of pulses is every m seasons, then the solutions are stabilized into an m-cycle. Moreover, we found easily verifiable conditions (see Proposition 1), which are met by a general family of maps usually employed in population dynamics and allow us to determine a range of values $[\mu_1(m), \mu_2(m))$ of μ for which the m-cycle of (2) is globally attracting. If condition (12) holds, then the point c is often a nondegenerated local maximum. If this is the case and we consider (2) with

$$\mu = \mu_1(m) = 1 - \frac{f^{-m+1}(c)}{f(c)},$$

then the attracting periodic orbit contains $f(c)$ and is superstable, that is, its multiplier is 0. Indeed, if $\mu = \mu_1(m)$ and $g(x) = f^m((1-\mu)x)$, then, since $f'(c) = 0$,

$$\begin{aligned}
g'\big(f(c)\big) &= (1-\mu)(f^m)'\big((1-\mu)f(c)\big) \\
&= (1-\mu)(f^m)'(c_m) \\
&= (1-\mu)\big(f \circ f^{m-1}\big)'(c_m) \\
&= (1-\mu)f'\big(f^{m-1}(c_m)\big)\big(f^{m-1}\big)'(c_m) \\
&= (1-\mu)f'(c)\big(f^{m-1}\big)'(c_m) = 0.
\end{aligned}$$

We notice that $\mu_1(m)$ is an increasing function of m. This means that, to stabilize the solutions of (1) about a globally stable m-periodic orbit using the control scheme (2), the removal rate μ must be larger as the time between pulses increases.

We emphasize that the PF method can be implemented as a strategy for suppressing chaos in population models, and even to enhance the average population stock. However, as usual some words of caution are in order since, unless the stock-recruitment relationship is very sharply known (which is in general difficult), this strategy is dangerous because the difference between the culling rates $\mu_1(m)$ and $\mu_2(m)$ is small, and the population is driven to extinction if we apply a harvesting rate greater than $\mu_2(m)$.

Table 1 Steps of the algorithm for stabilization of a 3-periodic orbit using proportional feedback control with pulses

Step	Action
1	Find numerically the critical point c
2	Determine the point $c_3 = f^{-2}(c)$, $c_3 \in (0, c)$
3	Choose $\mu^* \in [1 - c_3/f(c), 1 - 1/(f'(0))^3)$
4	Multiply by $(1 - \mu^*)$ every third iteration of (13)

Finally, in order to illustrate our results, we examine a case of study and give some numerical simulations for it.

Consider the following generalized Beverton–Holt equation (also referred to as Maynard Smith model [16]):

$$x_{n+1} = \frac{2.5x_n}{1 + x_n^5}, \tag{13}$$

whose dynamics is chaotic. We implement a PF method by pulses every third period, that is,

$$x_{n+1} = \begin{cases} f(x_n), & \text{if } n \neq 3k, \\ f((1-\mu)x_n), & \text{if } n = 3k, \ k \in \mathbb{Z}^+, \end{cases} \tag{14}$$

where $f(x) = 2.5x/(1 + x^5)$.

Function f is unimodal with a unique critical point $c \approx 0.758$, and $f(c) \approx 1.516$. We can find numerically the points $c_2 = f^{-1}(c) \approx 0.304 \in (0, c)$ and $c_3 = f^{-2}(c) \approx 0.121 \in (0, c)$.

Since f meets all conditions in the statement of Proposition 1, Theorem 2 ensures that the control scheme (14) stabilizes system (13) into a globally attracting cycle of prime period 3. If we choose

$$\mu = \mu_1 = 1 - \frac{c_3}{f(c)} = 1 - \frac{f^{-2}(c)}{f(c)} \approx 0.91979, \tag{15}$$

then the cycle is defined by

$$\Gamma = \{f^{-1}(c), c, f(c)\} = \{0.304, 0.758, 1.516\}. \tag{16}$$

Table 1 summarizes the steps of the algorithm used for the stabilization of the chaotic equation (13).

Starting at any $x_0 > 0$, the procedure explained in Table 1 produces a sequence converging to the globally attracting 3-cycle Γ.

In Fig. 1 we plot a cobweb diagram showing a chaotic orbit of the uncontrolled equation (13), and the globally attracting 3-cycle (represented by the dotted lines) for the controlled system (14).

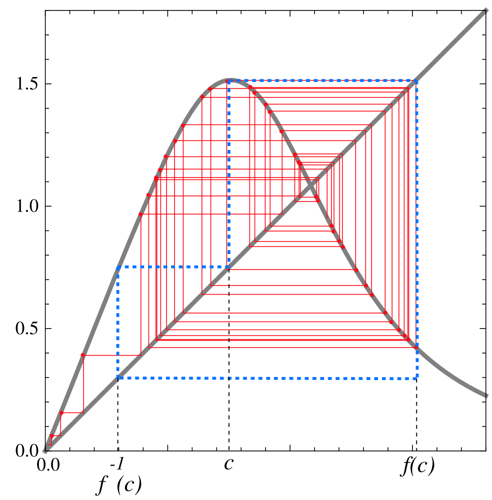

Fig. 1 Chaotic orbit of the uncontrolled equation (13), and the globally attracting 3-cycle (*dotted lines*) for the controlled system (14)

In Fig. 2(a) we plot a time series with 50 iterations of the chaotic system (13) starting at $x_0 = 1.4$, and the solution of the controlled system (14) with $\mu = \mu_1$ is displayed in Fig. 2(b).

The range of μ values for which Theorem 2 ensures a globally attracting 3-cycle of (14) is $[\mu_1, \mu_2)$, where μ_1 is given in (15), and

$$\mu_2 = 1 - \frac{1}{(f'(0))^3} = 1 - \left(\frac{2}{5}\right)^3 = 0.936.$$

At $\mu = \mu_2$, (14) has a transcritical bifurcation, after which all solutions converge to 0. In Fig. 3 we represent the bifurcation diagram of (14), for $\mu \in (0.85, 0.94)$. We notice that the 3-cycle arises after a period-halving bifurcation at $\mu = \mu_0 \approx 0.891$, where $g(x) = f^3((1-\mu)x)$ has a fixed point $\bar{x}$ with $g'(\bar{x}) = -1$. This allows to enlarge the interval where there is an asymptotically stable 3-cycle for (15) to (μ_0, μ_2); actually, in this example, it is not difficult to use the properties of the Schwarzian derivative to prove that the cycle is globally attracting for all $\mu \in (\mu_0, \mu_2)$. However, this can be a very difficult task if f is not a unimodal map with a negative Schwarzian derivative.

Finally, let us mention some open problems and directions for future research.

1. In this note and in many previous papers, for $m = 1$ stabilization was understood as the method of control leading to a stable equilibrium, and for

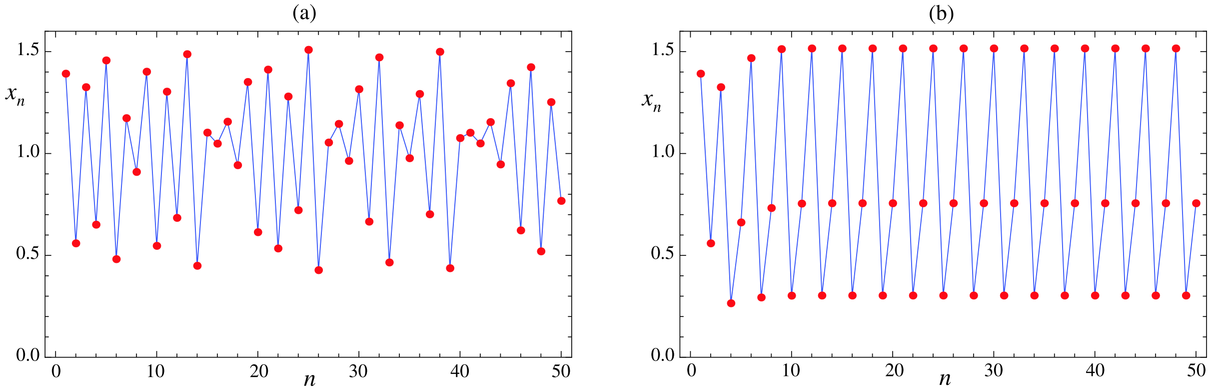

Fig. 2 Time series for **(a)** the uncontrolled equation (13), and **(b)** the controlled system (14), showing convergence to the attracting cycle of period 3 for $\mu = 0.92$

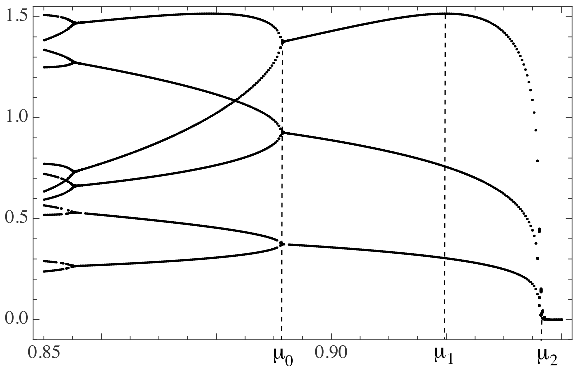

Fig. 3 Bifurcation diagram of (14), for $\mu \in (0.85, 0.94)$. There is a period-halving bifurcation at $\mu = \mu_0$. The points μ_1 and μ_2 are also indicated (see the text)

$m > 1$—where pulse controls are applied every m steps—to a stable m-cycle.

When the proportional feedback control cannot drive the system to a stable equilibrium, it would be interesting to study if a stable cycle can still be attained (for $m > 0$, a stable cycle of minimal period $k > m$).

2. Investigate the set of PF controls which do not lead necessarily to stable equilibria or cycles but keep the population within certain bounds $0 \le \alpha \le x_n \le M$. Here, for populations with a single positive equilibrium $K > 0$, the positive constant $\alpha < K$ can be chosen large enough for harvesting to avoid population extinction, and either $M > K$ or, for pest control, $M < K$ should be small enough.
3. For the general model (1), with f satisfying (A1), try to loosen the necessary conditions for global stabilization using other types of control, such as the predictive control considered in [11] or the constant feedback method addressed in [9]. For very high reproduction rates, a positive constant perturbation can lead to a stable cycle, see [4] and references therein.
4. Everywhere above, we assumed a proportional reduction at each or some selected steps. For systems with the Allee effect, enrichment can be required as well. Compared to [10], in the present paper we considered more general dynamical systems. However, it is still assumed that population grows for small values of x_n. The consideration of control and stabilization for systems with the Allee effect [13] is another interesting direction for future research.

Acknowledgements E. Braverman was partially supported by the NSERC grant. E. Liz was supported by the Spanish Ministry of Science and Innovation and FEDER, Grant MTM2010-14837.

The authors thank the useful comments of two anonymous reviewers of this paper.

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