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Bifurcations in Nonautonomous Dynamical Systems: Results and tools in discrete time

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Abstract

When extending the qualitative and dynamical theory from autonomous difference equations (mappings) to explicitly time-dependent problems, one is confronted with three intrinsic problems: One needs a more flexible notion of invariance, eigenvalues do not yield meaningful stability information, and generically such equations do not possess equilibria.

In this admittedly biased survey paper, we address the above aspects and discuss several approaches in the development of a corresponding bifurcation theory for nonautonomous difference equations. First, we present a spectral notion based on exponential dichotomies and give continuation results for entire bounded solutions. Second, we discuss so-called *solution*, as well as *attractor bifurcations* and illustrate them using various examples. Finally, to apply the above results in higher-dimensional problems, we tackle an applicable version of Pliss's reduction principle via nonautonomous center manifolds — so-called *center fiber bundles*.

Keywords: Nonautonomous difference equation, nonautonomous dynamical system, nonautonomous bifurcation, dichotomy spectrum, invariant fiber bundle

MSC Classification: 39A10; 39A28; 39A30; 37B55; 37C60; 37C70; 37G35; 37G99

Contents

1 An amble introduction

Bifurcation or branching theory as a part of nonlinear functional analysis deals with changes in the solution structure to abstract (nonlinear) equations under parameter variation (cf., e.g., the references [18, 25, 50, 90]). Applied to the theory of dynamical systems these equations are evolutionary differential or difference equations, and a bifurcation typically goes hand in hand with a change of stability properties to particular reference solutions. More specifically, classical dynamical bifurcation theory for discrete systems focusses on autonomous difference equations

$$x_{k+1} = g(x_k, \lambda) \tag{1.1}$$

with a smooth right-hand side $g : \mathbb{R}^d \times \Lambda \to \mathbb{R}^d$ depending on a parameter λ; here, the parameter space Λ is an ambient metric space — typically an open subset of $\mathbb{R}^n$ or of a some Banach space (cf., e.g., [42, 30, 88, 35, 60] or the survey paper [23]) but sometimes a more flexible setting is eligible. A central question is how stability and multiplicity properties of invariant sets for (1.1) change, when the parameter λ is varied? In the simplest, and most often considered situation, these invariant sets are fixed points or periodic solutions to a difference eqn. (1.1).

Given some fixed parameter value $\lambda^* \in \Lambda$, an equilibrium $x^* = g(x^*, \lambda^*)$ of (1.1) is called *hyperbolic*, provided the partial derivative $D_1 g(x^*, \lambda^*) \in \mathbb{R}^{d \times d}$ possesses no eigenvalue on the complex unit circle $\mathbb{S}^1$. Then the implicit function theorem (cf., e.g. [90, p. 150, Thm. 4.B]) allows a unique continuation $x(\lambda) \equiv g(x(\lambda), \lambda)$ of x^* in a neighborhood of λ^*. In particular, hyperbolicity rules out bifurcations understood as topological changes in the set $\{x \in \mathbb{R}^d : g(x, \lambda) = x\}$ near a reference pair (x^*, λ^*) or a stability change of x^*.

On the other hand, eigenvalues on the complex unit circle give rise to various well-understood autonomous bifurcation scenarios. Such classical examples include fold, transcritical or pitchfork bifurcations (eigenvalue 1), flip bifurcations (eigenvalue -1) or the Sacker-Neimark bifurcation (a pair of complex conjugate eigenvalues for $d \geq 2$). Via center manifold theory, higher dimensional problems can be reduced to the above situations. Moreover, normal form theory allows a classification of bifurcation scenarios by finding an algebraically most simple representation. It can be said that the dynamical bifurcation theory for autonomous equations has reached a remarkable maturity w.r.t. analytical as well as numerical aspects and various effective computational tools are available (cf., for instance, [31]).

Nevertheless, even in the time-invariant setting of eqn. (1.1), we will illustrate below that one easily encounters intrinsically nonautonomous problems, where neither the classical methods described above and presented in, for instance, [42, 30, 88, 35, 60, 23], nor the numerical routines of e.g. [31] apply. For this reason, we extend our perspective to the framework of general nonautonomous difference equations

$$\boxed{x_{k+1} = f_k(x_k, \lambda)} \tag{Δ_λ}$$

with a sufficiently smooth right-hand side $f_k : \mathbb{R}^d \times \Lambda \to \mathbb{R}^d$, $k \in \mathbb{Z}$. For our theory, we usually suppose that the function f_k and its derivatives map bounded subsets of $\mathbb{R}^d \times \Lambda$ into bounded sets uniformly in $k \in \mathbb{Z}$. Concrete applications and examples for nonautonomous equations (Δ_λ) are:

- Investigate the behavior of (1.1) along an entire reference solution $(\phi_k^*)_{k\in\mathbb{Z}}$, which is not constant or periodic. This is typically done using the (obviously nonautonomous) *equation of perturbed motion*

$$x_{k+1} = g(x_k + \phi_k^*, \lambda) - g(\phi_k^*, \lambda), \tag{1.2}$$

which evidently possesses the trivial solution; here, $f_k(x, \lambda) = g(x + \phi_k^*, \lambda) - g(\phi_k^*, \lambda)$.

- One replaces the constant parameter λ in (1.1) by a sequence $(\lambda_k)_{k\in\mathbb{Z}}$ in Λ, which varies in time. Also the resulting *parametrically perturbed* equation

$$x_{k+1} = g(x_k, \lambda_k)$$

becomes nonautonomous with $f_k(x, \lambda) = g(x, \lambda_k)$; note here the ambiguity that the parameter space in (Δ_λ) is an appropriate sequence space, while it is a subset of $\mathbb{R}^n$ in (1.1). This situation is highly relevant for applications, in order to mimic control or regulation strategies via the sequence $(\lambda_k)_{k\in\mathbb{Z}}$.

- Numerical discretizations of an autonomous ODE $\dot{x} = G(x, \lambda)$ with adaptive time-steps $h_k > 0$ yield nonautonomous difference equations. In the simplest case of the forward Euler-method, they are of the form

$$x_{k+1} = x_k + h_k G(x_k, \lambda)$$

and it is $f_k(x, \lambda) = x + h_k G(x, \lambda)$.

There is also a further source for nonautonomous dynamics: Given a so-called *base space* Ω and a map $f : \Omega \times \mathbb{R}^d \times \Lambda \to \mathbb{R}^d$, the concept of *driven difference equations*

$$x_{k+1} = f(\theta^k \omega, x_k, \lambda) \tag{1.3}$$

as nonautonomous problems with right-hand sides $f_k = f(\theta^k \omega, \cdot) : \mathbb{R}^d \times \Lambda \to \mathbb{R}^d$, is very fruitful from an applied point of view (see [20, 51]). For instance,

- a sequence $f_k : \mathbb{R}^d \times \Lambda \to \mathbb{R}^d$ is chosen periodically or, perhaps less regularly, from a finite family of maps $\{g_1, \dots, g_r\}$. A difference eqn. $x_{k+1} = g_{\omega_k}(x_k, \lambda)$, $\omega_k \in \{1, \dots, r\}$, can be written as (1.3) with Ω being the set of sequences from $\mathbb{Z}$ into $\{1, \dots, r\}$ and θ is the shift operator on Ω defined by $\theta((\omega_k)_{k\in\mathbb{Z}}) = (\omega_{k+1})_{k\in\mathbb{Z}}$. Then Ω becomes a compact metric space w.r.t. the metric

$$d(\omega, \bar{\omega}) := \sum_{k\in\mathbb{I}} (1 + r)^{-|k|} |\omega_k - \bar{\omega}_k| \quad \text{for all } \omega, \bar{\omega} \in \Omega.$$

- To incorporate random or stochastic influences, one considers metric dynamical systems, i.e. a probability space $(\Omega, F, \mathbb{P})$ with a measurable map $\theta : \Omega \to \Omega$ such that $\theta\mathbb{P} = \mathbb{P}$. Then a *random difference equation* (see [5]) is of the form

$$x_{k+1} = f(\theta^k \omega, x_k, \lambda),$$

where the mapping $f(\omega, \cdot, \lambda) : \mathbb{R}^d \to \mathbb{R}^d$, $\lambda \in \Lambda$, is assumed to be measurable. For a fixed event $\omega \in \Omega$, i.e. in a path-wise consideration, this is a nonautonomous difference equation (cf. [5, pp. 50ff, Sect. 2.1] or [87]).

Finally, we point out that these notes are based on [56, Chapt. 7] but provide a broader scope and various additional examples.

1.1 Nonautonomous dynamics

In the classical autonomous theory, the dynamical behavior of (1.1) only depends on the time elapsed since starting. For this reason, one chooses 0 as initial time and works with (1-parameter) semigroups

$$\phi_\lambda(k, \cdot) := \underbrace{g(\cdot, \lambda) \circ \ldots \circ g(\cdot, \lambda)}_{k \text{ times}} \quad \text{for all } k \in \mathbb{Z}_0^+$$

on the *state space* $\mathbb{R}^d$. Their dynamical behavior is captured by means of orbits $\{\phi_\lambda(k, \xi)\}_{k \in \mathbb{Z}_0^+}$, i.e. projections of solution sequences $\big(k, \phi_\lambda(k, \xi))\big)_{k \in \mathbb{Z}_0^+}$ to $\mathbb{R}^d$, where $\xi \in \mathbb{R}^d$ denotes an initial value. As a consequence, invariant and limit sets are subsets of the state space.

As opposed to this, for nonautonomous difference equations

$$\boxed{x_{k+1} = f_k(x_k)} \tag{Δ}$$

their dynamics depends on the starting time as well, and a vivid geometrical interpretation requires the *extended state space* $\mathbb{Z} \times \mathbb{R}^d$. An *entire solution* to (Δ) is a sequence $(\phi_k)_{k \in \mathbb{Z}}$ satisfying the identity $\phi_{k+1} \equiv f_k(\phi_k)$ on $\mathbb{Z}$, and the set $\big\{(k, \phi_k) \in \mathbb{Z} \times \mathbb{R}^d : \ k \in \mathbb{Z}\big\}$ is called *solution sequence*. In particular, the forward solution to (Δ) satisfying the initial condition $x_\kappa = \xi$ for given initial times $\kappa \in \mathbb{Z}$ and initial states $\xi \in \mathbb{R}^d$ is called *general solution*; it is denoted by $\varphi(\cdot; \kappa, \xi)$ and explicitly given by

$$\varphi(k; \kappa, \cdot) := \begin{cases} f_{k-1} \circ \ldots \circ f_\kappa, & \kappa < k, \\ \mathrm{id}, & k = \kappa. \end{cases} \tag{1.4}$$

Without invertibility assumptions on f_k, backward solutions to (Δ) might not exist or might not be unique. Hence, the maximal domain of definition for φ is $\big\{(k, \kappa, \xi) \in \mathbb{Z}^2 \times \mathbb{R}^d : \ \kappa \le k\big\}$. For bijective $f_k : \mathbb{R}^d \to \mathbb{R}^d$ with inverse f_k^{-1}, one additionally defines

$$\varphi(k; \kappa, \cdot) := f_k^{-1} \circ \ldots \circ f_{\kappa-1} \quad \text{for all } k < \kappa.$$

In the following lines, we briefly present an extension of the autonomous dynamical systems theory (see e.g. [30, 35]) to such 2-*parameter semigroups* or *processes* φ. Central for this endeavor is the notion of a *nonautonomous set* $\mathcal{A}$: At first, this is a subset of the extended state space $\mathbb{Z} \times \mathbb{R}^d$ and its *k-fiber* is defined to be

$$\mathcal{A}(k) := \Big\{x \in \mathbb{R}^d : \ (k, x) \in \mathcal{A}\Big\} \quad \text{for all } k \in \mathbb{Z}.$$

A *neighborhood* of $\mathcal{A}$ is a superset of the ϵ-neighborhood

$$\mathcal{B}_\epsilon(\mathcal{A}) := \left\{(k,x) \in \mathbb{Z} \times \mathbb{R}^d : \operatorname{dist}(x, \mathcal{A}(k)) < \varepsilon\right\}$$

with some $\varepsilon > 0$ and $\operatorname{dist}(x, A) := \inf_{a\in A} \|x - a\|$ for $x \in \mathbb{R}^d$ and $A \subseteq \mathbb{R}^d$. Such a nonautonomous set $\mathcal{A}$ is called

- *compact*, if every fiber $\mathcal{A}(k)$, $k \in \mathbb{Z}$, is compact
- *bounded*, if there exists a $R > 0$ such that $\mathcal{A}(k) \subseteq B_R(0)$ for all $k \in \mathbb{Z}$, where $B_R(0)$ denotes the open unit ball in $\mathbb{R}^d$ centered around 0
- *invariant*, if one has
$$\mathcal{A}(k+1) = f_k(\mathcal{A}(k)) \quad \text{for all } k \in \mathbb{Z}$$
and *forward invariant*, if $\mathcal{A}(k+1) \subseteq f_k(\mathcal{A}(k))$ holds for all $k \in \mathbb{Z}$, which is equivalent to $\mathcal{A}(k) = \varphi(k; \kappa, \mathcal{A}(\kappa))$ resp. the inclusion $\mathcal{A}(k) \subseteq \varphi(k; \kappa, \mathcal{A}(\kappa))$ for all $\kappa \leq k$
- *attractive*, if every bounded nonautonomous set $\mathcal{B}$ satisfies
$$\lim_{n\to\infty} h(\varphi(k; k-n, \mathcal{B}(k-n)), \mathcal{A}(k)) = 0 \quad \text{for all } k \in \mathbb{Z}, \tag{1.5}$$
where $h(A, B) := \sup_{b\in B} \operatorname{dist}(b, A)$ is the *Hausdorff semidistance* of subsets $A, B \subseteq \mathbb{R}^d$
- *repulsive*, if every bounded nonautonomous set $\mathcal{B}$ satisfies
$$\lim_{n\to\infty} h(\varphi(k-n; k, \mathcal{B}(k)), \mathcal{A}(k-n)) = 0 \quad \text{for all } k \in \mathbb{Z}, \tag{1.6}$$
where the right-hand side $f_k : \mathbb{R}^d \to \mathbb{R}^d$ of (Δ) is assumed to be bijective.

We furthermore speak of a *locally attractive* or *repulsive* set, if the respective relation (1.5) or (1.6) holds for all nonautonomous sets $\mathcal{B}$ contained in a neighborhood of $\mathcal{A}$.

Finally, a *global attractor* of (Δ) is defined as an invariant, compact and attractive nonautonomous set. Accordingly, a *local attractor* or *repeller* is invariant, compact and locally attractive resp. repulsive.

1.2 Examples for nonautonomous bifurcations

After these preliminaries on the process formulation of nonautonomous discrete dynamics, we return to a bifurcation theory and parameter-dependent difference eqns. (Δ_λ). Throughout, their general solution will be denoted by φ_λ.

Firstly, we observe that nonautonomous problems (Δ_λ) generically do not have constant solutions (equilibria), and in particular the fixed point sequences $x_k^* = f_k(x_k^*, \lambda^*)$ are normally not solutions to (Δ_λ). This gives rise to

Question 1: *If there are no equilibria, what bifurcates in a nonautonomous set-up?*

An adequate answer to this question forces us to enlarge the set of objects in which we look for bifurcating objects. For motivational purposes, consider the autonomous case (1.1) first and the problem of parametric perturbations.

Example 1.1. Let $b = (b_k)_{k\in\mathbb{Z}}$ be a bounded real sequence, λ be a real parameter and consider the scalar nonautonomous difference equation

$$x_{k+1} = \tfrac{1}{2}x_k + \lambda b_k. \tag{1.7}$$

For $\lambda = 0$ this equation is autonomous and has the unique fixed point $x^0 = 0$ resulting from the relation $x = \frac{1}{2}x$; this fixed point is the unique bounded and entire solution, as well as the global attractor for the autonomous problem $x_{k+1} = \frac{1}{2}x_k$. For parameters $\lambda \neq 0$ the unique solutions to $x = \frac{1}{2}x + \lambda b_k$ do not have a dynamical meaning. Nevertheless, the difference eqn. (1.7) still admits a unique bounded entire solution

$$\phi(\lambda)_k := \lambda \sum_{n=-\infty}^{k-1} \left(\tfrac{1}{2}\right)^{k-n-1} b_n \quad \text{for all } k \in \mathbb{Z}.$$

Hence, the fixed point of (1.7) for $\lambda = 0$ persists as an entire bounded solution $\phi(\lambda)$ (cf. Fig. 1). On the other hand, the global attractor to (1.7) is given by the nonautonomous set

$$\mathcal{A}_\lambda := \{(k, \phi(\lambda)_k) : k \in \mathbb{Z}\} \quad \text{for all } \lambda \in \mathbb{R}.$$

Its fibers consist of singletons and do not change their topological structure for arbitrary values of the parameter $\lambda \in \mathbb{R}$.

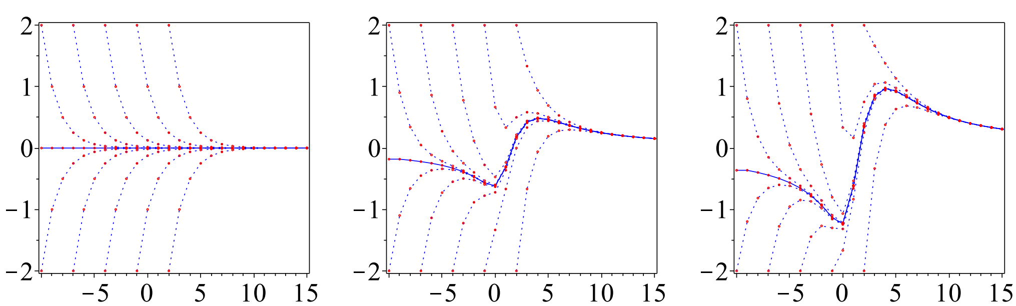

Figure 1: Solution sequences (dotted) of the linear difference eqn. (1.7) with $b_k = \frac{k}{2+k^2}$ and $\lambda = 0$ (left), $\lambda = 1$ (middle), $\lambda = 2$ (right) and the unique bounded solution $\phi(\lambda)$ (solid)

This facile linear example and Fig. 1 yield the conjecture that equilibria of autonomous difference eqns. (1.1) persist as bounded entire solutions under parametric perturbations and that this behavior can also be observed for nonlinear equations. It will be shown below in Thm. 3.4 (or in [69, Thm. 3.4]) that this conjecture is generically true in the sense that a fixed point of (1.1) has to be hyperbolic in order to persist under parametric perturbations.

In the following, we study various scenarios where the hyperbolicity condition is violated and persistence cannot be guaranteed.

Example 1.2. As above, consider a scalar nonautonomous difference equation

$$x_{k+1} = \lambda x_k + \tfrac{1}{1+|k|}. \tag{1.8}$$

Depending on the real parameter λ we obtain the following behavior:

- $|\lambda| < 1$: The eqn. (1.8) has a unique bounded entire solution $\phi(\lambda)_k := \sum_{n=-\infty}^{k-1} \frac{\lambda^{k-n-1}}{1+|n|}$ for all $k \in \mathbb{Z}$. It is uniformly asymptotically stable and accordingly the nonautonomous set $\mathcal{A}_\lambda := \{(k, \phi(\lambda)_k) : k \in \mathbb{Z}\}$ is the global attractor.

- $\lambda = 1$: Due to the variation of constants formula (cf., e.g., [1, p. 59]), the general solution to (1.8) has the form of a harmonic series

$$\varphi_1(k; \kappa, \xi) = \xi \begin{cases} +\sum_{n=\kappa}^{k-1} \frac{1}{1+|n|}, & k \geq \kappa, \\ -\sum_{n=k}^{\kappa-1} \frac{1}{1+|n|}, & k < \kappa \end{cases}$$

and thus there exist no bounded entire solutions to (1.8).

- $\lambda = -1$: Reasoning as above, the general solution is

$$\varphi_{-1}(k; \kappa, \xi) = (-1)^{k-\kappa} \xi \begin{cases} -\sum_{n=\kappa}^{k-1} \frac{(-1)^{k-n}}{1+|n|}, & k \geq \kappa, \\ +\sum_{n=k}^{\kappa-1} \frac{(-1)^{k-n}}{1+|n|}, & k < \kappa \end{cases}$$

and consequently every solution of (1.8) is bounded.

- $|\lambda| > 1$: The eqn. (1.8) has a unique bounded entire solution $\phi(\lambda)_k := -\sum_{n=k}^{\infty} \frac{\lambda^{k-n-1}}{1+|n|}$, which is unstable; the nonautonomous set $\mathcal{A}_\lambda := \{(k, \phi(\lambda)_k) : k \in \mathbb{Z}\}$ is a repeller.

For the critical and nonhyperbolic parameter values $\lambda = \pm 1$ the linear eqn. (1.8) changes its stability. At $\lambda = -1$ the number of bounded entire solutions explodes, while there exists a unique bounded entire solution in the vicinity of $\lambda = -1$. Also close to the parameter value $\lambda = -1$ there are unique bounded entire solutions, while there is none for $\lambda = 1$.

Example 1.3 (pitchfork bifurcation). For $\lambda > 0$ consider the autonomous difference equation

$$x_{k+1} = f_k(x_k, \lambda), \qquad f_k(x, \lambda) := \frac{\lambda x}{1 + |x|}. \tag{1.9}$$

It is a prototype example featuring an autonomous pitchfork bifurcation (cf., e.g. [60, pp. 119ff, Sect. 4.4]), where the unique asymptotically stable fixed point $x^* = 0$ for $\lambda \in (0, 1)$ bifurcates into two asymptotically stable equilibria $x_\pm := \pm(\lambda - 1)$ for $\lambda > 1$.

Along the trivial solution the linearization $x_{k+1} = \lambda x_k$ to (1.9) is nonhyperbolic for $\lambda = 1$. From a nonautonomous perspective, this loss of hyperbolicity causes an attractor bifurcation:

- $\lambda \in (0, 1]$: The set $\mathcal{A}_\lambda = \mathbb{Z} \times \{0\}$ is the global attractor which consists of the unique bounded entire solution to (1.9) (see Fig. 2 left, middle)

- $\lambda > 1$: The zero solution becomes unstable and the global attractor $\mathcal{A}_\lambda = \mathbb{Z} \times [1 - \lambda, \lambda - 1]$ is nontrivial. Here, a whole family of bounded entire solutions exists, connecting the equilibrium 0 with $\pm(\lambda - 1)$ (see Fig. 2 right)

While the above example shows how (autonomous) bifurcations can be understood as attractor alternations, the following scenario is intrinsically nonautonomous. Here, we generate a nonhyperbolic situation by concatenating two hyperbolic systems:

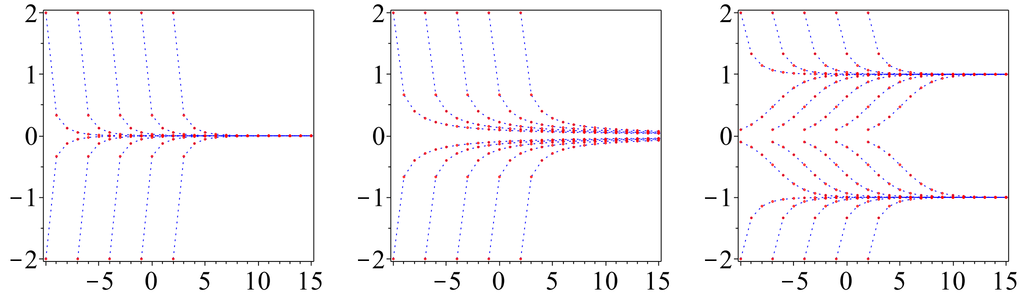

Figure 2: Solution sequences (dotted) of the difference eqn. (1.9) with $\lambda = \frac{1}{2}$ (left), $\lambda = 1$ (middle), $\lambda = 2$ (right)

Example 1.4 (shovel bifurcation). Consider a scalar linear difference equation

$$x_{k+1} = a_k(\lambda)x_k, \qquad a_k(\lambda) := \begin{cases} \frac{1}{2}+\lambda, & k<0, \\ \lambda, & k \geq 0, \end{cases} \tag{1.10}$$

depending on a real $\lambda > 0$. In order to understand the dynamics of (1.10), we distinguish three parameter constellations:

- $\lambda \in (0, \frac{1}{2})$: The unique bounded entire solution is the trivial one and (1.10) is uniformly asymptotically stable. Its global attractor reads as $\mathcal{A}_\lambda = \mathbb{Z} \times \{0\}$ (cf. Fig. 3 left).
- $\lambda \in (\frac{1}{2}, 1)$: For this parameter regime, every solution of (1.10) is bounded. Moreover, (1.10) is asymptotically stable, but not uniformly asymptotically stable on the whole time axis $\mathbb{Z}$. There exists no global attractor (cf. Fig. 3, middle).
- $\lambda > 1$: The unique bounded entire solution is 0, (1.10) is unstable, there is no global attractor, but the trivial solution is a repeller (cf. Fig. 3, right).

The parameter values $\lambda \in \{\frac{1}{2}, 1\}$ are critical: $\lambda = \frac{1}{2}$ yields a uniformly stable and $\lambda = 1$ a merely stable eqn. (1.10). In both situations, the number of bounded entire solutions to the difference eqn. (1.10) changes drastically. Furthermore, there is a loss of stability in two steps: From uniformly asymptotically stable to asymptotically stable, and finally to unstable, as λ increases through the values $\frac{1}{2}$ and 1. Hence, both values can be considered as bifurcation values, since the number of bounded entire solutions changes, as well as their stability properties.

The next example requires the state space to be at least two-dimensional, but also concatenates two hyperbolic autonomous problems. Here, for the first time, we use the notation ℓ^∞ for the space of all bounded sequences $\phi = (\phi_k)_{k\in\mathbb{Z}}$.

Example 1.5 (fold solution bifurcation). Consider the planar difference equation

$$x_{k+1} = f_k(x_k, \lambda) := \begin{pmatrix} b_k & 0 \\ 0 & c_k \end{pmatrix} x_k + \begin{pmatrix} 0 \\ (x_k^1)^2 \end{pmatrix} - \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.11}$$

with components $x_k = (x_k^1, x_k^2)$, a parameter $\lambda \in \mathbb{R}$ and asymptotically constant sequences

$$b_k := \begin{cases} 2, & k<0, \\ \frac{1}{2}, & k \geq 0, \end{cases} \qquad c_k := \begin{cases} \frac{1}{2}, & k<0, \\ 2, & k \geq 0. \end{cases} \tag{1.12}$$

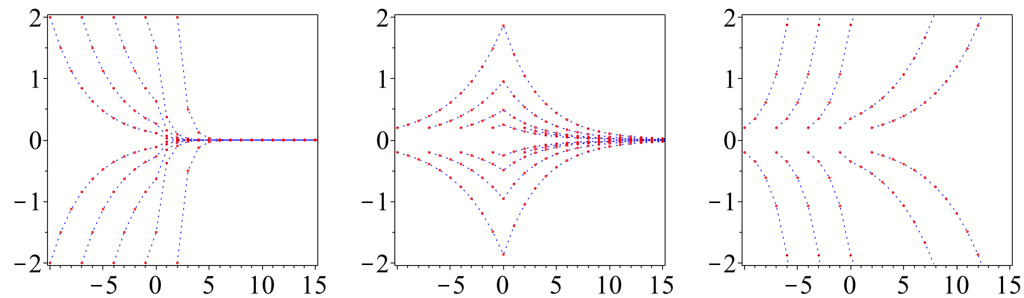

Figure 3: Solution sequences (dotted) of the linear difference eqn. (1.10) with $\lambda = \frac{1}{4}$ (left), $\lambda = \frac{3}{4}$ (middle), $\lambda = \frac{5}{4}$ (right)

Let $\varphi_\lambda(\cdot, 0, \eta)$ be the general solution to (1.11). Its first component φ_λ^1 reads as

$$\varphi_\lambda^1(k, 0, \eta) = 2^{-|k|}\eta_1 \quad \text{for all } k \in \mathbb{Z}, \tag{1.13}$$

while the variation of constants formula (cf., e.g., [1, p. 59]) can be used to deduce the asymptotic representation

$$\varphi_\lambda^2(k, 0, \eta) = \begin{cases} 2^k\left(\eta_2 + \frac{4}{7}\eta_1^2 - \lambda\right) + O(1), & k \to \infty, \\ \frac{1}{2^k}\left(\eta_2 - \frac{1}{2}\eta_1^2 + 2\lambda\right) + O(1), & k \to -\infty. \end{cases}$$

Therefore, the sequence $\varphi_\lambda(\cdot; 0, \eta)$ is bounded if and only if $\eta_2 = -\frac{4}{7}\eta_1^2 + \lambda$ and $\eta_2 = \frac{1}{2}\eta_1^2 - 2\lambda$ holds, i.e., $\eta_1^2 = \frac{7}{2}\lambda$, $\eta_2 = -\lambda$. From the first relation, one sees that there exist two bounded solutions if $\lambda > 0$, the trivial solution is the unique bounded solution for $\lambda = 0$ and there are no bounded solutions for parameters $\lambda < 0$; see Fig. 4 (left) for an illustration. For this reason, one can interpret $\lambda^* = 0$ as bifurcation value, since the number of bounded entire solutions increases from 0 to 2 as λ increases through 0.

This method of explicit solutions can also be applied to the related difference equation

$$x_{k+1} = f_k(x_k, \lambda) := \begin{pmatrix} b_k & 0 \\ 0 & c_k \end{pmatrix} x_k + \begin{pmatrix} 0 \\ (x_k^1)^3 \end{pmatrix} - \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.14}$$

with a cubic, rather than a quadratic nonlinearity as previously in (1.11). Again using the variation of constants formula (cf. [1, p. 59]), it is possible to derive that the crucial second component for the general solution $\varphi_\lambda(\cdot; 0, \eta)$ to (1.14) fulfills

$$\varphi_\lambda^2(k, 0, \eta) = \begin{cases} 2^k\left(\eta_2 + \frac{8}{15}\eta_1^3 - \lambda\right) + O(1), & k \to \infty, \\ \frac{1}{2^k}\left(\eta_2 - \frac{2}{15}\eta_1^3 + 2\lambda\right) + O(1), & k \to -\infty. \end{cases}$$

Since its first component is also given by (1.13), the sequence $\varphi_\lambda(\cdot; 0, \eta)$ is bounded if and only if $\eta_2 = -\frac{8}{15}\eta_1^3 + \lambda$ and $\eta_2 = \frac{2}{15}\eta_1^3 - 2\lambda$, which in turn is equivalent to

$$\eta_1 = \sqrt[3]{\tfrac{9}{2}\lambda}, \qquad \eta_2 = -\tfrac{7}{5}\lambda.$$

Hence, these particular initial values $\eta \in \mathbb{R}^2$ on the cusp shaped curve depicted in Fig. 4 (right) lead to bounded entire solutions of (1.14). The number of these solutions does not change and there is no bifurcation.

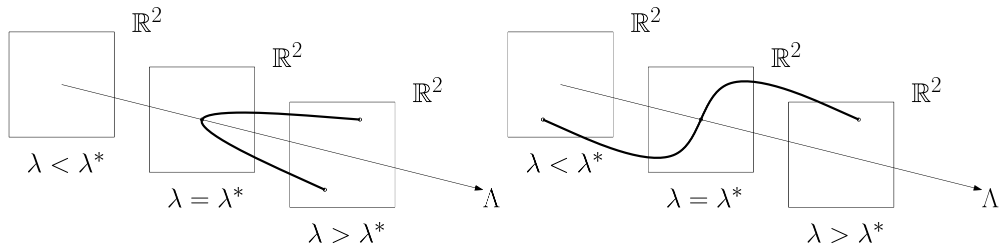

Figure 4: Bifurcation diagram for Exam. 1.5 with $\lambda^* = 0$:
Left (supercritical fold): Initial values $\eta \in \mathbb{R}^2$ which guarantee that an entire bounded solution $\varphi_\lambda(\cdot; 0, \eta)$ of (1.11) exists for different parameter values λ
Right (cusp): Initial values $\eta \in \mathbb{R}^2$ yielding an entire bounded solution $\varphi_\lambda(\cdot; 0, \eta)$ of (1.14) for different parameter values λ

To conclude this subsection, we observed in our Exams. 1.2–1.5 that parameter variation lead to a change in the number of bounded entire solutions for the respective nonautonomous difference equations — we denote this behavior as *solution bifurcation*. In the Exams. 1.2–1.4 we additionally observed a "topological" change in the attractor $\mathcal{A}_\lambda$ as follows:

- From a nonautonomous set $\mathcal{A}_\lambda$ consisting of singleton fibers, over the empty set to a repeller as λ was increasing through 1 in Exam. 1.2
- A continuous transition of $\mathcal{A}_\lambda$ from having singleton to interval fibers as λ increases through the value 1 in Exam. 1.3
- In Exam. 1.4 the trivial solution changes from being an attractor for $\lambda \in (0, \frac{1}{2})$ to a repeller for $\lambda > 1$, while there is no attractor for $\lambda \in (\frac{1}{2}, 1)$

One can understand such a phenomenon as *attractor bifurcation*; in our examples this also went hand in hand with a change in stability. In the other side, entire solutions can bifurcate while staying unstable (cf. Exam. 5.3).

Remarks

Our, by nature, biased survey on existing tools and concepts in bifurcation theory for nonautonomous difference equations relies on their more intuitive process formulation (1.4), rather than a skew-product formalism (cf., e.g. [56, Sect. 4]) to describe nonautonomous dynamics. Nevertheless, we will hint to further and alternative results throughout the remarks supplementing each section.

The particular form of attraction considered here is also denoted as *pullback attraction* and dates back to at least [52]. This kind of convergence guarantees for instance that limit sets become invariant and inherit various canonical properties from their autonomous special cases (cf. [70, p. 1ff, Chapt. 1]). Yet, we do not conceal the fact that pullback convergence strongly emphasizes backward behavior and lacks to capture forward dynamics (see the note [57] for a more detailed discussion). A comparison of different attractor notions can be found in [21]. Our repeller concept is taken from [81, p. 13, Def. 2.6]. Finally, as a general source for nonautonomous dynamical systems, we refer to the recent monograph [58] or the survey paper [56] with a focus on discrete dynamics.

One of the earliest contribution to nonautonomous bifurcations in time-discrete equations we are aware of, is [80], relying on attractor bifurcation. The alternative approach via solution bifurcations arose later from [71, 75] and both concepts were featured above in form of various examples.

2 Spectral theory

Typical examples of time-variant difference equations having the trivial solution are equations of perturbed motion (1.2). Their variational equation along nonconstant solutions $(\phi_k^*)_{k\in\mathbb{Z}}$ to (1.1) is intrinsically nonautonomous and given by $x_{k+1} = D_1 g(\phi_k^*, \lambda) x_k$. Investigating the stability properties as well as the behavior of ϕ^* under variation of λ requires an appropriate nonautonomous spectral and hyperbolicity notion.

For this purpose, we investigate finite-dimensional linear but nonautonomous difference equations. Precisely, suppose $A_k \in \mathbb{R}^{d\times d}$, $k \in \mathbb{Z}$, is a bounded sequence of matrices, and consider

$$\boxed{x_{k+1} = A_k x_k} \tag{L}$$

with the *transition matrix* (cf. (1.4))

$$\Phi(k,l) := \begin{cases} A_{k-1}\cdots A_l, & l < k, \\ \mathrm{id}, & k = l. \end{cases}$$

If the *coefficient matrices* A_k are invertible, we moreover set $\Phi(k,l) := A_k^{-1}\cdots A_{l-1}^{-1}$ for $k < l$. Typically, one obtains (L) as variational equation along a reference solution to (Δ) in ℓ^∞. For this reason our boundedness assumption on the sequence A_k is barely restrictive.

Differing from the autonomous situation, the k-dependent eigenvalues of A_k are of no use in stability investigations. Thereto, let us consider an example from [26, pp. 190–191, Ex. 4.17]:

Example 2.1. The 2×2-matrices

$$A_k := \tfrac{1}{2}\begin{pmatrix} 0 & 2+(-1)^k \\ 2-(-1)^k & 0 \end{pmatrix} \quad \text{for all } k\in\mathbb{Z}$$

have constant eigenvalues $\pm\frac{\sqrt{3}}{2}$ with modulus less than 1. Nevertheless, this does not allow us to deduce (asymptotic) stability of the nonautonomous difference eqn. (L) with A_k as coefficient matrices. Indeed, (L) has the transition matrix

$$\Phi(k,0) := \tfrac{1}{2}\begin{pmatrix} 2^{1-k}-(-2)^{1-k} & \left(\frac{3}{2}\right)^k - \left(-\frac{3}{2}\right)^k \\ 2^{-k}-(-2)^{-k} & \left(\frac{3}{2}\right)^k - \left(-\frac{3}{2}\right)^k \end{pmatrix} \quad \text{for all } k\geq 0$$

and therefore unbounded solutions. Hence, (L) is unstable, which is also indicated by the corresponding Floquet multipliers $\frac{1}{4}, \frac{9}{4}$ (cf. the following remark).

Remark 2.1 (periodic equations). Let $p\in\mathbb{N}$. A satisfying spectral theory exists for p-periodic difference eqns. (L), where we have $A_{k+p} = A_k$ for all $k\in\mathbb{Z}$. In this set-up, the eigenvalues to A_k have to be replaced by *Floquet multipliers*, i.e. eigenvalues of the *period map*

$$\Pi := \Phi(p,0) = A_{p-1}\cdots A_0.$$

This yields a classical perturbation and stability theory for periodic difference equations. In particular, with $\sigma(\Pi)\subseteq\mathbb{C}$ denoting the set of eigenvalues of Π, stability criteria read as follows:

(a) If $\sigma(\Pi) \subset B_1(0)$, then (L) is uniformly asymptotically stable.

(b) If there exists a $\lambda \in \sigma(\Pi)$ with $|\lambda| > 1$, then (L) is unstable.

Nevertheless, since our aim is to capture general time-dependencies beyond being periodic, one is still confronted with

Question 2: *If eigenvalues are of no use, what indicates stability in a nonautonomous set-up?*

A first and frequently met guess is the concept of *characteristic* or *Lyapunov exponents* yielding criteria for merely asymptotic stability. Yet, as classical examples show (cf. [61], or [70, p. 128, Ex. 3.4.1]), without the assumption of regularity this is not a robust stability notion. In fact, asymptotic stability of a linear equation can be destroyed by perturbations of order $o(x)$.

Keeping this in mind, we argue that a much more feasible concept is uniform asymptotic stability or its natural generalization in form of exponential dichotomies: Thereto, let $\mathbb{I}$ be the intersection of a real interval with the integers $\mathbb{Z}$, a so-called *discrete interval*, and define the *lapped interval* $\mathbb{I}' := \{k \in \mathbb{I} : \ k+1 \in \mathbb{I}\}$. Then an *invariant projector* for (L) is a sequence $P_k \in \mathbb{R}^{d\times d}$, $k \in \mathbb{I}$, of projections $P_k = P_k^2$ such that

$$A_{k+1}P_k = P_kA_k, \qquad A_k|_{N(P_k)} : N(P_k) \to N(P_{k+1}) \quad \text{is invertible for all } k \in \mathbb{I}'. \tag{2.1}$$

This assumption guarantees that the restriction $\Phi(k,l)|_{N(P_l)} : N(P_l) \to N(P_k)$, $l \leq k$, is well-defined and invertible with inverse denoted by $\Phi(l,k)$. Moreover, it ensures that the kernels $N(P_k)$, $k \in \mathbb{I}$, share the same dimension.

Definition 2.1 (exponential dichotomy). A linear difference eqn. (L) is said to admit an *exponential dichotomy* on $\mathbb{I}$ (for short, ED), if there exists an invariant projector P_k with complementary projector $Q_k := \mathrm{id} - P_k$ and reals $K \geq 1$, $\alpha \in (0,1)$ such that for $k, l \in \mathbb{I}$ one has

$$\|\Phi(k,l)P_l\| \leq K\alpha^{k-l} \quad \text{if } l \leq k, \qquad\qquad \|\Phi(k,l)Q_l\| \leq K\alpha^{l-k} \quad \text{if } k \leq l. \tag{2.2}$$

Remark 2.2. (1) In case $\mathbb{I}$ is unbounded above, then the *stable vector bundle*

$$\mathcal{V}^+ := \left\{(\kappa,\xi) \in \mathbb{I}\times\mathbb{R}^d : \lim_{k\to\infty}\Phi(k,\kappa)\xi = 0\right\} = \left\{(\kappa,\xi) \in \mathbb{I}\times\mathbb{R}^d : \sup_{\kappa\leq k}\|\Phi(k,\kappa)\xi\| < \infty\right\}$$

satisfies $R(P_k) = \mathcal{V}^+(k)$ for all $k \in \mathbb{I}$ and the range of P_k is uniquely determined. In a similar fashion, for $\mathbb{I}$ unbounded below the *unstable vector bundle*

$$\begin{aligned}\mathcal{V}^- :=& \left\{(\kappa,\xi) \in \mathbb{I}\times\mathbb{R}^d : \begin{array}{l}\text{there exists a solution } \phi = (\phi_k)_{k\in\mathbb{I}} \\ \text{with } \phi_\kappa = \xi \text{ and } \lim_{k\to-\infty}\phi_k = 0\end{array}\right\} \\ =& \left\{(\kappa,\xi) \in \mathbb{I}\times\mathbb{R}^d : \begin{array}{l}\text{there exists a solution } \phi = (\phi_k)_{k\in\mathbb{I}} \\ \text{with } \phi_\kappa = \xi \text{ and } \sup_{k\leq\kappa}\|\phi_k\| < \infty\end{array}\right\}\end{aligned}$$

allows the characterization $R(Q_k) = N(P_k) = \mathcal{V}^-(k)$ for all $k \in \mathbb{I}$ determining $R(Q_k)$. Therefore, for EDs on $\mathbb{Z}$ the invariant projector P_k is uniquely determined (cf. Fig. 5).

(2) The *Green's function* associated to an exponentially dichotomous eqn. (L) is given by

$$\Gamma_A(k,l) := \begin{cases}\Phi(k,l)P_l, & l \leq k, \\ -\Phi(k,l)Q_l, & k < l.\end{cases}$$

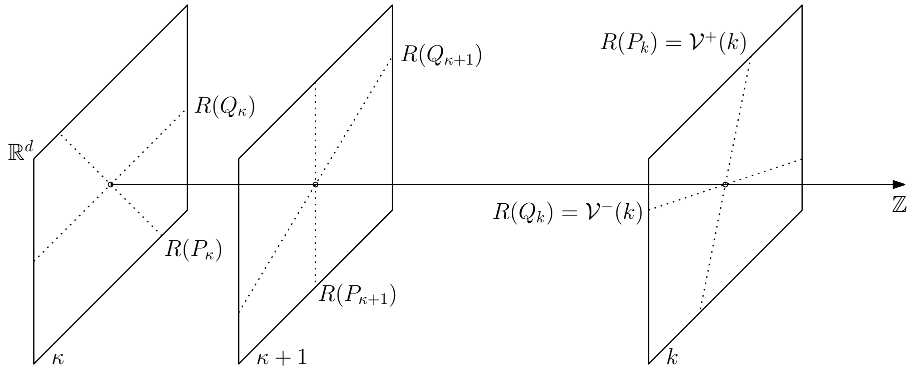

Figure 5: Fibers for the stable and unstable vector bundle $\mathcal{V}^+$ resp. $\mathcal{V}^-$ to (L)

Example 2.2. An autonomous difference equation

$$x_{k+1} = Ax_k \tag{2.3}$$

possesses an ED on $\mathbb{Z}$, if and only if it is hyperbolic. This means the coefficient matrix $A \in \mathbb{R}^{d\times d}$ has no eigenvalues on the complex unit circle $\mathbb{S}^1$. More precisely, for a spectral splitting

$$\sigma(A) = \sigma^+ \cup \sigma^-, \qquad \sigma^+ \subset B_1(0), \qquad \sigma^- \cap \bar{B}_1(0) = \emptyset$$

one can choose any growth rate α satisfying $\alpha \in \big(\max_{\lambda\in\sigma^+} |\lambda|, 1\big)$, $\alpha^{-1} \in \big(1, \min_{\lambda\in\sigma^-} |\lambda|\big)$. The corresponding invariant projector P_k is constant in k and, following [48, pp. 67ff], determined by the *Riesz projection*

$$P := -\frac{1}{2\pi i}\int_{\mathbb{S}^1} [A - z\,\mathrm{id}]^{-1} dz.$$

The associated vector bundles $\mathcal{V}^+$ and $\mathcal{V}^-$ have constant fibers given by the stable resp. unstable subspace of (2.3). A similar result holds for periodic difference equations by means of a spectral splitting for the period map Π.

It is well-known that hyperbolicity is a generic property in the class of autonomous or periodic linear difference equation, i.e. hyperbolic systems are open and dense among autonomous/periodic problems in $\mathbb{R}^d$. When passing to nonautonomous eqns. (L), it follows from the roughness theorem (cf. [34, p. 232, Thm. 7.6.7] or [77, p. 165 Thm. 3.6.5]) that an ED is merely an open property. However, the exponentially dichotomous systems are not dense, and consequently not generic in the class of difference eqns. (L) with bounded coefficient sequences. For an example we refer to [77, p. 149, Ex. 3.4.34].

2.1 Dichotomy spectrum

With the notion of an ED available, we now introduce an appropriate spectral notion. Indeed, there is an elegant connection between the dynamical notion of an ED and operator theory. Given $\gamma > 0$, the *scaled difference equation*

$$x_{k+1} = \gamma^{-1} A_k x_k \tag{L_γ}$$

admits an ED on $\mathbb{Z}$ if and only if the linear operator $S_\gamma : \ell^\infty \to \ell^\infty$,

$$(S_\gamma \phi)_k := \phi_{k+1} - \gamma^{-1} A_k \phi_k$$

has a bounded inverse, i.e. $S_\gamma \in GL(\ell^\infty)$ (cf. [34, p. 230, Thm. 7.6.5]). On this basis, it is convenient to introduce the following dichotomy spectra

- $\Sigma(A) := \{\gamma > 0 : (L_\gamma) \text{ does not have an ED on } \mathbb{Z}\} = \{\gamma > 0 : S_\gamma \notin GL(\ell^\infty)\}$,
- $\Sigma_\kappa^+(A) := \{\gamma > 0 : (L_\gamma) \text{ does not have an ED on } \mathbb{Z}_\kappa^+\}$,
- $\Sigma_\kappa^-(A) := \{\gamma > 0 : (L_\gamma) \text{ does not have an ED on } \mathbb{Z}_\kappa^-\}$

and the *dichotomy resolvent* $\rho(A) := \mathbb{R} \setminus \Sigma(A)$.

Next we indicate various properties of the above dichotomy spectra, which we denote by Σ for convenience. First, the boundedness of the sequence $(A_k)_{k\in\mathbb{Z}}$ carries over to the spectra Σ. As shown in [12, 13, 6] we know that every dichotomy spectrum $\Sigma \subseteq (0,\infty)$ is the disjoint union of $n \le d$ nonempty *spectral intervals* $\sigma_1, \dots, \sigma_n \subseteq (0,\infty)$, i.e. of the form

$$\Sigma = \bigcup_{i=1}^n \sigma_i, \qquad \sigma_1 = \begin{cases} [a_1, b_1] \\ \text{or} \\ (0, b_1] \end{cases}, \qquad \sigma_i = [a_i, b_i] \quad \text{for all } 2 \le i \le n$$

with reals $0 < a_1 \le b_1 < a_2 \le \ldots < b_n$. Here, σ_n is called *dominant spectral interval* of (L) and the additional assumption $A_k \in GL(\mathbb{R}^d)$ with $\sup_{k\in\mathbb{I}} \left\| A_k^{-1} \right\| < \infty$ ensures $\sigma_1 = [a_1, b_1]$. As illustrated in Exam. 2.4 below, the spectral intervals can be considered as a nonautonomous counterpart to the eigenvalue moduli for autonomous problems.

Remark 2.3 (properties of dichotomy spectra). (1) One has the inclusion $\Sigma_\kappa^\pm(A) \subseteq \Sigma(A)$.

(2) For invertible coefficient matrices A_k, $k \in \mathbb{I}'$, is not difficult to see that $\Sigma_\kappa^\pm(A)$ are independent of the starting time $\kappa \in \mathbb{Z}$.

(3) Both the dichotomy spectra $\Sigma_\kappa^\pm(A)$ are invariant under ℓ_0-perturbations, i.e. one has the relation $\Sigma_\kappa^\pm(A) = \Sigma_\kappa^\pm(A+B)$ for matrix sequences $B_k \in \mathbb{R}^{d\times d}$ satisfying $\lim_{k\to\pm\infty} B_k = 0$ (for this, see [13, Thm. 2.3]). As we will demonstrate in Exam. 2.7 this is not true for $\Sigma(A)$. Indeed, one has to impose additional assumptions (cf. [72, Thm. 4]) to ensure the invariance of $\Sigma(A)$ under additive perturbations decaying to 0.

(4) The dichotomy spectra $\Sigma_\kappa^\pm(A)$ and $\Sigma(A)$ depend upper-semicontinuously on perturbations of the coefficients $(A_k)_{k\in\mathbb{I}'}$ in the ℓ^∞-topology (cf. [68, Cor. 4] and [73, Cor. 3.24]). We again refer to Exam. 2.7 for an explicit example illustrating this. Hence, it is difficult to establish a smooth perturbation theory for spectral intervals like it is possible for eigenvalues in the autonomous or periodic case. Nevertheless, the set of discontinuity points for the set-valued functions $\Sigma^\pm, \Sigma$ is of first category (cf. [73, Rem. 4.26(1)]).

On the full integer line $\mathbb{I} = \mathbb{Z}$ the dichotomy spectrum $\Sigma(A)$ also provides a geometric insight into the dynamics of (L). This means we can establish a "nonautonomous linear algebra". Thereto, given a growth rate $\gamma > 0$, we define the

- *γ-stable vector bundle*

$$\mathcal{V}_\gamma^+ := \left\{ (\kappa, \xi) \in \mathbb{Z} \times \mathbb{R}^d : \sup_{\kappa \le k} |\Phi(k,\kappa)\xi| \, \gamma^{\kappa - k} < \infty \right\}$$

- and the γ-*unstable vector bundle*

$$\mathcal{V}_\gamma^- := \left\{(\kappa,\xi)\in\mathbb{Z}\times\mathbb{R}^d : \begin{array}{l}\text{there exists a solution } \phi=(\phi_k)_{k\in\mathbb{I}} \text{ with}\\ \phi_\kappa=\xi \text{ and } \sup_{k\le\kappa}\|\phi_k\|\gamma^{\kappa-k}<\infty\end{array}\right\},$$

whose fibers are linear subspaces of $\mathbb{R}^d$; in particular, it is $\mathcal{V}^\pm=\mathcal{V}_1^\pm$. Furthermore, choose rates $\gamma_i\in(b_i,a_{i+1})$ for $1\le i<n$, in case $\sigma_1=[a_1,b_1]$ choose $\gamma_0\in(0,a_1)$ and define

$$\mathcal{V}_0 := \begin{cases}\mathcal{V}_{\gamma_0}^+, & \text{if } \sigma_1=[a_1,b_1],\\ \mathbb{Z}\times\{0\}, & \text{else}\end{cases} \qquad \mathcal{V}_1 := \mathcal{V}_{\gamma_1}^+\cap\begin{cases}\mathcal{V}_{\gamma_0}^-, & \text{if } \sigma_1=[a_1,b_1],\\ \mathbb{Z}\times\mathbb{R}^d, & \text{else,}\end{cases}$$

$$\mathcal{V}_i := \mathcal{V}_{\gamma_i}^+\cap\mathcal{V}_{\gamma_{i-1}}^- \quad \text{for all } 1<i\le n, \qquad \mathcal{V}_{n+1} := \mathcal{V}_{\gamma_n}^-.$$

These vector bundles $\mathcal{V}_0,\dots,\mathcal{V}_{n+1}$ are forward invariant nonautonomous sets whose fibers possess constant dimension $\dim\mathcal{V}_i$, which is also called the *multiplicity* of the corresponding spectral interval σ_i for indices $1\le i\le n$. In addition, one has the *Whitney sum*

$$\mathbb{Z}\times\mathbb{R}^d=\mathcal{V}_0\oplus\ldots\oplus\mathcal{V}_{n+1},$$

which reduces to the well-known direct decomposition of the state space $\mathbb{R}^d$ into generalized eigenspaces for autonomous eqns. (2.3) (cf. [38, pp. 110ff, Chapt. 6]).

Referring to [12], the boundary points of the spectral intervals are *Bohl exponents*. For real sequences $(a_k)_{k\in\mathbb{I}}$ they are defined as limits

$$\beta_{\mathbb{I}}^+(a)=\lim_{j\to\infty}\sqrt[j]{\inf_{n\in\mathbb{I}}\prod_{k=n}^{n+j-1}|a_k|}, \qquad \beta_{\mathbb{I}}^+(a)=\lim_{j\to\infty}\sqrt[j]{\sup_{n\in\mathbb{I}}\prod_{k=n}^{n+j-1}|a_k|}. \tag{2.4}$$

Next we illustrate the dichotomy spectra using a combination of results from [12, Sect. 4], [9] and [68] to deduce the following examples in which $\mathbb{I}=\mathbb{Z}$.

Example 2.3 (scalar equations). For scalar eqns. $x_{k+1}=a_kx_k$ with coefficients $a_k\in\mathbb{R}\setminus\{0\}$ and $\sup_{k\in\mathbb{Z}}\left\{|a_k|,\left|a_k^{-1}\right|\right\}<\infty$, the dichotomy spectrum is related to Bohl exponents in terms of

$$\Sigma(A)=[\beta_{\mathbb{Z}}^-(a),\beta_{\mathbb{Z}}^+(a)]$$

(cf. [12, Thm. 4.6]). In particular, for the asymptotically constant special case $a_k=a^+$ for $k\ge\kappa$ and $a_k=a^-$ for $k<\kappa$, $a^+,a^-\in\mathbb{R}\setminus\{0\}$, one deduces

$$\beta_{\mathbb{Z}}^-(a)=\min\left\{\left|a^+\right|,\left|a^-\right|\right\}, \qquad \beta_{\mathbb{Z}}^+(a)=\max\left\{\left|a^+\right|,\left|a^-\right|\right\}.$$

Furthermore, the dichotomy spectra extend the autonomous and periodic situation studied in Exam. 2.2, where moduli of eigenvalues determine stability properties. More general, the spectral intervals measure exponential growth of solutions, but we do not want to conceal that they are useless to indicate rotational behavior.

Example 2.4 (autonomous equations). For autonomous linear difference eqns. (2.3) with coefficient matrix $A\in\mathbb{R}^{d\times d}$ one has

$$\Sigma(A)=\Sigma_\kappa^+(A)=\Sigma_\kappa^-(A)=\{|\lambda| : \lambda\in\sigma(A)\}\setminus\{0\},$$

which can be seen using Exam. 2.2 or [42, p. 6, Technical lemma 1].

Example 2.5 (periodic equations). For a p-periodic difference eqn. (L) with the period mapping $\Pi = \Phi(p, 0) \in GL(\mathbb{R}^d)$ one has (cf. [12, Thm. 4.1]

$$\Sigma(A) = \Sigma_\kappa^+(A) = \Sigma_\kappa^-(A) = \left\{ \sqrt[p]{|\lambda|} > 0 : \lambda \in \sigma(\Pi) \right\}.$$

Example 2.6 (asymptotically autonomous equations). If the coefficient sequence $A_k \in GL(\mathbb{R}^d)$ in eqn. (L) satisfies

$$A^+ := \lim_{k \to \infty} A_k, \qquad A^- := \lim_{k \to -\infty} A_k$$

with invertible limits $A^+, A^- \in \mathbb{R}^{d \times d}$, then one obtains the dichotomy spectra

$$\Sigma_\kappa^+(A) = \left\{ |\lambda| > 0 : \lambda \in \sigma(A^+) \right\}, \quad \Sigma_\kappa^-(A) = \left\{ |\lambda| > 0 : \lambda \in \sigma(A^-) \right\} \quad \text{for all } \kappa \in \mathbb{Z}.$$

To determine the spectrum $\Sigma(A)$ is more involved and we restrict to difference eqns. (L) with piecewise constant coefficient matrices $A_k = A^-$ for $k < \kappa$ and $A_k = A^+$ for $k \geq \kappa$. Thereto, given $\rho > 0$ denote by $N_\rho(A^-)$ (resp. $R_\rho(A^+)$) the kernel (resp. range) of the Riesz projection associated to the closed disk $\{z \in \mathbb{C} : |z| \leq \rho\}$. Let us suppose that

$$\sigma(A^+) \cup \sigma(A^-) = \{\lambda_1, \dots, \lambda_{2d}\},$$

where the $\lambda_i \in \mathbb{C}$ are ordered according to

$$|\lambda_1| = \dots = |\lambda_{n_1}| < |\lambda_{n_1+1}| = \dots = |\lambda_{n_k}| < |\lambda_{n_k+1}| = \dots = \left|\lambda_{n_{k+1}}\right|,$$

i.e., the indices $n_1 < \dots < n_k$ indicate one of the $k < 2d$ jumps in the moduli of the elements in the union $\sigma(A^+) \cup \sigma(A^-)$, and we set $n_{k+1} := 2d$. Moreover, choose indices $i_1 < \dots < i_{l-1}$ from $\{1, \dots, k\}$ such that $N_{|\lambda_{n_{i_m}}|}(A^-) \oplus R_{|\lambda_{n_{i_m}}|}(A^+) = \mathbb{R}^d$ holds for $0 \leq m < l$. This yields $l \leq d + 1$ and, with $i_0 = 0$, $i_l = k + 1$, such a piecewise constant difference eqn. (L) has the dichotomy spectrum (cf. [12, Thm. 4.8])

$$\Sigma(A) = \bigcup_{m=0}^{l-1} \left[\left|\lambda_{n_{i_m+1}}\right|, \left|\lambda_{n_{i_{m+1}}}\right| \right].$$

In our following considerations it will be particularly important to understand the nonhyperbolic situation $1 \in \Sigma(A)$, i.e. the case when (L) does not have an ED on $\mathbb{Z}$. Here, the following characterization of an ED turns out to be helpful.

Theorem 2.1. *Let $\kappa \in \mathbb{Z}$. A linear eqn. (L) has an ED on $\mathbb{Z}$, if and only if the following conditions are fulfilled:*

(i) *(L) has an ED on $\mathbb{Z}_\kappa^+$ with projector P_k^+, as well as an ED on $\mathbb{Z}_\kappa^-$ with projector P_k^-,*

(ii) $R(P_\kappa^+) \oplus N(P_\kappa^-) = \mathbb{R}^d$.

Proof. See [15, Lemma 2.4]. □

Our next example fulfills two purposes. First, it illustrates that the dichotomy spectrum can suddenly shrink under arbitrarily small perturbations (cf. Rem. 2.3(4)). Second, it shows that in contrast to the one-sided dichotomy spectra (cf. Rem. 2.3(3)), the spectrum on the whole integer axis is not invariant under perturbations decaying to 0.

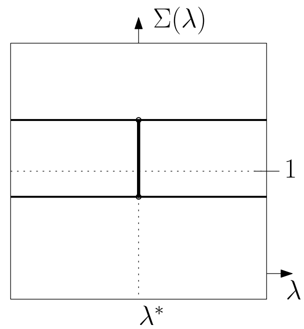

Figure 6: Dichotomy spectrum $\Sigma(\lambda)$ in Exam. 2.7 being discrete for parameters $\lambda \neq \lambda^* = 0$ and an interval for $\lambda = \lambda^*$

Example 2.7. Let $\kappa = 0$ and suppose that δ, ε are nonzero reals satisfying $|\varepsilon| < 1 < |\delta|$. We consider a λ-dependent difference eqn. $x_{k+1} = A_k^\lambda x_k$ with

$$A_k^\lambda := \begin{pmatrix} a_k & \lambda\varepsilon_k \\ 0 & a_k^{-1} \end{pmatrix}, \qquad a_k := \begin{cases} \delta, & k \geq 0, \\ \frac{1}{\delta}, & k < 0, \end{cases} \qquad \varepsilon_k := \begin{cases} \varepsilon^k, & k \geq 0, \\ 0, & k < 0 \end{cases}$$

and $\lambda \in \mathbb{R}$. Using Exam. 2.6 we easily deduce the dichotomy spectrum

$$\Sigma(A^0) = \left[\tfrac{1}{|\delta|}, |\delta|\right]$$

and consider the matrix sequence $\lambda\left(\begin{smallmatrix} 0 & \varepsilon_k \\ 0 & 0 \end{smallmatrix}\right)$, $k \in \mathbb{Z}$, as perturbation of $x_{k+1} = A_k^0 x_k$. Thanks to $\sup_{k\in\mathbb{Z}} |\lambda\varepsilon_k| = |\lambda|$ this perturbation can be made arbitrarily small. Moreover it decays to 0 even exponentially, but *does* effect the dichotomy spectrum $\Sigma(A^\lambda)$. This can be seen as follows: For rates $\gamma > 0$ the transition matrix Φ_γ of the scaled perturbed eqn. (L_γ) with $\lambda \neq 0$ reads as

$$\Phi_\gamma(k,0) = \gamma^{-k} \begin{pmatrix} \delta^k & \frac{\lambda\delta}{\delta^2-\varepsilon}\left(\delta^k - \left(\frac{\varepsilon}{\delta}\right)^k\right) \\ 0 & \delta^{-k} \end{pmatrix} \qquad \text{for all } k \geq 0$$

yielding the γ-stable resp. γ-unstable fibers

$$\mathcal{V}_\gamma^+(0) = \begin{cases} \mathbb{R}^2, & |\delta| \leq \gamma, \\ \mathbb{R}\begin{pmatrix} \delta\lambda \\ \varepsilon - \delta^2 \end{pmatrix}, & \frac{1}{|\delta|} \leq \gamma < |\delta|, \\ \{0\}, & \gamma < \frac{1}{|\delta|}, \end{cases} \qquad \mathcal{V}_\gamma^-(0) = \begin{cases} \{0\}, & |\delta| < \gamma, \\ \{0\} \times \mathbb{R}, & \frac{1}{|\delta|} < \gamma \leq |\delta|, \\ \mathbb{R}^2, & \gamma \leq \frac{1}{|\delta|}. \end{cases}$$

Hence, for values $\gamma \notin \left\{|\delta|, \frac{1}{|\delta|}\right\}$ we obtain the direct sum $\mathcal{V}_\gamma^+(0) \oplus \mathcal{V}_\gamma^-(0) = \mathbb{R}^2$ and Rem. 2.2(1) combined with Thm. 2.1 shows that (L_γ) admits an ED on the whole axis $\mathbb{Z}$. This manifests a change in the dichotomy spectrum under the above perturbations, since we can conclude

$$\Sigma(A^\lambda) = \begin{cases} \left\{\frac{1}{|\delta|}, |\delta|\right\}, & \lambda \neq 0, \\ \left[\frac{1}{|\delta|}, |\delta|\right], & \lambda = 0. \end{cases}$$

2.2 Fine structure

We already pointed out that an ED of the linear difference eqns. (L) or (L_γ) on $\mathbb{Z}$ can be characterized in terms of invertibility of the shift operators $S_1 \in L(\ell^\infty)$ resp. $S_\gamma \in L(\ell^\infty)$ with

$$(S_\gamma\phi)_k := \phi_{k+1} - \gamma^{-1}A_k\phi_k.$$

In the subsequent Sects. 4 and 5 it will be apparent that bifurcations in nonlinear problems (Δ_λ) can only occur in absence of an ED. Therefore, it is crucial to investigate different forms of non-invertibility for S_1. This gives rise to the following subsets of the dichotomy spectrum $\Sigma(A)$:

- The *point spectrum* $\Sigma_p(A) := \left\{\gamma > 0 : \dim S_\gamma^{-1}(\{0\}) > 0\right\}$
- the *surjectivity spectrum* $\Sigma_s(A) := \{\gamma > 0 : S_\gamma \text{ is not onto}\}$
- the *Fredholm spectra* $\Sigma_F(A) := \{\gamma > 0 : S_\gamma \text{ is not Fredholm}\}$ and

$$\Sigma_{F_0}(A) := \{\gamma > 0 : S_\gamma \text{ is not Fredholm or of nonzero index}\}.$$

It turns out that also the set-valued mappings $\Sigma_s, \Sigma_F, \Sigma_{F_0}$ are upper-semicontinuous on the set of linear eqns. (L) with bounded coefficient sequences (cf. [73, Cors. 4.21(c) and 4.26(c)]).

As illustrated below, these different spectra allow a classification of nonautonomous bifurcations already on a linear level. Moreover, one can deduce the following relations between them:

Corollary 2.2. *For every $\kappa \in \mathbb{Z}$ and coefficient sequences $A_k \in GL(\mathbb{R}^d)$, $k \in \mathbb{Z}$, one has*

$$\begin{array}{ccccccccc}
 & & & & \Sigma_p(A) & \subseteq & \Sigma_p(A)\cup\Sigma_s(A) & & \\
 & & & & & & \| & & \\
\Sigma_\kappa^+(A)\cup\Sigma_\kappa^-(A) & = & \Sigma_F(A) & \subseteq & \Sigma_{F_0}(A) & \subseteq & \Sigma(A) & = & \Sigma_s(A)\cup\Sigma_{F_0}(A) \\
 & & \cap\!| & & & & \| & & \\
\partial\Sigma(A) & \subseteq & \Sigma_s(A) & & \subseteq & & \Sigma_p(A)\cup\Sigma_{F_0}(A) & &
\end{array}$$

Proof. See [73, Cor. 4.31]. □

Remarks

A historically first reference for exponential dichotomies in discrete time might be [24], but we also refer to the more recent and approachable contribution [66] with applications to shadowing and Smale's theorem; a generalization to noninvertible difference equations is due to [47]. Further related results can be found in [84, 36, 2, 4], [34, pp. 229ff], [70, pp. 128ff]. The connection between structural stability and an exponential dichotomy was studied in [54, 55, 11]. An elegant technique to investigate dichotomies using operator theory was introduced in [10, 9] (see also [12, 13, 14]).

Our nonautonomous spectral theory based on dichotomies dates back to [85], who consider differential equations (and linear skew-product flows), while corresponding discrete time results can be found in [7, 6], whereas [82] features an interesting alternative approach via the Morse spectrum. The ideas from [9] were continued in [68, 72] and the fine structure of the dichotomy spectrum was investigated in [73]. Due to results from [67, 86] the scheme of inclusions from Cor. 2.2 and the overall spectral theory drastically simplifies for almost periodic difference equations (cf. [73, Cor. 4.34]).

As a word of caution: Despite the above examples, it is difficult to verify an exponential dichotomy rigorously or to compute dichotomy spectra analytically. For a numerical approach to such problems we refer to the work of Thorsten Hüls in [40, 41].

3 Continuation and stability

We argued above that the dichotomy spectrum yields the correct hyperbolicity notion in a nonautonomous context. In this section we are about to specify this statement.

First of all, nonetheless, the dichotomy spectrum is a crucial tool to determine stability properties of solutions. At the moment, it suffices to retreat to parameter-free difference equations

$$\boxed{x_{k+1} = f_k(x_k)} \tag{Δ}$$

with a smooth right-hand side $f_k : \mathbb{R}^d \to \mathbb{R}^d$. A solution $\phi^* = (\phi_k^*)_{k\in\mathbb{I}}$ of (Δ) is called *hyperbolic* (on $\mathbb{I}$), provided the variational equation

$$\boxed{x_{k+1} = Df_k(\phi_k^*)x_k} \tag{V}$$

has an ED on the discrete interval $\mathbb{I}$. If P_k denotes the corresponding invariant projector, then the constant dimension of $N(P_k)$, $k \in \mathbb{I}$, is called the *Morse index* of the solution ϕ^*. Indicating the number of unstable directions, it is a measure of instability for ϕ^*. We write Σ_κ^+, Σ for the associate dichotomy spectra and obtain the following stability criteria:

Proposition 3.1. *(a) If* $\max \Sigma_\kappa^+ < 1$*, then* ϕ *is asymptotically stable.*

(b) If $\max \Sigma < 1$*, then* ϕ *is uniformly asymptotically stable.*

Proof. See [75, Prop. 3.9]. □

Proposition 3.2. *If the dominant spectral interval* σ *of* Σ_κ^+ *fulfills* $\min \sigma > 1$*, then* ϕ *is unstable.*

Proof. See [75, Prop. 3.10(a)]. □

Referring to Exam. 2.4 and 2.5, both Prop. 3.1 and 3.2 canonically generalize the well-known stability conditions in an autonomous resp. periodic setting. Stability criteria for scalar difference equations in the nonhyperbolic situation $\Sigma_\kappa^+ = \{1\}$ can be found in [78, Prop. 5.4].

Throughout the remaining section we return to parameter-dependent difference eqns. (Δ_λ), suppose that the parameter space Λ is an open subset of a Banach space Y and denote the general solution to (Δ_λ) by φ_λ. This flexible parameter setting allows us to consider parametric perturbations, as well as perturbations of the right-hand side itself. We furthermore write $\Sigma(\lambda)$ for the associate dichotomy spectrum of the variational equation

$$\boxed{x_{k+1} = D_1 f_k(\phi_k^*, \lambda)x_k.} \tag{V_λ}$$

Our interest is centered around the behavior of a hyperbolic bounded solution $\phi^* = (\phi_k^*)_{k\in\mathbb{I}}$ to eqn. (Δ_{λ^*}) when the system parameter λ is varied. For discrete intervals $\mathbb{I}$ of the form $\mathbb{Z}_\kappa^\pm$ the situation is as follows: Since the variational eqn. (V_{λ^*}) along ϕ^* admits an ED on $\mathbb{I}$, there exists a corresponding so-called stable fiber bundle $\mathcal{W}_{\lambda^*}^+$ (for $\mathbb{I}$ unbounded above), as well as an unstable fiber bundle $\mathcal{W}_{\lambda^*}^-$ (for $\mathbb{I}$ unbounded below). Hence, a hyperbolic solution ϕ^* on a semiaxis is embedded into a whole family of forward resp. backward bounded solutions given by $\mathcal{W}_{\lambda^*}^\pm$.

More precisely, for every $\lambda \in \Lambda$ the *stable set* of ϕ^* is defined as

$$\hat{\mathcal{W}}_\lambda^+ := \Big\{(\kappa, \xi) \in \mathbb{I} \times \mathbb{R}^d : \varphi_\lambda(k; \kappa, \xi) - \phi_k^* \xrightarrow[k\to\infty]{} 0\Big\},$$

while the corresponding *unstable set* reads as

$$\hat{\mathcal{W}}_\lambda^- := \left\{ (\kappa,\xi) \in \mathbb{I} \times \mathbb{R}^d : \begin{array}{l} \text{there exists a solution } \phi = (\phi_k)_{k\in\mathbb{I}} \text{ of } (\Delta_\lambda) \\ \text{such that } \phi_\kappa = \xi \text{ and } \phi_k - \phi_k^* \xrightarrow[k\to-\infty]{} 0 \end{array} \right\},$$

where the interval $\mathbb{I}$ is assumed to be unbounded above resp. below. The local structure of these nonautonomous sets can be described as follows, which also yields a nonautonomous version of the stable manifold theorem — we speak of *invariant fiber bundles*:

Theorem 3.3 (continuation of solutions on half-lines). *Let $\lambda^* \in \Lambda$, $\kappa \in \mathbb{Z}$, $\mathbb{I} = \mathbb{Z}_\kappa^\pm$ and suppose that f_k is of class C^m, $m \geq 1$. If $\phi^* = (\phi_k^*)_{k\in\mathbb{I}}$ is a bounded solution of (Δ_{λ^*}) with*

$$1 \notin \Sigma_\kappa^\pm(\lambda^*) \quad \textit{and associated invariant projector } P_k^\pm, \tag{3.1}$$

then there exist $\rho, \varepsilon > 0$ and a unique C^m-function $w_\kappa^\pm : B_\rho(0,\lambda^*) \subseteq R(P_\kappa^\pm) \times \Lambda \to N(P_\kappa^\pm)$ such that for all $\lambda \in B_\rho(\lambda^*)$ the following holds:

$$\mathcal{W}_\lambda^\pm(\kappa) = \hat{\mathcal{W}}_\lambda^\pm(\kappa) \cap B_\varepsilon(\phi_\kappa^*) = \left\{ \phi_\kappa^* + \xi + w_\kappa^\pm(\xi,\lambda) \in \mathbb{R}^d : \xi \in B_\rho(0) \subseteq R(P_\kappa^\pm) \right\}.$$

Proof. We refer to [76, Cor. 2.23] for a simple proof on basis of the surjective implicit function theorem (cf. [90, p. 177, Thm. 4.H]). □

In a geometric language, for parameters λ near λ^*, Thm. 3.3 states that the stable/unstable set $\hat{\mathcal{W}}_\lambda^\pm$ is locally graph of a smooth function over the stable/unstable vector bundle $\mathcal{V}^\pm$ to (V_{λ^*}).

Now we tackle entire bounded solutions $\phi^* = (\phi_k^*)_{k\in\mathbb{Z}}$ to (Δ_{λ^*}), i.e. the situation of an ED on the whole axis $\mathbb{I} = \mathbb{Z}$. It turns out that ϕ^* allows a unique smooth continuation near λ^*. Moreover, also the saddle-point structure consisting of stable and unstable fiber bundles around ϕ^* persists under variation of λ:

Theorem 3.4 (continuation of entire solutions). *Let $\lambda^* \in \Lambda$ and suppose that f_k is of class C^m, $m \geq 1$. If $\phi^* = (\phi_k^*)_{k\in\mathbb{Z}}$ is an entire bounded solution of (Δ_{λ^*}) with*

$$1 \notin \Sigma(\lambda^*),$$

then there exists an open neighborhood $\Lambda_0 \subseteq \Lambda$ of λ^* and a unique C^m-function $\phi : \Lambda_0 \to \ell^\infty$ such that

(a) $\phi(\lambda^*) = \phi^*$,

(b) $\phi(\lambda)$ *is a bounded entire and hyperbolic solution of* (Δ_λ) *with the same Morse index as* ϕ^*.

Remark 3.1. This result naturally generalizes the autonomous situation: If x^* is a fixed point to $x_{k+1} = g(x_k, \lambda^*)$ with $1 \notin \sigma(D_1 g(x^*, \lambda^*))$, then x^* can be uniquely continued in the parameter λ; formally, only the spectrum $\sigma(\lambda^*)$ has to be replaced by the dichotomy spectrum $\Sigma(\lambda^*)$. Moreover, Thm. 3.4 ensures that such hyperbolic fixed points x^* persist as entire bounded solutions under ℓ^∞-small parametric perturbations.

Proof. See [76, Thm. 2.11]. □

Due to the C^m-dependence of the perturbed solution $\phi(\lambda) \in \ell^\infty$ on the parameter $\lambda \in \Lambda_0$, one can approximate $\phi(\lambda)$ using a finite Taylor series in λ. Here, a phenomenon typical for the nonautonomous theory occurs: Algebraic problems in an autonomous setting become dynamical problems, i.e., instead of solving algebraic equations, one has to find bounded solutions of a linear nonautonomous difference equation, in order to obtain the Taylor coefficients. We will make a similar observation in Sect. 6 when dealing with invariant fiber bundles.

In order to formulate this problem algorithmically, we have to introduce the following notations: With given $j, n \in \mathbb{N}$ we write

$$P_j^<(n) := \left\{ (N_1, \dots, N_j) \,\middle|\, \begin{array}{l} N_i \subseteq \{1, \dots, n\} \text{ and } N_i \neq \emptyset \text{ for } i \in \{1, \dots, j\}, \\ N_1 \cup \ldots \cup N_j = \{1, \dots, l\}, \\ N_i \cap N_k = \emptyset \text{ for } i \neq k,\ i, k \in \{1, \dots, j\}, \\ \max N_i < \max N_{i+1} \text{ for } i \in \{1, \dots, j-1\} \end{array} \right\}$$

for the set of *ordered partitions* of $\{1, \dots, n\}$ with *length* j and $\#N$ for the cardinality of a finite set $N \subset \mathbb{N}$. In case $N = \{n_1, \dots, n_k\} \subseteq \{1, \dots, n\}$ for $k \in \mathbb{N}$, $k \leq n$, we abbreviate $D^k g(x) x_N := D^k g(x) x_{n_1} \cdots x_{n_k}$ and

$$D^k g(x) x_1^{(k)} := D^k g(x) \underbrace{x_1 \cdots x_1}_{k \text{ times}}$$

for vectors $x, x_1, \dots, x_n \in \mathbb{R}^d$. Here, the mapping $g : \mathbb{R}^d \to \mathbb{R}^d$ is assumed to be n-times continuously differentiable with derivatives $D^k g(x) \in L_k(\mathbb{R}^d)$.[1)]

As a result of Taylor's theorem (cf., e.g., [90, p. 148, Thm. 4.A]) we can write

$$\phi(\lambda) = \phi^* + \sum_{n=1}^{m} \frac{1}{n!} D^n \phi(\lambda^*)(\lambda - \lambda^*)^{(n)} + R_m(\lambda) \tag{3.2}$$

with coefficients $D^n \phi(\lambda^*) \in L_n(Y, \mathbb{R}^d)$ and remainder R_m satisfying $\lim_{\lambda \to 0} \frac{R_m(\lambda)}{|\lambda|^m} = 0$. For $n = 1, \dots, m$ we apply the higher order chain rule (see [78, Lemma 4.1] for a reference in our notation) to the solution identity

$$\phi(\lambda)_{k+1} \equiv f_k(\phi(\lambda)_k, \lambda) \quad \text{on } \Lambda_0$$

for all $k \in \mathbb{Z}$. This yields the relation

$$\begin{aligned} D^n \phi(\lambda)_{k+1} y_1 \cdots y_n &= D_1 f_k(\phi(\lambda)_k, \lambda) D^n \phi(\lambda)_k y_1 \cdots y_n \\ &\quad + \sum_{j=2}^{n} \sum_{(N_1, \dots, N_j) \in P_j^<(n)} D^j f_k(\phi(\lambda)_k, \lambda) g_k^{\#N_1}(\lambda) y_{N_1} \cdots g_k^{\#N_j}(\lambda) y_{N_j} \end{aligned}$$

for all $y_1, \dots, y_n \in Y$, where we abbreviate $g_k^{\#N_1}(\lambda) := \frac{d^{\#N_1}(\phi(\lambda)_k, \lambda)}{d\lambda^{\#N_1}}$. Setting $\lambda = \lambda^*$ in this relation yields that the Taylor coefficients $D^n \phi(\lambda^*) \in L_n(Y, \ell^\infty) \cong \ell^\infty(L_n(Y, \mathbb{R}^d))$ fulfill the linearly inhomogeneous difference equation

$$X_{k+1} = D_1 f_k(\phi_k^*, \lambda^*) X_k + H_n(k) \tag*{$(I)_n$}$$

1) Given Banach spaces X, Y and $k \in \mathbb{N}_0$, we write $L_k(X, Y)$ for the linear space of all symmetric linear k-forms, and often abbreviate $L_k(X) := L_k(X, X)$, $L_0(X, Y) := Y$.

in $L_n(Y, \mathbb{R}^d)$, where the inhomogeneity $H_n : \mathbb{Z} \to L_n(Y, \mathbb{R}^d)$ reads as

$$H_n(k)y_1 \cdots y_n := \sum_{j=2}^{n} \sum_{(N_1,\dots,N_j)\in P_j^<(n)} D^j f_k(\phi_k^*, \lambda^*) g_k^{\#N_1}(\lambda^*) y_{N_1} \cdots g_k^{\#N_j}(\lambda^*) y_{N_j}$$

and in particular $H_1(k) = D_2 f_k(\phi_k^*, \lambda^*)$. Having these preparations at hand, we deduce

Corollary 3.5. *The coefficients $D^n\phi(\lambda^*) : \mathbb{Z} \to L_n(Y, \mathbb{R}^d)$, $1 \le n \le m$, in the Taylor expansion* (3.2) *can be determined recursively from the* Lyapunov-Perron sums

$$D^n\phi(\lambda^*)_k = \sum_{l\in\mathbb{Z}} \Gamma_{\lambda^*}(k, l+1) H_n(l) \quad \textit{for all } 1 \le n \le m,$$

where Γ_{λ^*} is the Green's function associated to (V_{λ^*}).

Proof. See [76, Cor. 2.20]. □

Example 3.1. (1) In the linear eqns. (1.7) or (1.8) from Exam. 1.1 resp. 1.2 it is possible to obtain the continuation $\phi(\lambda) \in \ell^\infty$ explicitly, where the latter example requires the hyperbolicity assumption $|\lambda| \ne 1$.

(2) With a bounded sequence $(a_k)_{k\in\mathbb{Z}}$ in $\mathbb{R}$ we consider the parametrically perturbed scalar difference equation

$$x_{k+1} = \frac{4}{\pi} \arctan x_k + \lambda a_k \tag{3.3}$$

depending on $\lambda \in \mathbb{R}$ and an arbitrarily smooth right-hand side. For the parameter value $\lambda = 0$ the eqn. (3.3) is autonomous and admits the three equilibria $x_0 = 0$ and $x_\pm = \pm 1$ (cf. Fig. 7 (left)). Next we investigate the behavior of these fixed points for values $\lambda \ne 0$.

- $x_0 = 0$: The variational eqn. at $\lambda^* = 0$ reads as $x_{k+1} = \frac{4}{\pi} x_k$ and is therefore unstable. Thus, Thm. 3.4 ensures that x_0 persists for small values of λ as an entire bounded solution $\phi_0(\lambda)$ to (3.3). Thanks to Cor. 3.5 its derivatives can be recursively computed as

$$D\phi_0(0)_k = -\left(\tfrac{4}{\pi}\right)^{k-1} \sum_{n=k}^{\infty} \left(\tfrac{\pi}{4}\right)^n a_n, \qquad D^2\phi_0(0)_k = 0,$$
$$D^3\phi_0(0)_k = 2\left(\tfrac{4}{\pi}\right)^{k} \sum_{n=k}^{\infty} \left(\tfrac{\pi}{4}\right)^n D\phi_0(0)_n^3, \qquad D^4\phi_0(0)_k = 0 \quad \text{for all } k \in \mathbb{Z}.$$

- $x_\pm = \pm 1$: Here the variational eqn. with $\lambda^* = 0$ becomes $x_{k+1} = \frac{2}{\pi} x_k$ and so $x_\pm$ are uniformly asymptotically stable. Their unique continuation $\phi_\pm(\lambda)$ for $\lambda \ne 0$ can be approximated as above.

See Fig. 7 for the solution portrait with different values of λ.

Remarks

Both the proof of Thm. 3.3 and 3.4 is essentially based on the implicit function theorem. Hence, using a quantitative version of this result (cf., for instance, [37]), one can obtain estimates for the size of the neighborhoods occurring. This, in turn, yields robustness results on the magnitude of parametric perturbations.

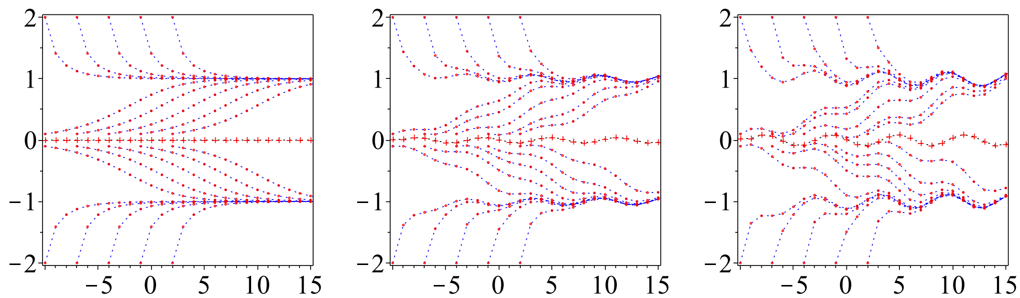

Figure 7: Solution sequences (dotted) of the difference eqn. (3.3) with $a_k = \sin(k)$ and $\lambda = 0$ (left), $\lambda = 0.05$ (middle) and $\lambda = 0.1$ (right). The solution $\phi_0(\lambda)$ is marked with crosses.

4 Attractor bifurcation

As already indicated in Subsect. 1.2, this survey distinguishes between attractor and solution bifurcation. Hence, an easy example for a bifurcation of an attractor was discussed already in Exam. 1.3. In the following, a general bifurcation pattern will be derived, which ensures that under certain conditions on Taylor coefficients of the right-hand side f_k in (Δ_λ), an attractor changes qualitatively under variation of the parameter. This allows us to extend autonomous bifurcation patterns of transcritical and pitchfork type. Although the attractor discussed in Exam. 1.3 was a *global* attractor, the bifurcation scenarios presented here only yield properties for *local* attractors.

The results of this section are due to Martin Rasmussen [80, 81] and are formulated for scalar equations. By means of the nonautonomous center manifold reduction presented in Sect. 6 they can be extended to higher-dimensional problems.

We retreat to one-parameter bifurcations, i.e. the parameter space $\Lambda \subseteq \mathbb{R}$ is open. Suppose that our difference eqn. (Δ_λ) is scalar ($d = 1$), the right-hand sides $f_k(\cdot, \lambda) : \mathbb{R} \to \mathbb{R}$, $k \in \mathbb{Z}$, $\lambda \in \Lambda$, are invertible and possesses a family (or a *branch*) $\phi(\lambda)$, $\lambda \in \Lambda$, of bounded entire solutions, i.e.

$$\phi(\lambda)_{k+1} \equiv f_k(\phi(\lambda)_k, \lambda) \quad \text{on } \mathbb{Z}.$$

Then the general solution $\varphi_\lambda(k; \kappa, \cdot)$ exists for all $k, \kappa \in \mathbb{Z}$. Given a fixed parameter $\lambda^* \in \Lambda$, the solution $\phi^* = \phi(\lambda^*)$ is called

- *all-time attractive*, if there exists a $\rho > 0$ such that

$$\lim_{k\to\infty} \sup_{n\in\mathbb{Z}} h(\varphi_{\lambda^*}(n + k, n, B_\rho(\phi^*_n)), \{\phi^*_{n+k}\}) = 0$$

  and the supremum of all such $\rho > 0$ is called the *attraction radius* $\rho^+_{\lambda^*}(\phi^*) > 0$,

- *all-time repulsive*, if there exists a $\rho > 0$ such that

$$\lim_{k\to\infty} \sup_{n\in\mathbb{Z}} h(\varphi_{\lambda^*}(n - k, n, B_\rho(\phi^*_n)), \{\phi^*_{n-k}\}) = 0$$

  and the supremum of all such $\rho > 0$ is called the *repulsion radius* $\rho^-_{\lambda^*}(\phi^*) > 0$. As shown in [81, p. 21, Def. 2.24] one can also define repulsivity for noninvertible equations.

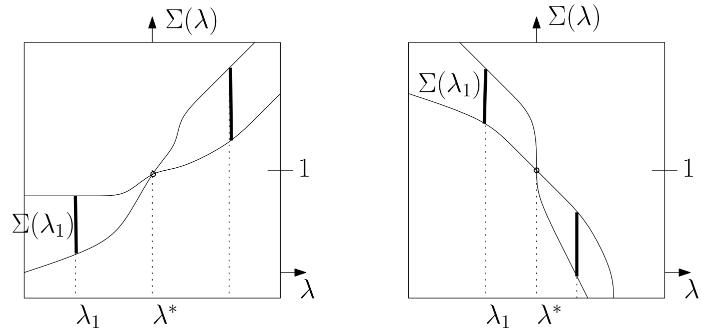

Figure 8: Dichotomy spectra $\Sigma(\lambda)$ under Hyp. 4.1 degenerating to a singleton $\{1\}$ in the limit $\lambda \to \lambda^*$ for functions $\gamma_\pm$ increasing (left) or $\gamma_\pm$ decreasing (right)

The transition matrix of the corresponding variational equation

$$x_{k+1} = D_1 f_k(\phi(\lambda)_k, \lambda) x_k \tag{4.1}$$

is denoted by $\Phi_\lambda(k,l) \in \mathbb{R}$. We assume

Hypothesis 4.1. Suppose that $D_1 f_k(\phi(\lambda)_k, \lambda) > 0$ for all $k \in \mathbb{Z}$, $\lambda \in \Lambda$ and that there exist constants $K \geq 1$ and functions $\gamma_+, \gamma_- : \Lambda \to (0,\infty)$ which are either both increasing or decreasing with $\lim_{\lambda\to\lambda^*} \gamma_+(\lambda) = \lim_{\lambda\to\lambda^*} \gamma_-(\lambda) = 1$ and

$$\Phi_\lambda(k,l) \leq K\gamma_+(\lambda)^{k-l}, \qquad \Phi_\lambda(l,k) \leq K\gamma_-(\lambda)^{l-k} \quad \text{for all } l \leq k,\ \lambda \in \Lambda.$$

Remark 4.1. The above assumptions have various consequences:

(1) They ensure the bounded growth estimate $|\Phi_\lambda(k,l)| \leq K\gamma(\lambda)^{|k-l|}$ for all $k,l \in \mathbb{Z}$ with the function $\gamma(\lambda) := \max\{\gamma_-(\lambda), \gamma_+(\lambda)\}$ and consequently (see [6, Thm. 3.5])

$$\Sigma(\lambda) \subseteq \begin{cases} \left[\gamma(\lambda)^{-1}, \gamma(\lambda)\right], & \gamma(\lambda) > 1, \\ \left[\gamma(\lambda), \gamma(\lambda)^{-1}\right], & \gamma(\lambda) \leq 1. \end{cases}$$

Thus, the dichotomy spectrum $\Sigma(\lambda)$ of the variational eqn. (4.1) fulfills (cf. Fig. 8)

$$\lim_{\lambda\to\lambda^*} h(\Sigma(\lambda), \{1\}) = 0.$$

(2) The variational difference eqn. (4.1) at $\lambda = \lambda^*$ does not have an ED on both semiaxes $\mathbb{Z}_\kappa^-$ and $\mathbb{Z}_\kappa^+$, and hyperbolicity condition (i) in Thm. 2.1 will be violated. Hence, the subsequent attractor bifurcations occur under the nonhyperbolicity condition

$$1 \in \Sigma_F(\lambda^*).$$

This yields a nonautonomous counterpart to the classical pitchfork bifurcation pattern:

Theorem 4.1 (transcritical attractor bifurcation, cf. [80]). *Suppose Hyp. 4.1 holds and that the right-hand side $f_k(\cdot,\lambda) : \mathbb{R} \to \mathbb{R}$, $k \in \mathbb{Z}$, $\lambda \in \Lambda$, is of class C^3. If there exists a $\lambda^* \in \Lambda$ such that*

$$-\infty < \liminf_{\lambda\to\lambda^*} \inf_{k\in\mathbb{Z}} D_1^2 f_k(\phi(\lambda)_k, \lambda) \leq \limsup_{\lambda\to\lambda^*} \sup_{k\in\mathbb{Z}} D_1^2 f_k(\phi(\lambda)_k, \lambda) < 0 \tag{4.2}$$

is satisfied and the remainder fulfills

$$\lim_{x\to 0} \sup_{\lambda\in(\lambda^*-|x|,\lambda^*+|x|)} \sup_{k\in\mathbb{Z}} x \int_0^1 (1-t)^2 D^3 f_k(\phi(\lambda)_k + tx, \lambda)\, dt = 0,$$

$$\limsup_{\lambda \to \lambda^*} \limsup_{x \to 0} \sup_{k \in \mathbb{Z}} \frac{Kx^2}{1 - \min\{\gamma_+(\lambda), \gamma_-(\lambda)^{-1}\}} \int_0^1 (1-t)^2 D^3 f_k(\phi(\lambda)_k + tx, \lambda)\, dt < 1,$$

then there exist $\lambda_- < \lambda^* < \lambda_+$ *so that the following statements hold:*

(a) *For increasing functions* γ_+, γ_- *the solution* $\phi(\lambda)$ *is all-time attractive for* $\lambda \in (\lambda_-, \lambda^*)$ *and all-time repulsive for* $\lambda \in (\lambda^*, \lambda_+)$. *At* $\lambda = \lambda^*$, *a difference eqn.* (Δ_λ) *admits an attractor bifurcation with*

$$\lim_{\lambda \nearrow \lambda^*} \rho_\lambda^+(\phi(\lambda)) = 0, \qquad \lim_{\lambda \searrow \lambda^*} \rho_\lambda^-(\phi(\lambda)) = 0. \tag{4.3}$$

(b) *For decreasing functions* γ_+, γ_- *the solution* $\phi(\lambda)$ *is all-time repulsive for* $\lambda \in (\lambda_-, \lambda^*)$ *and all-time attractive for* $\lambda \in (\lambda^*, \lambda_+)$. *At* $\lambda = \lambda^*$, *a difference eqn.* (Δ_λ) *admits an attractor bifurcation with*

$$\lim_{\lambda \searrow \lambda^*} \rho_\lambda^+(\phi(\lambda)) = 0, \qquad \lim_{\lambda \nearrow \lambda^*} \rho_\lambda^-(\phi(\lambda)) = 0.$$

Remark 4.2. (1) Dual assertions as in Thm. 4.1 hold under the assumption (cf. [80, Thm. 5.1])

$$0 < \liminf_{\lambda \to \lambda^*} \inf_{k \in \mathbb{Z}} D_1^2 f_k(\phi(\lambda)_k, \lambda) \le \limsup_{\lambda \to \lambda^*} \sup_{k \in \mathbb{Z}} D_1^2 f_k(\phi(\lambda)_k, \lambda) < \infty.$$

(2) A version of Thm. 4.1 can also be formulated for difference equations on a semiaxis $\mathbb{I} = \mathbb{Z}_\kappa^+$ or $\mathbb{I} = \mathbb{Z}_\kappa^-$, where the concepts of all-time attraction/repulsion has to be replaced by future resp. past attractivity and repulsivity (cf. [80]).

Proof. First of all, we pass over to the equation of perturbed motion

$$x_{k+1} = f_k(x + \phi(\lambda)_k, \lambda) - f_k(\phi(\lambda)_k, \lambda) =: F_k(x_k, \lambda) \tag{4.4}$$

which clearly has the trivial solution for all $\lambda \in \Lambda$. Thus, we can apply [80, Thm. 5.1] to the corresponding second order Taylor expansion

$$F_k(x, \lambda) = D_1 F_k(0, \lambda)x + \tfrac{1}{2} D_1^2 F_k(0, \lambda)x^2 + \int_0^1 \tfrac{(1-t)^2}{2!} D_1^3 F_k(tx, \lambda)\, dt\, x^3$$

(cf. [90, p. 148, Thm. 4.A]) of the right-hand side of (4.4). □

Example 4.1. Let $(a_k)_{k \in \mathbb{Z}}$ be a bounded real sequence with $0 < \inf_{k \in \mathbb{Z}} a_k$. We consider the scalar difference equation

$$x_{k+1} = (1 + \lambda a_k)(1 - e^{-x_k}). \tag{4.5}$$

If we denote the right-hand side of (4.5) by f_k, then for parameters λ in a neighborhood of zero the mapping $f_k(\cdot, \lambda)$ is invertible. It has the family of trivial solutions $\phi(\lambda) = 0$ for every $\lambda \in \mathbb{R}$ and the corresponding variational equation is

$$x_{k+1} = (1 + \lambda a_k)x_k. \tag{4.6}$$

The eqn. (4.5) does satisfy Hyp. 4.1 with $K = 1$ and increasing functions γ_+, γ_- given by

$$\gamma_+(\lambda) := 1 + \lambda \begin{cases} \sup_{k\in\mathbb{Z}} a_k, & \lambda \geq 0, \\ \inf_{k\in\mathbb{Z}} a_k, & \lambda < 0, \end{cases} \qquad \gamma_-(\lambda) := 1 + \lambda \begin{cases} \inf_{k\in\mathbb{Z}} a_k, & \lambda \geq 0, \\ \sup_{k\in\mathbb{Z}} a_k, & \lambda < 0 \end{cases}$$

and possesses the dichotomy spectrum $\Sigma(\lambda) = 1 + \lambda\left[\beta_{\mathbb{Z}}^-(a), \beta_{\mathbb{Z}}^+(a)\right]$. Evaluated at the critical parameter value $\lambda^* = 0$ we use

$$D_1^2 f_k(0, \lambda) = -(1 + \lambda a_k) \quad \text{for all } k \in \mathbb{Z},\ \lambda \in \mathbb{R}$$

to deduce $\lim_{\lambda\to\lambda^*} D_1^2 f_k(0, \lambda) < 0$. Finally, we compute

$$x \int_0^1 (1-t)^2 D^3 f_k(tx, \lambda)\, dt = (1 + \lambda a_k) \frac{x^2 - 2x + 2 - 2e^{-x}}{x^2}$$

and thus verify that the assumptions of Thm. 4.1(a) are satisfied. Hence, as λ is growing through the critical value $\lambda^* = 0$ the trivial solution to (4.5) becomes unstable and bifurcates in the sense of (4.3). An illustration is given in Fig. 9.

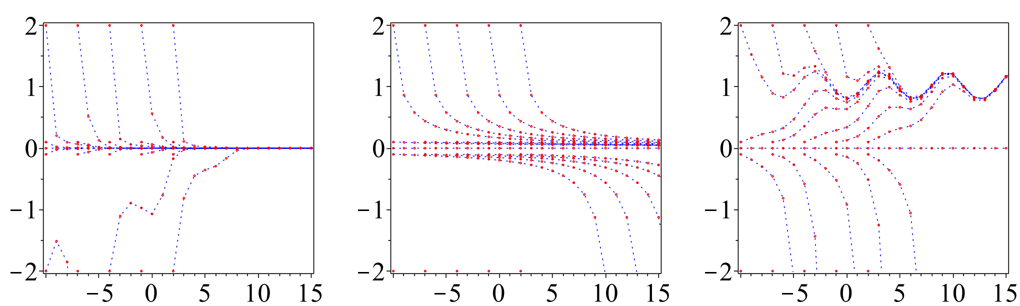

Figure 9: Solution sequences (dotted) of the difference eqn. (4.5) with $a_k = 2 + \sin(k)$ and $\lambda = -0.3$ (left), $\lambda = 0$ (middle) and $\lambda = 0.4$ (right), indicating a stability change

The nongeneric situation where (4.2) is violated, leads to

Theorem 4.2 (pitchfork attractor bifurcation, cf. [80]). *Suppose Hyp. 4.1 holds and that the right-hand side $f_k(\cdot, \lambda) : \mathbb{R} \to \mathbb{R}$ is of class C^4 with*

$$D_1^2 f_k(\phi(\lambda)_k, \lambda) = 0 \quad \textit{for all } k \in \mathbb{Z} \textit{ and } \lambda \in \Lambda.$$

If there exists a $\lambda^* \in \Lambda$ such that the following hypotheses hold:

- *Provided the functions γ_+ and γ_- are increasing, then*

$$-\infty < \liminf_{\lambda\to\lambda^*} \inf_{k\in\mathbb{Z}} D_1^3 f_k(\phi(\lambda)_k, \lambda) \leq \limsup_{\lambda\to\lambda^*} \sup_{k\in\mathbb{Z}} D_1^3 f_k(\phi(\lambda)_k, \lambda) < 0$$

- *Provided the functions γ_+ and γ_- are decreasing, then*

$$0 < \liminf_{\lambda\to\lambda^*} \inf_{k\in\mathbb{Z}} D_1^3 f_k(\phi(\lambda)_k, \lambda) \leq \limsup_{\lambda\to\lambda^*} \sup_{k\in\mathbb{Z}} D_1^3 f_k(\phi(\lambda)_k, \lambda) < \infty.$$

If the remainder fulfills

$$\lim_{x\to 0}\sup_{\lambda\in(\lambda^*-x^2,\lambda^*+x^2)}\sup_{k\in\mathbb{Z}} x\int_0^1 (1-t)^3 D^4 f_k(\phi(\lambda)_k+tx,\lambda)\,dt = 0,$$
$$\limsup_{\lambda\to\lambda^*}\limsup_{x\to 0}\sup_{k\in\mathbb{Z}} \frac{Kx^3}{1-\min\{\gamma_+(\lambda),\gamma_-(\lambda)^{-1}\}}\int_0^1 (1-t)^3 D^4 f_k(\phi(\lambda)_k+tx,\lambda)\,dt < 3,$$

then there exist $\lambda_- < \lambda^* < \lambda_+$ so that the following statements hold:

(a) For increasing functions γ_+,γ_- the solution $\phi(\lambda)$ is a local attractor for $\lambda\in(\lambda_-,\lambda^*)$ bifurcating into a nontrivial local attractor $\mathcal{A}_\lambda$, $\lambda\in(\lambda^*,\lambda^+)$, and fulfilling the limit relation

$$\lim_{\lambda\to\lambda^*}\sup_{k\in\mathbb{Z}} h(\mathcal{A}_\lambda(k),\{\phi(\lambda)_k\}) = 0.$$

(b) For decreasing functions γ_+,γ_- the solution $\phi(\lambda)$ is a local attractor for $\lambda\in(\lambda^*,\lambda^+)$ bifurcating into a nontrivial local attractor $\mathcal{A}_\lambda$, $\lambda\in(\lambda_-,\lambda^*)$, and fulfilling the limit relation

$$\lim_{\lambda\to\lambda^*}\sup_{k\in\mathbb{Z}} h(\mathcal{A}_\lambda(k),\{\phi(\lambda)_k\}) = 0.$$

Remark 4.3. (1) A dual version to Thm. 4.2 for pitchfork bifurcations into nontrivial repellers was given in [80, Thm. 6.1].

(2) The global invertibility of $f_k(\cdot,\lambda)$ is not given in various applications. Yet, without this restriction, $D_1 f_k(\phi(\lambda)_k,\lambda) > 0$ implies at least local invertibility. Under the assumptions

$$\sup_{k\in\mathbb{Z}} |D_1 f_k(\phi(\lambda)_k,\lambda)|^{-1} < \infty,$$
$$\lim_{x\to 0}\sup_{k\in\mathbb{Z}} [D_1 f_k(x+\phi(\lambda)_k,\lambda) - D_1 f_k(\phi(\lambda)_k,\lambda)] = 0 \quad \text{for all } \lambda\in\Lambda,$$

one can apply Thm. A.1 to construct a globally invertible extension of (4.4) to the whole state space $\mathbb{R}^d$. It coincides with the equation of perturbed motion (4.4) on a neighborhood of 0 which is uniform in $k\in\mathbb{Z}$. To this modification, [80, Thms. 5.1 and 6.1] are applicable yielding noninvertible versions of Thms. 4.1 and 4.2.

Proof. As in the proof of Thm. 4.1 the claim essentially follows from [80, Thm. 6.1]. □

Example 4.2. Let $(a_k)_{k\in\mathbb{Z}}$ be a bounded real sequence again with $0 < \inf_{k\in\mathbb{Z}} a_k$. We consider the scalar difference equation

$$x_{k+1} = (1+\lambda a_k)\arctan x_k. \tag{4.7}$$

For parameters λ in a neighborhood of $\lambda^* = 0$ its right-hand side is invertible. Moreover, it has the family of bounded entire solutions $\phi(\lambda) = 0$ for every $\lambda\in\mathbb{R}$; the corresponding variational equation coincides with (4.6) and Hyp. 4.1 holds with the same data as in Exam. 4.1. If we denote the right-hand side of (4.7) by f_k and investigate the critical parameter $\lambda^* = 0$, one obtains from

$$D_1^2 f_k(0,\lambda) = 0, \qquad D_1^3 f_k(0,\lambda) = -2-2\lambda a_k \quad \text{for all } k\in\mathbb{Z},\ \lambda\in\mathbb{R}$$

that $\lim_{\lambda\to\lambda^*} D_1^3 f_k(0,\lambda) < 0$. Furthermore, it is

$$x^3 \int_0^1 (1-t)^3 D^4 f_k(tx,\lambda) = 2(1+\lambda a_k)\left(x^2 - 3x + \frac{\arctan x}{x}\right)$$

and therefore the assumptions of Thm. 4.2(a) are satisfied. Hence, as λ is growing through the critical value $\lambda^* = 0$ the trivial solution to (4.7) becomes unstable and bifurcates into a nontrivial attractor. An illustration is given in Fig. 10.

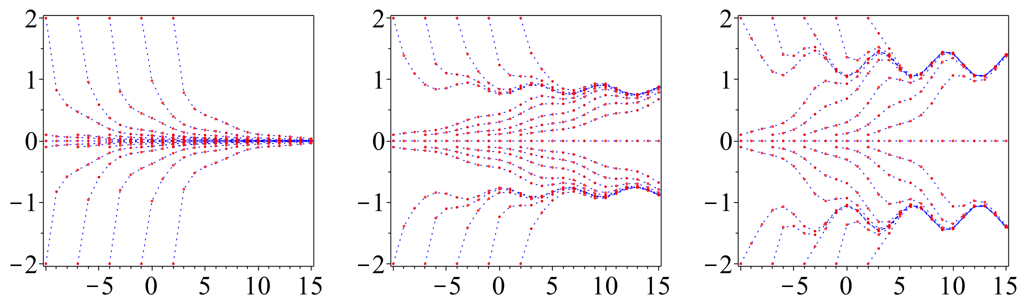

Figure 10: Solution sequences (dotted) of the difference eqn. (4.7) with $a_k = 2 + \sin k$ and $\lambda = -0.1$ (left), $\lambda = 0.1$ (middle) and $\lambda = 0.2$ (right).

Remarks

Both the attractor bifurcation Thms. 4.1 and 4.2 also hold for difference equations defined only on half-lines with appropriately modified attraction/repulsion notions (cf. [80, 81]). In addition, note that particularly autonomous transcritical and pitchfork bifurcations fit into the framework of Thms. 4.1 and 4.2 (for this, see [80, Ex. 5.3] resp. [80, Rem. 6.2(vi)]). However, it seems an autonomous fold bifurcation is not suitable for a formulation in terms of an attractor bifurcation.

We refer to [39] for a further detailed explicit bifurcation analysis in a population dynamics model involving the above results.

5 Solution bifurcation

In the previous section on attractor bifurcations, the first hyperbolicity condition (i) in Thm. 2.1, given by EDs on both semiaxes, has been violated. The present concept of solution bifurcation is based on the assumption that merely Thm. 2.1(ii) does not hold. Yet, the existence of EDs on both semiaxes enables us to employ an abstract analytical branching theory based on Fredholm linearizations. Rather than using dynamical systems tools, we consider difference equations as abstract equations in sequence spaces. In this sense our approach resembles Sect. 3, where invertibility is weakened to being Fredholm with index 0 now. Of particular importance in this functional analytical approach will be bounded sequences ℓ^∞ and limit-zero sequences ℓ_0.

We again restrict to one-parameter bifurcations, where $\Lambda \subseteq \mathbb{R}$ is open. The concept of *solution bifurcation* is classical in branching theory (cf., e.g., [25, 50, 90]) and understood as follows: Suppose that for a fixed parameter $\lambda^* \in \Lambda$, (Δ_{λ^*}) possesses an entire bounded reference solution

$\phi^* = \phi(\lambda^*)$. Then one says a difference eqn. (Δ_λ) undergoes a *bifurcation* at $\lambda = \lambda^*$ along ϕ^*, or ϕ^* *bifurcates* at λ^*, if there exist a convergent parameter sequence $(\lambda_n)_{n\in\mathbb{N}}$ in Λ with limit λ^* so that each (Δ_{λ_n}) has two distinct entire solutions $\phi^1_{\lambda_n}, \phi^2_{\lambda_n} \in \ell^\infty$ both satisfying

$$\lim_{n\to\infty} \phi^1_{\lambda_n} = \lim_{n\to\infty} \phi^2_{\lambda_n} = \phi^*.$$

In this context, λ^* is called *bifurcation value* for (Δ_λ). One speaks of a *subcritical* or a *supercritical bifurcation*, if the sequence $(\lambda_n)_{n\in\mathbb{N}}$ can be chosen according to $\lambda_n < \lambda^*$ or $\lambda_n > \lambda^*$, respectively.

This definition immediately yields a necessary condition for bifurcation:

Proposition 5.1. *Let $\lambda^* \in \Lambda$. If an entire bounded solution ϕ^* of (Δ_λ) bifurcates at λ^*, then ϕ^* is nonhyperbolic.*

Proof. If we suppose $1 \notin \Sigma(\lambda^*)$, then Thm. 3.4 yields neighborhoods $\Lambda_0 \subseteq \Lambda$ for λ^* and $U \subseteq \ell^\infty$ for ϕ^*, so that (Δ_λ) has a unique entire solution $\phi(\lambda) \in U$ for all $\lambda \in \Lambda_0$. Hence, ϕ^* cannot bifurcate at λ^*. □

Consequently, in order to ensure nonhyperbolicity $1 \in \Sigma(\lambda^*)$, we now make the following crucial and standing assumption:

Hypothesis 5.1. Let $\kappa \in \mathbb{Z}$. Suppose (Δ_{λ^*}) has an ED both on $\mathbb{Z}^+_\kappa$ (with projector P^+_k) and on $\mathbb{Z}^-_\kappa$ (with projector P^-_k) such that there exist nonzero $\xi_1 \in \mathbb{R}^d$, $\xi_1' \in \mathbb{R}^d$ satisfying

$$R(P^+_\kappa) \cap N(P^-_\kappa) = \mathbb{R}\xi_1, \qquad (R(P^+_\kappa) + N(P^-_\kappa))^\perp = \mathbb{R}\xi_1'. \tag{5.1}$$

Remark 5.1. (1) One has the orthogonality relation $\xi_1 \perp \xi_1'$.

(2) Note that Hyp. 5.1 cannot hold for the trivial projector $P^+_\kappa = \mathrm{id}$. Hence, $\Sigma^+_\kappa(\lambda^*)$ has a spectral interval in $(1, \infty)$ and Prop. 3.2 guarantees that the solution ϕ^* is unstable. It is also impossible to fulfill (5.1) for the zero projector $P^+_\kappa = 0$.

(3) In order to satisfy Hyp. 5.1 one needs state spaces of dimension $d > 1$. Otherwise the only possible projections are $P^\pm_k \in \{0, 1\}$ and for them (5.1) cannot hold.

(4) The assumption (5.1) guarantees that the point spectrum fulfills $1 \in \Sigma_p(\lambda^*)$. Moreover, we have EDs on both semiaxes and therefore it is (see [73, Prop. 4.9])

$$1 \in \Sigma(\lambda^*) \setminus \Sigma_{F_0}(\lambda^*).$$

(5) Under Hyp. 5.1 the variational eqn. (V_λ) is intrinsically nonautonomous: Indeed, if (V_λ) is almost periodic (or autonomous or periodic), then an ED on a semiaxis extends to the whole integer axis (cf. [86, Thm. 2]) and the reference solution $\phi = (\phi^*_k)_{k\in\mathbb{Z}}$ becomes hyperbolic. For this reason the following bifurcation scenarios cannot occur for almost periodic equations.

From a functional-analytical perspective the above Hyp. 5.1 implies Fredholm properties and, thus, allows to employ a Lyapunov-Schmidt reduction technique (see, for example, [50, 90]). Here, Hyp. 5.1 enables a geometrical insight into the following abstract bifurcation results using *invariant fiber bundles*, i.e., nonautonomous counterparts to invariant manifolds: Since the variational eqn. (V_λ) has an ED on $\mathbb{Z}^+_\kappa$, there exists a stable fiber bundle $\phi^* + \mathcal{W}^+_\lambda$ consisting of all solutions to (Δ_λ) approaching ϕ^* in forward time. In particular, $\mathcal{W}^+_\lambda$ is locally a graph over the stable vector bundle $\mathcal{V}^+$. Analogously, an ED on the negative half line $\mathbb{Z}^-_\kappa$ guarantees an unstable fiber bundle $\phi^* + \mathcal{W}_\lambda^-$ consisting of solutions decaying to ϕ^* in backward time (cf. Thm. 3.3). Then bounded entire solutions to (Δ_λ) are contained in the set $(\phi^* + \mathcal{W}_\lambda^+) \cap (\phi^* + \mathcal{W}_\lambda^-)$. One concludes that the intersection of the fibers

$$S_\lambda := \phi_\kappa^* + \mathcal{W}_\lambda^+(\kappa) \cap \phi_\kappa^* + \mathcal{W}_\lambda^-(\kappa) \subseteq \mathbb{R}^d$$

yields initial values (at initial time $k = \kappa$) for bounded entire solutions (see Fig. 11).

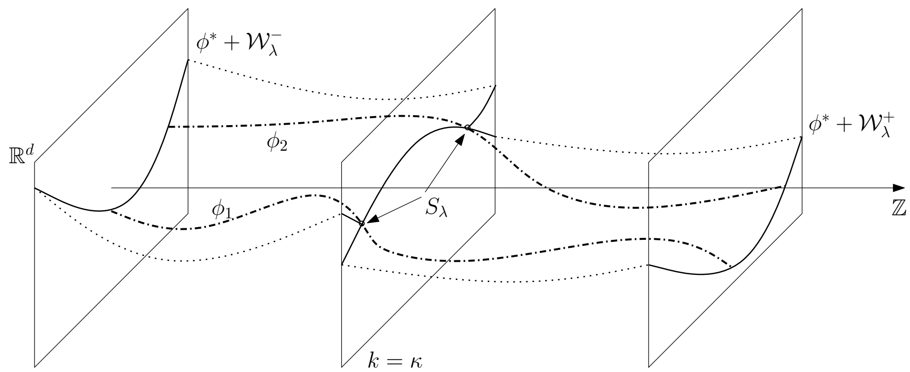

Figure 11: Intersection $S_\lambda \subseteq \mathbb{R}^d$ of the stable fiber bundle $\phi^* + \mathcal{W}_\lambda^+ \subseteq \mathbb{Z}_\kappa^+ \times \mathbb{R}^d$ with the unstable fiber bundle $\phi^* + \mathcal{W}_\lambda^- \subseteq \mathbb{Z}_\kappa^- \times \mathbb{R}^d$ at time $k = \kappa$ yields two bounded entire solutions ϕ_1, ϕ_2 to eqn. (Δ_λ) indicated as dotted dashed lines

5.1 Fold bifurcation

At first we study a fold bifurcation scenario already encountered in Exam. 1.5.

Thereto, we interpret $\mathbb{R}^d$ as Euclidean space equipped with inner product $\langle x, y\rangle = \sum_{n=1}^d x_n y_n$.

Theorem 5.2 (fold solution bifurcation). *Suppose Hyp. 5.1 holds and that the right-hand side f_k, $k \in \mathbb{Z}$, is of class C^m, $m \geq 2$. If*

$$g_{01} := \sum_{j\in\mathbb{Z}} \langle \Phi_{\lambda^*}(\kappa, j+1)^T \xi_1', D_2 f_j(\phi_j^*, \lambda^*)\rangle \neq 0,$$

then there exists a real $\rho > 0$, open convex neighborhoods $U \subseteq \ell^\infty$ of ϕ^*, $\Lambda_0 \subseteq \Lambda$ of λ^* and C^m-functions $\phi : (-\rho, \rho) \to U$, $\lambda : (-\rho, \rho) \to \Lambda_0$ such that

(a) $\phi(0) = \phi^*$, $\lambda(0) = \lambda^*$ and $\dot\phi(0) = \Phi_{\lambda^*}(\cdot, \kappa)\xi_1$, $\dot\lambda(0) = 0$,

(b) each $\phi(s)$ is an entire solution of $(\Delta_{\lambda(s)})$ in ℓ^∞.

Moreover, under the additional assumption

$$g_{20} := \sum_{j\in\mathbb{Z}} \langle \Phi_{\lambda^*}(\kappa, j+1)^T \xi_1', D_1^2 f_j(\phi_j^*, \lambda^*)[\Phi_{\lambda^*}(j, \kappa)\xi_1]^2\rangle \neq 0,$$

the solution $\phi^* \in \ell^\infty$ of (Δ_{λ^*}) bifurcates at λ^*, one has $\ddot\lambda(0) = -\frac{g_{20}}{g_{01}}$ and the following holds locally in $U \times \Lambda_0$:

(c) Subcritical case*: If $g_{20}/g_{01} > 0$, then (Δ_λ) has no entire solution in ℓ^∞ for $\lambda > \lambda^*$, ϕ^* is the unique entire solution of (Δ_{λ^*}) in ℓ^∞ and (Δ_λ) has exactly two distinct entire bounded solutions for $\lambda < \lambda^*$.*

(d) Supercritical case*: If $g_{20}/g_{01} < 0$, then (Δ_λ) has no entire solution in ℓ^∞ for $\lambda < \lambda^*$, ϕ^* is the unique entire solution of (Δ_{λ^*}) in ℓ^∞ and (Δ_λ) has exactly two distinct entire bounded solutions for $\lambda > \lambda^*$.*

Proof. See [71, Thm. 2.13]. □

Example 5.1 (fold solution bifurcation). We return to eqn. (1.11) studied in Exam. 1.5 and verify that its assertion can be deduced on the basis of Thm. 5.2 as well. First, the variational equation for (1.11) corresponding to the trivial solution and the critical parameter $\lambda^* = 0$ reads as

$$x_{k+1} = D_1 f_k(0,0)x_k := \begin{pmatrix} b_k & 0 \\ 0 & c_k \end{pmatrix} x_k$$

with the sequences b_k, c_k given in (1.12). It admits an ED on $\mathbb{Z}_0^+$, as well as on $\mathbb{Z}_0^-$ with corresponding invariant projectors $P_k^+ \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $P_k^- \equiv \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. This yields

$$R(P_0^+) \cap N(P_0^-) = \mathbb{R}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad R(P_0^+) + N(P_0^-) = \mathbb{R}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and thus condition (ii) of Thm. 2.1 is violated. Hence, the trivial solution to (1.11) for $\lambda = 0$ is not hyperbolic. On the other hand, Hyp. 5.1 holds with $\xi_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, $\xi_1' = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ and $\kappa = 0$. Therefore, we can compute

$$g_{01} = -\sum_{j\in\mathbb{Z}} \left(\tfrac{1}{2}\right)^{|j+1|} = -3, \qquad g_{20} = \tfrac{12}{7}$$

and Thm. 5.2 yields that the bounded solutions to (1.11) exhibit a supercritical fold bifurcation. This corresponds to the explicitly computed results from Exam. 1.5.

5.2 Crossing-curve bifurcation

Further prototype bifurcation patterns for equations possessing a trivial branch of solutions, are of transcritical and pitchfork type. In this context it is clear that any branch $\phi(\lambda)$ of solutions to (Δ_λ) can be transformed into the trivial one, as long as $\phi(\lambda)$ is known beforehand. The following result does not require such global information and contains pitchfork and transcritical bifurcations as special cases:

Theorem 5.3 (crossing curve solution bifurcation). *Suppose Hyp. 5.1 holds and that the right-hand side f_k, $k \in \mathbb{Z}$, is of class C^m, $m \geq 2$. If*

$$D_2 f_k(\phi_k^*, \lambda^*) \equiv 0 \quad \text{on } \mathbb{Z}, \qquad g_{02} := \sum_{j\in\mathbb{Z}} \langle \Phi_{\lambda^*}(\kappa, j+1)^T \xi_1', D_2^2 f_j(\phi_j^*, \lambda^*)\rangle = 0 \tag{5.2}$$

and the transversality condition

$$g_{11} := \sum_{j\in\mathbb{Z}} \langle \Phi_{\lambda^*}(\kappa, j+1)^T \xi_1', D_1 D_2 f_j(\phi_j^*, \lambda^*)\Phi_{\lambda^*}(j,\kappa)\xi_1\rangle \neq 0 \tag{5.3}$$

hold, then the entire solution ϕ^* of (Δ_{λ^*}) bifurcates at λ^*. In detail, there exist open convex neighborhoods $S \subseteq \mathbb{R}$ of 0, $U_1 \times U_2 \subseteq \ell^\infty \times \Lambda$ of (ϕ^*, λ^*) and C^{m-1}-curves $\gamma_1, \gamma_2 : S \to U_1 \times U_2$ with the following properties:

(a) The set of bounded entire solutions for (Δ_λ) in the neighborhood U_1 is given by the intersection $(\gamma_1(S) \cup \gamma_2(S)) \cap \ell^\infty \times \{\lambda\}$ (see Fig. 12).

(b) $\gamma_1(s) = (\gamma(s), \lambda^* + s)$ with $\gamma_1(0) = (\phi^*, \lambda^*)$, $\dot{\gamma}(0) = 0$ and

$$\gamma_2(0) = \begin{pmatrix} \phi^* \\ \lambda^* \end{pmatrix}, \qquad \dot{\gamma}_2(0) = \begin{pmatrix} \Phi_{\lambda^*}(\cdot, \kappa)\xi_1 \\ -\frac{g_{20}}{2g_{11}} \end{pmatrix}$$

where $g_{20} := \sum_{j \in \mathbb{Z}} \langle \Phi_{\lambda^*}(\kappa, j+1)^T \xi_1', D_1^2 f_j(\phi_j^*, \lambda^*)[\Phi_{\lambda^*}(j, \kappa)\xi_1]^2 \rangle$.

Remark 5.2. If the entire solution ϕ^* is embedded into a branch of trivial solutions to (Δ_λ), then (5.2) is automatically fulfilled and γ_1 resp. γ represents the zero branch. In this sense, Thm. 5.3 generalizes [71, Thm. 3.14 and Cors. 3.15, 3.16]. Moreover, the direction of the crossing curve bifurcation from Thm. 5.3 is given by the coefficient $\frac{g_{20}}{2g_{11}}$:

(1) For $g_{20} \neq 0$ there are locally exactly two entire solutions to (Δ_λ) in ℓ^∞ for $\lambda \neq \lambda^*$. This yields a transcritical pattern (see Fig. 12 (left)).

(2) In the degenerate case $g_{20} = 0$ we assume $m \geq 3$ and a higher order condition

$$\begin{aligned} g_{30} := &\sum_{j \in \mathbb{Z}} \langle \Phi_{\lambda^*}(\kappa, j+1)^T \xi_1', D_2^3 f(s, \phi_j^*, \lambda^*)[\Phi_{\lambda^*}(j, \kappa)\xi_1]^3 \rangle \\ &- 3 \sum_{j \in \mathbb{Z}} \langle \Phi_{\lambda^*}(\kappa, j+1)^T \xi_1', D_2^2 f(s, \phi_j^*, \lambda^*) \Phi_{\lambda^*}(j, \kappa)\xi_1 \overline{D_2^2 f(s, \phi_j^*, \lambda^*)[\Phi_{\lambda^*}(j, \kappa)\xi_1]^2} \rangle, \end{aligned}$$

yielding a pitchfork pattern (see Fig. 12 (right)):

(a) For $g_{30}/g_{11} < 0$ (*supercritical case*) there is a unique entire solution of (Δ_λ) in ℓ^∞ for parameters $\lambda \leq \lambda^*$ and (Δ_λ) has exactly three entire solutions in ℓ^∞ for $\lambda > \lambda^*$.

(b) For $g_{30}/g_{11} > 0$ (*subcritical case*) there is a unique entire solution of (Δ_λ) in ℓ^∞ for parameters $\lambda \geq \lambda^*$ and (Δ_λ) has exactly three entire solutions in ℓ^∞ for $\lambda < \lambda^*$.

Here, given a sequence $\psi = (\psi_k)_{k \in \mathbb{Z}}$, we use the notation

$$\overline{\psi_k} := \begin{cases} \Phi_{\lambda^*}(k, \kappa) P_\kappa^+ \xi_\kappa^* + \sum_{j=\kappa}^\infty \Gamma_{P^+}(k, j+1)\psi_j, & k \geq \kappa, \\ \Phi_{\lambda^*}(k, \kappa)[\mathrm{id} - P_\kappa^-]\xi_\kappa^* + \sum_{j=-\infty}^{\kappa-1} \Gamma_{P^-}(k, j+1)\psi_j, & k < \kappa, \end{cases} \tag{5.4}$$

$$\xi_\kappa^* := [P_\kappa^+ + P_\kappa^- - \mathrm{id}]^\dagger \left(\sum_{j=-\infty}^{\kappa-1} \Phi_{\lambda^*}(\kappa, j+1) P_j^- \psi_j + \sum_{j=\kappa}^\infty \Phi_{\lambda^*}(\kappa, j+1)[\mathrm{id} - P_j^+]\psi_j \right)$$

and $[P_\kappa^+ + P_\kappa^- - \mathrm{id}]^\dagger \in \mathbb{R}^{d \times d}$ denotes the pseudo-inverse to $P_\kappa^+ + P_\kappa^- - \mathrm{id}$ (cf., e.g., [22]).

Proof. This is a discrete time counterpart to [74, Thm. 4.1] and can be shown along the same lines. □

The following example illustrates Thm. 5.3:

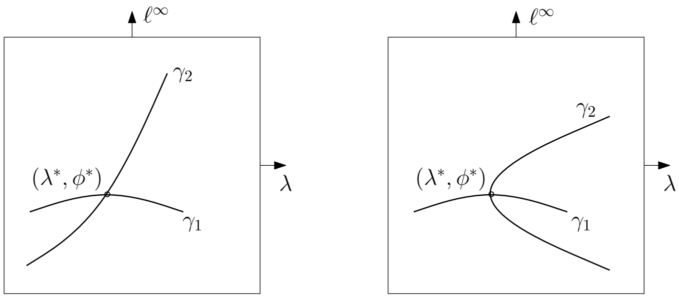

Figure 12: Schematic crossing curve bifurcation from Thm. 5.3: Generic case of a transcritical situation (left) and the degenerate case of a supercritical pitchfork situation (right)

Example 5.2 (transcritical solution bifurcation). Consider the nonlinear difference equation

$$x_{k+1} = f_k(x_k, \lambda) := \begin{pmatrix} b_k & 0 \\ \lambda & c_k \end{pmatrix} x_k + \begin{pmatrix} 0 \\ (x_k^1)^2 \end{pmatrix} \tag{5.5}$$

depending on a bifurcation parameter $\lambda \in \mathbb{R}$ and sequences b_k, c_k defined in (1.12). As in the previous examples, our assumptions hold with $\lambda^* = 0$ and

$$g_{11} = \tfrac{4}{3} \neq 0, \qquad\qquad g_{20} = \tfrac{12}{7} \neq 0.$$

Hence, Rem. 5.2(1) can be applied in order to see that the trivial solution of (5.5) has a transcritical bifurcation at $\lambda = 0$. This bifurcation can be described quantitatively. While the first component of the general solution $\varphi_\lambda(\cdot; 0, \eta)$ given by (1.13) is homoclinic, the second component fulfills

$$\varphi_\lambda^2(k; 0, \eta) = \begin{cases} 2^k \left(\eta_2 + \frac{4}{7}\eta_1^2 + \frac{2\lambda}{3}\eta_1\right) + o(1), & k \to \infty, \\ 2^{-k} \left(\eta_2 - \frac{2}{7}\eta_1^2 - \frac{2\lambda}{3}\eta_1\right) + o(1), & k \to -\infty; \end{cases}$$

in conclusion, one sees that $\varphi_\lambda(\cdot; 0, \eta)$ is bounded if and only if $\eta = (0, 0)$ or

$$\eta_1 = -\tfrac{14}{9}\lambda, \qquad\qquad \eta_2 = \tfrac{28}{81}\lambda^2.$$

Therefore, besides the zero solution there is a unique nontrivial entire bounded solution to (5.5) passing through the initial point $\eta = (\eta_1, \eta_2)$ at time $k = 0$ for $\lambda \neq 0$. This means the solution bifurcation pattern sketched in Fig. 13 (left) holds.

Example 5.3 (pitchfork solution bifurcation). Suppose that δ is a fixed nonzero real. Here, consider the nonlinear difference equation

$$x_{k+1} = f_k(x_k, \lambda) := \begin{pmatrix} b_k & 0 \\ \lambda & c_k \end{pmatrix} x_k + \delta \begin{pmatrix} 0 \\ (x_k^1)^3 \end{pmatrix} \tag{5.6}$$

depending on a bifurcation parameter $\lambda \in \mathbb{R}$ and the sequences b_k, c_k from (1.12). As in Exam. 5.2, the assumptions of Thm. 5.3 hold with $\lambda^* = 0$. The transversality condition reads as $g_{11} = \frac{4}{3} \neq 0$. Moreover, $D_1^2 f_k(0, 0) \equiv 0$ on $\mathbb{Z}$ implies $g_{20} = 0$, whereas the relation $D_1^3 f_k(0, 0)\zeta^3 = \left(\begin{smallmatrix} 0 \\ 6\delta\zeta_1^3 \end{smallmatrix}\right)$ for all $k \in \mathbb{Z}$, $\zeta \in \mathbb{R}^2$ leads to $g_{30} = 4\delta \neq 0$; having this available, one arrives at the crucial quotient $\frac{g_{30}}{g_{11}} = 3\delta$. By Rem. 5.3(2) one deduces a subcritical (supercritical) pitchfork bifurcation of the

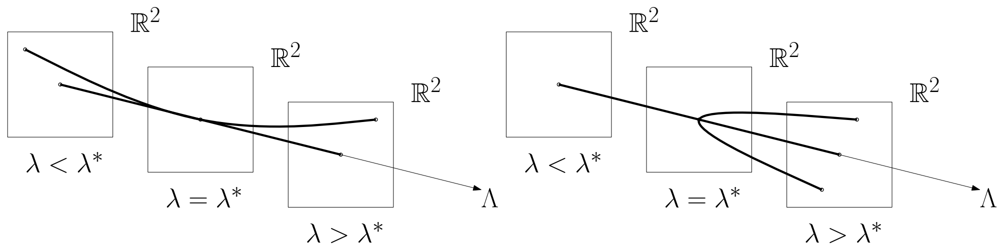

Figure 13: Bifurcation diagram for Exam. 5.2 (left) resp. 5.3 (right) with $\lambda^* = 0$:
Left (transcritical): Initial values $\eta \in \mathbb{R}^2$ yielding a homoclinic solution $\varphi_\lambda(\cdot; 0, \eta)$ of eqn. (5.5) for different parameter values λ.
Right (supercritical pitchfork): Initial values $\eta \in \mathbb{R}^2$ yielding a homoclinic solution $\varphi_\lambda(\cdot; 0, \eta)$ of eqn. (5.6) for different parameter values λ

trivial solution to (5.6) at $\lambda^* = 0$, provided $\delta > 0$ (resp. $\delta < 0$). Anew, we can illustrate this result using the general solution $\varphi_\lambda(\cdot; 0, \eta)$ to (5.6). As above, the first component is given by (1.13), which helps to compute for the second component that

$$\varphi_\lambda^2(k; 0, \eta) = \begin{cases} 2^k \left(\eta_2 + \frac{8\delta}{15}\eta_1^3 + \frac{2\lambda}{3}\eta_1\right) + o(1), & k \to \infty, \\ 2^{-k} \left(\eta_2 - \frac{2\delta}{15}\eta_1^3 - \frac{4\lambda}{3}\eta_1\right) + o(1), & k \to -\infty. \end{cases}$$

This asymptotic representation shows that $\varphi_\lambda(\cdot; 0, \eta)$ is homoclinic to 0 (or bounded) if and only if $\eta = 0$ or $\eta_1^2 = -\frac{2}{\delta}\lambda$ and $\eta_2 = \frac{4}{15}\frac{(5\delta+16\lambda)}{\delta^2}\lambda^2$. Hence, there is a correspondence to the pitchfork solution bifurcation from Rem. 5.2(2). An illustration is given in Fig. 13 (right).

5.3 Shovel bifurcation

The solution bifurcation patterns discussed in Thm. 5.2 and 5.3 were flawed by the fact that only unstable solutions can bifurcate (cf. Rem. 5.1(2)). This somehow contradicts the folklore understanding that a bifurcation goes hand in hand with a change in stability. Actually, we did impose Hyp. 5.1, whose assumption (5.1) is not robust under parameter variation. This causes the dichotomy spectrum to behave as illustrated in Fig. 6, i.e. to suddenly shrink for parameters $\lambda \neq \lambda^*$.

Now we investigate the somehow more "generic" situation of a dominant spectral interval crossing the stability boundary.

Hypothesis 5.2. Suppose that (Δ_λ) has a branch $\phi(\lambda) \in \ell^\infty$, $\lambda \in \Lambda$, of entire solutions. Let $D_1 f_k(\phi(\lambda)_k, \lambda) \in GL(\mathbb{R}^d)$ for all $k \in \mathbb{Z}$, $\lambda \in \Lambda$ and suppose the dichotomy spectra of (V_λ) allow a splitting

$$\Sigma(\lambda) = \Sigma_-(\lambda)\dot\cup\sigma(\lambda), \qquad \Sigma^\pm(\lambda) = \Sigma_-^\pm(\lambda)\dot\cup\sigma^\pm(\lambda) \quad \text{for all } \lambda \in \Lambda$$

into dominant intervals $\sigma(\lambda)$, $\sigma^\pm(\lambda)$ and a remaining spectral part with $\sup_{\lambda\in\Lambda} \max \Sigma_-(\lambda) < 1$. In addition, let m be the multiplicity of $\sigma(\lambda)$.

Remark 5.3. In case $\max \Sigma_-(\lambda) < 1$, a nonautonomous difference eqn. (Δ_λ) possesses a center fiber bundle (cf. the subsequent Thm. 6.1) and the stability analysis for the bounded entire solution $\phi(\lambda)$ reduces to an m-dimensional problem, where a corresponding nonautonomous reduction principle will be given in Thm. 6.2. In the remaining, we neglect the situation $\max \sigma^+(\lambda) = 1$. Here, the stability behavior of the reference solution ϕ^* is determined by the restriction of (Δ_λ)

on a center fiber bundle and particularly on Taylor coefficients of nonlinear terms (cf. [78]). As opposed to this setting, in the following, stability and bifurcation results are determined by the linear part alone.

In the autonomous (or periodic) situation the classical (or Floquet) spectrum consists of eigenvalues with a powerful perturbation theory available, yielding their differentiable dependence on the parameters (see, for instance, [48, Chapt. 7]). Since the dichotomy spectrum depends only upper-semicontinuously on parameters (cf. Rem. 2.3(4)), one cannot expect a similar smooth behavior for the boundary points of spectral intervals and instead we have to assume certain monotonicity properties for them. In this context, given a function $\sigma : \Lambda \to \mathbb{R}$, a convenient terminology is as follows: We briefly say $\sigma(\lambda^*) = 1$ *increases* (*decreases*), if $\sigma(\lambda^*) = 1$ and the function σ is strictly increasing (decreasing) in a neighborhood of λ^*.

Theorem 5.4 (shovel solution bifurcation). *Suppose that Hyp. 5.2 holds. If*

$$\max \Sigma(\lambda^*) = \max \sigma(\lambda^*) = 1$$

and the dominant spectral interval $\sigma^-(\lambda)$ has constant multiplicity m, then there exists a neighborhood $\Lambda_1 \subseteq \Lambda$ of λ^* such that for all $\lambda \in \Lambda_1$ it is:

(a) Subcritical case*: If $\max \sigma$ is decreasing at λ^*, then*

- (a_1) *for $\lambda < \lambda^*$ one has*
  - $\bullet_1$ *if $\max \sigma^+(\lambda^*) < 1$ or $\max \sigma^+(\lambda^*) = 1$ increases, then $\phi(\lambda)$ is asymptotically stable, and if also $\min \sigma^-(\lambda^*) = 1$ decreases, then $\phi(\lambda)$ is embedded into an m-parameter family of bounded entire solutions to (Δ_λ),*
  - $\bullet_2$ *if $\min \sigma^+(\lambda^*) = 1$ decreases, then $\phi(\lambda)$ is unstable,*
- (a_2) *for $\lambda = \lambda^*$ and $\max \sigma^+(\lambda^*) < 1$ the solution $\phi(\lambda^*)$ is asymptotically stable,*
- (a_3) *for $\lambda > \lambda^*$ the unique entire bounded solution of (Δ_λ) is $\phi(\lambda)$; it is uniformly asymptotically stable*

(b) Supercritical case*: If $\max \sigma$ is increasing at λ^*, then*

- (b_1) *for $\lambda < \lambda^*$ the unique entire bounded solution of (Δ_λ) is $\phi(\lambda)$; it is uniformly asymptotically stable,*
- (b_2) *for $\lambda = \lambda^*$ and $\max \sigma^+(\lambda^*) < 1$ the solution $\phi(\lambda^*)$ is asymptotically stable,*
- (b_3) *for $\lambda > \lambda^*$ one has*
  - $\bullet_1$ *if $\max \sigma^+(\lambda^*) < 1$ or $\max \sigma^+(\lambda^*) = 1$ decreases, then $\phi(\lambda)$ is asymptotically stable, and if also $\min \sigma^-(\lambda^*) = 1$ increases, then $\phi(\lambda)$ is embedded into an m-parameter family of bounded entire solutions to (Δ_λ),*
  - $\bullet_2$ *if $\min \sigma^+(\lambda^*) = 1$ increases, then $\phi(\lambda)$ is unstable.*

The dominant spectral intervals are illustrated in Fig. 14.

Furthermore, we refer to Fig. 15 for a schematic illustration of the bifurcation patterns described in Thm. 5.4. To explain our terminology, the set of solutions in ℓ^∞ for different values of the parameter λ resembles a shovel rather than e.g. a pitchfork. The shape of the shovel depends on the nonlinearity (see the discussion in Exam. 1.4). For linear difference equations, the bifurcating family of bounded solutions fills the whole half-plane left (subcritical case) resp. right (supercritical case) of the critical parameter λ^*.

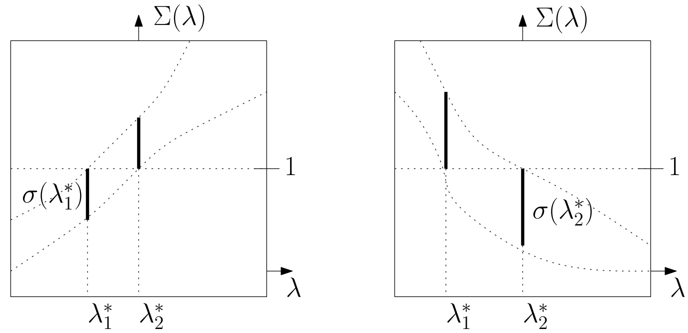

Figure 14: Dominant spectral intervals $\sigma(\lambda)$ as required in Thm. 5.4, where $\max \sigma(\lambda_1^*) = 1$ is increasing (left) or $\min \sigma(\lambda_2^*) = 1$ decreasing (right)

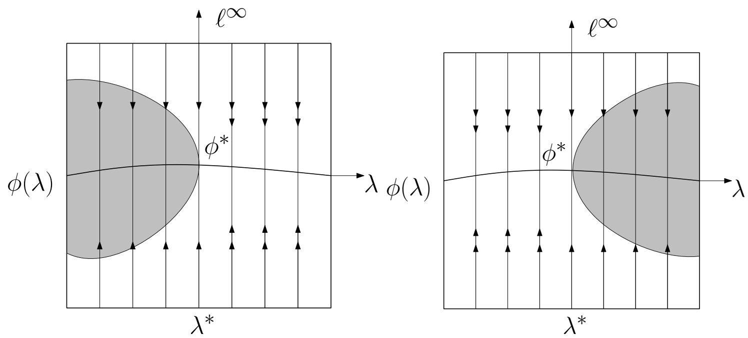

Figure 15: Schematic bifurcation diagram for Thm. 5.4$\bullet_1$ with a subcritical shovel bifurcation (left) and a supercritical shovel bifurcation (right) of an entire solution ϕ^* (double arrows indicate uniform asymptotic stability).

Remark 5.4. (1) A corresponding bifurcation scenario when the dominant spectral interval $\sigma(\lambda)$ leaves the stability boundary, i.e. under the assumption

$$\min \sigma(\lambda^*) = 1$$

was described in [75, Thm. 3.16]. See Fig. 14 for an illustration.

(2) The phenomenon of a shovel bifurcation is somewhat based on the assumption that the reference solution first becomes "unstable" on the negative axis $\mathbb{Z}_\kappa^-$, while it stays stable on the positive half line $\mathbb{Z}_\kappa^+$, as λ increases through the critical value λ^*. In the complementary situation where $\max \sigma^+(\lambda^*) = 1$ one can reduce (Δ_λ) to a center fiber bundle (cf. the following Sect. 6) and, provided the resulting bifurcation equation is scalar and the corresponding assumptions hold, possibly apply a version of Thm. 4.1 or 4.2 (cf. also Rem. 4.2(2)).

(3) In terms of the dichotomy spectrum, a supercritical shovel bifurcation allows the following interpretation: For $\lambda < \lambda^*$ one is in a hyperbolic situation $1 \notin \Sigma(\lambda^*)$, which becomes nonhyperbolic $1 \in \Sigma(\lambda)$ for $\lambda > \lambda^*$ in such a way that $1 \notin \Sigma_s(\lambda)$ (cf. [73, Thm. 4.20]).

Proof. See [75, Thm. 3.15]. □

A linear example exhibiting a shovel bifurcation was already discussed in Exam. 1.4 — now we give a nonlinear version.

Example 5.4. Let $(a_k)_{k\in\mathbb{Z}}$ be a bounded real sequence and consider the scalar difference equation

$$x_{k+1} = (\lambda + a_k) \arctan x_k. \tag{5.7}$$

With the reference solution $\phi^* = 0$ we obtain the variational eqn. $x_{k+1} = (\lambda + a_k)x_k$; it is invertible for sequences $(a_k)_{k\in\mathbb{Z}}$ never equal to $-\lambda$. The dichotomy spectra reduce to the dominant intervals given by

$$\sigma(\lambda) = \lambda + \left[\min\left\{\beta_{\mathbb{Z}}^-(a), \beta_{\mathbb{Z}}^+(a)\right\}, \max\left\{\beta_{\mathbb{Z}}^-(a), \beta_{\mathbb{Z}}^+(a)\right\}\right],$$
$$\sigma^+(\lambda) = \lambda + \left[\min\left\{\beta_{\mathbb{Z}_\kappa^+}^-(a), \beta_{\mathbb{Z}_\kappa^+}^+(a)\right\}, \max\left\{\beta_{\mathbb{Z}_\kappa^+}^-(a), \beta_{\mathbb{Z}_\kappa^+}^+(a)\right\}\right]$$

involving the Bohl exponents introduced in (2.4). In particular, for the sequence

$$a_k := \begin{cases} \frac{1}{2}, & k < 0, \\ 0, & k \geq 0 \end{cases} \tag{5.8}$$

one obtains $\sigma(\lambda) = \lambda + [0, \frac{1}{2}]$ and $\sigma^+(\lambda) = \{\lambda\}$. This yields a supercritical shovel bifurcation at $\lambda^* = \frac{1}{2}$. We refer to Fig. 16 for a solution portrait of eqn. (5.7) with a_k given by (5.8)

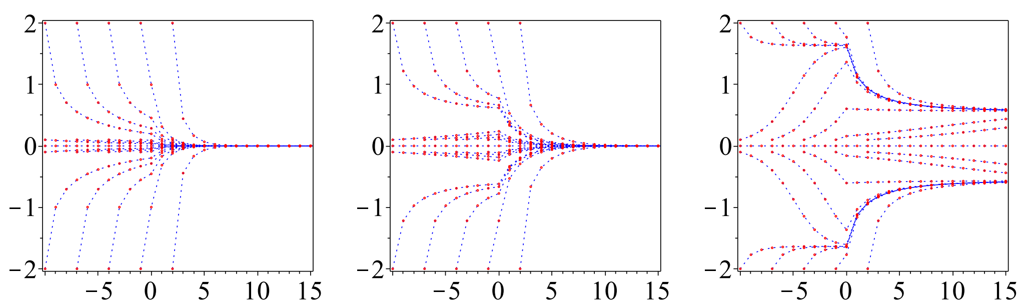

Figure 16: Solution sequences (dotted) of the difference eqn. (5.7) with a_k given in (5.8) and $\lambda = 0.5$ (left), $\lambda = 0.6$ (middle) and $\lambda = 1.1$ (right). It is indicated that the trivial solution becomes unstable in two steps.

Remarks

Under the assumption $\lim_{k\to\pm\infty} f_k(0, \lambda) = 0$ for all $\lambda \in \Lambda$ one can show that the bifurcating solutions in Thm. 5.2 and 5.3 are indeed homoclinic to 0, i.e. contained in the sequence space ℓ_0.

Persistence results, i.e. the behavior of the above solution bifurcations and their bifurcation diagrams under perturbation, was investigated in [77].

Also the classical autonomous transcritical and pitchfork patterns can be interpreted as "shovel bifurcations" in the sense that a whole family of bounded entire solutions branches, namely the heteroclinic connections of the bifurcating fixed points. However, the corresponding assumptions significantly differ from Hyp. 5.2.

Finally, we refer to [32] for an interesting result on almost periodic variational equations.

6 Center fiber bundles and reduction

In this section, we finally introduce a nonautonomous counterpart to a center manifold — a so-called *center fiber bundle*. It serves as a dynamically meaningful tool to reduce the dimension of bifurcation problems (Δ_λ). Thereto, let us suppose $\mathbb{I}$ is a discrete interval unbounded below and initially consider a parameter-free nonautonomous difference equation

$$\boxed{x_{k+1} = f_k(x_k)} \tag{Δ}$$

with a C^m-right-hand side $f_k : \mathbb{R}^d \to \mathbb{R}^d$, $k \in \mathbb{I}$, and $m \geq 2$. The well-established procedure how to include parameters into center manifold theory will be reviewed at the end of this section.

We suppose that (Δ) admits a fixed reference solution $\phi^* = (\phi_k^*)_{k\in\mathbb{I}} \in \ell^\infty$ and pass over to the corresponding equation of perturbed motion

$$x_{k+1} = A_k x_k + F_k(x_k) \tag{6.1}$$

with $A_k := Df_k(\phi_k^*)$ and the C^m-nonlinearity $F_k : \mathbb{R}^d \to \mathbb{R}^d$,

$$F_k(x) := f_k(x + \phi_k^*) - f_k(\phi_k^*) - Df_k(\phi_k^*)x$$

fulfilling the limit relation

$$\lim_{x\to 0} DF_k(x) = 0 \quad \text{uniformly in } k \in \mathbb{I}. \tag{6.2}$$

Moreover, we assume the dichotomy spectrum Σ of the variational difference eqn. (V) satisfies

$$\Sigma \cap (\alpha_-, \alpha_+) = \emptyset$$

with reals $0 < \alpha_- < \alpha_+$. The invariant projector associated to this spectral gap is called P_k, while $Q_k = \mathrm{id} - P_k$ denotes its complementary projector.

Our next aim is to describe a nonautonomous counterpart of an invariant manifold for (Δ) along ϕ^* resp. (6.1) along the trivial solution. To that end, let $U \subseteq \mathbb{R}^d$ be an open convex neighborhood of 0. Suppose $c_k : U \to \mathbb{R}^d$, $k \in \mathbb{I}$, are continuously differentiable mappings satisfying

$$c_k(0) \equiv 0 \quad \text{on } \mathbb{I}, \qquad \lim_{x\to 0} Dc_k(x) = 0 \quad \text{uniformly in } k \in \mathbb{I}, \tag{6.3}$$

$$c_k(x) = c_k(Q_k x) \in R(P_k) \quad \text{for all } k \in \mathbb{I},\ x \in U. \tag{6.4}$$

Then the graph (cf. Fig. 17)

$$\phi^* + \mathcal{C} := \left\{(\kappa, \phi_\kappa^* + \xi + c_\kappa(\xi)) \in \mathbb{I} \times \mathbb{R}^d : \xi \in R(Q_\kappa) \cap U\right\}$$

is called a *locally invariant fiber bundle* for the solution ϕ^* to (Δ), if one has the implication

$$(k_0, x_0) \in \phi^* + \mathcal{C} \quad \Rightarrow \quad (k, \varphi(k; k_0, x_0)) \in \phi^* + \mathcal{C} \quad \text{for all } k_0 \leq k \tag{6.5}$$

as long as the general solution to (Δ) satisfies $\varphi(k; k_0, x_0) \in \phi + U$.

Specifically, one speaks of a C^m-*fiber bundle*, if the derivatives $D^n c_k$ exist and are continuous for $1 \leq n \leq m$, and of a *center fiber bundle* provided $0 < \alpha_- < 1$.

Theorem 6.1 (existence of locally invariant fiber bundles). *There exist real numbers $\rho_0 > 0$ and $\gamma_0, \dots, \gamma_m \geq 0$ such that the following holds with $U = B_{\rho_0}(0)$: If the spectral gap condition*

$$\alpha_- < \alpha_+^m \tag{6.6}$$

is satisfied, then the solution ϕ^* to (Δ) possesses a locally invariant C^m-fiber bundle $\mathcal{C}$ with the following properties:

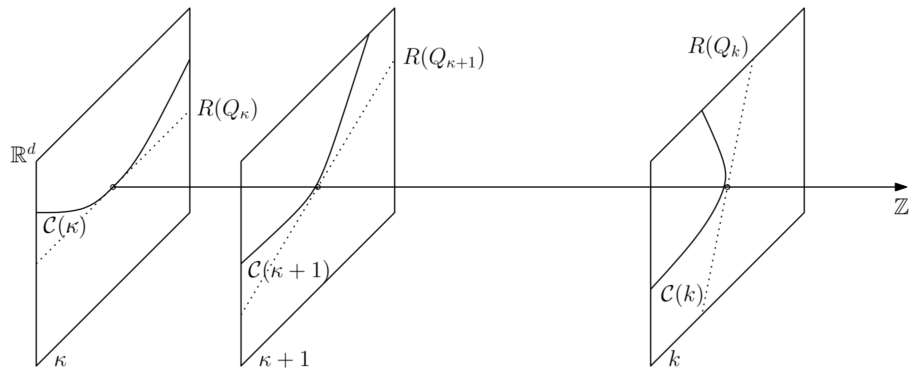

Figure 17: Fibers $\mathcal{C}(k)$ of an invariant fiber bundle $\mathcal{C} \subseteq \mathbb{I} \times \mathbb{R}^d$ along the trivial solution being smooth curves tangential to the ranges $R(Q_k)$, $k \in \mathbb{I}$

(a) The corresponding mappings $c_k : U \to \mathbb{R}^d$, $k \in \mathbb{I}$, satisfy

$$\|D^n c_k(x)\| \le \gamma_n \quad \textit{for all } x \in U,\ k \in \mathbb{I},\ n \in \{0, \dots, m\}, \tag{6.7}$$

(b) if the right-hand side f_k and the solution ϕ^* are periodic with period $p \in \mathbb{N}$, then

$$c_{k+p} = c_k \quad \textit{for all } k \in \mathbb{Z};$$

for an autonomous (Δ) and constant ϕ^* there is a mapping $c : U \to \mathbb{R}^d$ with $c \equiv c_k$ on $\mathbb{I}$, i.e., the set $\{\phi^* + \xi + c(\xi) \in \mathbb{R}^d : \xi \in R(Q) \cap U\}$ is a locally invariant manifold of (Δ).

Remark 6.1. (1) The invariant fiber bundles share the well-known properties of invariant manifolds. In particular, they need not to be unique but have the same Taylor coefficients (see [78, Thm. 3.4]). Furthermore, even for analytical right-hand sides f_k they need not to be of class C^∞.

(2) Besides (α_-, α_+) being disjoint from the dichotomy spectrum Σ, we made no further assumption on the growth rates $\alpha_- < \alpha_+$. For this reason, $\mathcal{C}$ is also denoted as *pseudo-unstable fiber bundle*. Given a discrete interval $\mathbb{I}$ being unbounded above, one can dually introduce pseudo-stable fiber bundles which are given as graphs over $R(P_k)$, $k \in \mathbb{I}$.

Proof. See [78, Thm. 3.2]. □

The usefulness of center fiber bundles is due to the fact that they allow a dimension reduction in critical stability situations:

Theorem 6.2 (reduction principle). *Let $\mathbb{I} = \mathbb{Z}$ and $\alpha_- < 1$. A solution ϕ^* of (Δ) is stable (uniformly stable, asymptotically stable, uniformly asymptotically stable, exponentially stable, or unstable), if and only if the* reduced equation

$$x_{k+1} = A_k x_k + Q_{k+1} F_k(x_k + c_k(x_k)) \tag{6.8}$$

in the unstable vector bundle $\mathcal{V}^-$ has the respective stability property.

Proof. See [70, p. 267, Thm. 4.6.14]. □

We are interested in local approximations of such a mapping $c_k : U \to \mathbb{R}^d$, $k \in \mathbb{I}$, describing a C^m-invariant fiber bundle for the solution ϕ^* to (Δ). Taylor's Theorem (cf. [90, p. 148, Thm. 4.A]) together with (6.3) implies the representation

$$c_k(x) = \sum_{n=2}^{m} \frac{1}{n!} c_k^n x^{(n)} + R_k^m(x) \tag{6.9}$$

with coefficient sequences $c_k^n \in L_n(\mathbb{R}^d)$ given by $c_k^n := D^n c_k(0)$ and a remainder R_k^m satisfying the limit relation $\lim_{x \to 0} \frac{R_k^m(x)}{\|x\|^m} = 0$. We know from [78, Thm. 3.4] that c_k^n is uniquely determined by the mappings c_k from Thm. 6.1. Due to (6.7) the coefficient sequences $(c_k^n)_{k \in \mathbb{I}}$ are bounded, i.e., one has $\|c_k^n\| \leq \gamma_n$ for $k \in \mathbb{I}$, $2 \leq n \leq m$ with reals $\gamma_2, \dots, \gamma_m \geq 0$. We need further notational preparations:

- It is convenient to introduce $C_k : U \to \mathbb{R}^d$, $C_k(x) := Q_k x + c_k(x)$, satisfying

$$DC_k(0) \overset{(6.3)}{=} Q_k, \qquad D^n C_k(0) = D^n c_k(0) \quad \text{for all } k \in \mathbb{I} \tag{6.10}$$

  and $n \in \{2, \dots, m\}$. Hence, for the derivatives $C_k^n := D^n C_k(0)$ we have the estimates

$$\left\| C_k^1 \right\| \overset{(2.2)}{\leq} K, \qquad \|C_k^n\| \overset{(6.7)}{\leq} \gamma_n \quad \text{for all } n \in \{2, \dots, m\}. \tag{6.11}$$

- We abbreviate $g_k(x) := Q_{k+1}\left[A_k x + F_k(Q_k x + c_k(x))\right]$ and the corresponding partial derivatives $g_k^n := D^n g_k(0)$ are given by (cf. (6.2)–(6.3))

$$\begin{aligned} g_k^1 x_1 &\overset{(2.1)}{=} A_k Q_k x_1, \\ g_k^n x_1 \cdots x_n &= \sum_{l=2}^{n} \sum_{(N_1, \dots, N_l) \in P_l^<(n)} Q_{k+1} D^l F_k(0) C_k^{\# N_1}|_{Q_k} x_{N_1} \cdots C_k^{\# N_l}|_{Q_k} x_{N_l} \end{aligned}$$

  for all $x_1, \dots, x_n \in \mathbb{R}^d$ and $n \in \{2, \dots, m\}$.

Given (multi-)linear mappings $X \in L_n(\mathbb{R}^d)$ and $T \in L(\mathbb{R}^d)$ it is convenient to introduce the brief notation $X|_T x_1 \cdots x_n := X(Tx_1, \dots, Tx_n)$ for $x_1, \dots, x_n \in \mathbb{R}^d$. In [78] we show that each Taylor coefficient sequence c^n is a solution to the so-called *homological equation* for $\mathcal{C}$ given by

$$X_{k+1}|_{A_k Q_k} = A_k X_k|_{Q_k} + H_k^n|_{Q_k}; \tag{6.12}$$

this is a linear difference equation in $L_n(\mathbb{R}^d)$ with inhomogeneities $H_k^n \in L_n(\mathbb{R}^d)$ defined by

$$\begin{aligned} H_k^n x_1 \cdots x_n := & P_{k+1}\Big[D^n F_k(0)|_{Q_k} x_1 \cdots x_n \\ & + \sum_{l=2}^{n-1} \sum_{(N_1, \dots, N_l) \in P_l^<(n)} \big(D^l F_k(0) C_k^{\# N_1}|_{Q_k} x_{N_1} \cdots C_k^{\# N_l}|_{Q_k} x_{N_l} \\ & - c_{k+1}^l g_k^{\# N_1}|_{Q_k} x_{N_1} \cdots g_k^{\# N_l}|_{Q_k} x_{N_l} \big) \Big]. \end{aligned} \tag{6.13}$$

Obviously, one has $H_2^k = P_{k+1} D^2 F_k(x)|_{Q_k}$ and for $n \in \{3, \dots, m\}$ the values H_k^n only depend on the sequences $c^2, \dots, c^{n-1}$. This leads to the following

Theorem 6.3. *The coefficients $c_k^n \in L_n(\mathbb{R}^d)$, $2 \leq n \leq m$, in the Taylor expansion* (6.9) *of the mapping $c_k : U \to \mathbb{R}^d$ can be determined recursively from the* Lyapunov-Perron sums

$$c_k^n = \sum_{j=-\infty}^{k-1} \Phi(k, j+1) H_j^n|_{\Phi(j,k)Q_k} \quad \text{for all } 2 \leq n \leq m. \tag{6.14}$$

Proof. See [78, Thm. 4.2(b)]. □

As an application we study a discrete epidemic model from [16]. Our present analysis extends corresponding earlier results from [78, Exams. 5.1 and 5.5].

Example 6.1. Let $(\alpha_k)_{k\in\mathbb{I}}$, $(\beta_k)_{k\in\mathbb{I}}$ denote bounded real sequences and let $\lambda \in (-1, \infty)$ be the bifurcation parameter. Consider the scalar second-order nonautonomous problem

$$y_{k+2} = (1 - \alpha_k y_{k+1} - \beta_k y_k)\left(1 - e^{-(\lambda+1)y_{k+1}}\right), \tag{6.15}$$

which is equivalent to the planar first-order system (Δ_λ) with

$$f_k(x, \lambda) := \begin{pmatrix} x_2 \\ (1 - \alpha_k x_2 - \beta_k x_1)\left(1 - e^{-(\lambda+1)x_2}\right) \end{pmatrix}.$$

The linear transformation $x \mapsto Tx$ with $T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & \lambda+1 \end{smallmatrix}\right)$, $T^{-1} = \begin{pmatrix} 1 & -\frac{1}{\lambda+1} \\ 0 & \frac{1}{\lambda+1} \end{pmatrix}$ applied to (Δ_λ) yields

$$x_{k+1} = \begin{pmatrix} 0 & 0 \\ 0 & \lambda+1 \end{pmatrix} x_k + F_k(x_k, \lambda), \tag{6.16}$$

where we have abbreviated

$$F_k(x_1, x_2, \lambda) := \left[(\lambda+1)x_2 - (1 - \alpha_k(\lambda+1)x_2 - \beta_k(x_1 + x_2))\,\frac{1 - e^{-(\lambda+1)x_2}}{\lambda+1}\right]\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This planar system satisfies our assumptions with dichotomy data given by $\alpha_- \in (0, \lambda+1)$, $\alpha_+ = \lambda+1$, $K = 1$ and $P_+ = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$; hence, Thm. 6.1 applies with associate dichotomy spectrum $\Sigma(\lambda) = \{\lambda + 1\}$. In case $\mathbb{I} = \mathbb{Z}$, formula (6.14) from Thm. 6.3 implies that the coefficients c_k^n for the fiber bundle $\mathcal{C}$ of (6.16) can be computed explicitly; the first three are given by

$$\begin{aligned}
c_k^2 =& \tfrac{1}{\lambda+1}\left((\lambda+1)^2 + 2\alpha_{k-1}(\lambda+1) + 2\beta_{k-1}\right), \qquad (6.17)\\
c_k^3 =& \tfrac{3\beta_{k-1}}{(\lambda+1)^2} c_{k-1}^2 + \tfrac{3(\lambda+1)^3 + 6\alpha_{k-1}(\lambda+1)^2 + 6(\lambda+1)\beta_{k-1}}{(\lambda+1)^2} c_k^2 - 3\alpha_{k-1}(\lambda+1) - 3\beta_{k-1} - (\lambda+1)^2,\\
c_k^4 =& \tfrac{12\beta_{k-1}}{(\lambda+1)^2} c_{k-1}^2 c_k^2 - \tfrac{6\beta_{k-1}}{\lambda+1} c_{k-1}^2\\
&- \tfrac{24(\lambda+1)^3\beta_{k-1} + 12(\lambda+1)\beta_{k-1}^2 + 7(\lambda+1)^5 + 24(\lambda+1)^4\alpha_{k-1} + 12(\lambda+1)^3{\alpha_{k-1}}^2 + 24(\lambda+1)^2\alpha_{k-1}\beta_{k-1}}{(\lambda+1)^3} c_k^2\\
&+ \tfrac{4\beta_{k-1}}{(\lambda+1)^3} c_{k-1}^3 + \tfrac{12\beta_{k-1}(\lambda+1)^2 + 6(\lambda+1)^4 + 12\alpha_{k-1}(\lambda+1)^3}{(\lambda+1)^3} c_k^3 + (\lambda+1)^3 + 4\alpha_{k-1}(\lambda+1)^2\\
&+ 4\beta_{k-1}(\lambda+1).
\end{aligned}$$

The stability properties of the zero solution of (6.16) (or (6.15)) depend on the parameter λ. We have asymptotic stability for $\lambda \in (-1, 0)$ (cf. Prop. 3.1), instability for $\lambda \in (0, \infty)$ (cf. Prop. 3.2).

In the critical situation $\lambda = 0$ the stability behavior depends on nonlinear terms involving the center fiber bundle $\mathcal{C}$. If we reduce (6.16) to $\mathcal{C}$ we obtain the scalar difference equation

$$x_{k+1} = x_k - (1 + 2\alpha_k + 2\beta_k)x_k^2 + (1 - 3\beta_k c_k^2 + 3\alpha_k + 3\beta_k)x_k^3 + O(x_k^4). \tag{6.18}$$

Hence, due to the nonautonomous reduction principle from Thm. 6.2, the stability of the zero solution for (6.18) involves the sequence $(1 + 2\alpha_k + 2\beta_k)_{k\in\mathbb{Z}}$. In fact, [78, Prop. 5.4] yields

- asymptotic left stability for $\limsup_{k\to\infty} (\alpha_k + \beta_k) < -\frac{1}{2}$,
- asymptotic right stability for $\liminf_{k\to\infty} (\alpha_k + \beta_k) > -\frac{1}{2}$;

in any case, the zero solution of (6.16) is unstable in the above situation. In the degenerate case where $1 + 2\alpha_k + 2\beta_k \equiv 0$ on $\mathbb{Z}$, one has to take the center fiber bundle $\mathcal{C}$ of (6.16) into account. Keeping in mind (6.17), the reduced equation reads as

$$x_{k+1} = x_k + \left[1 - 3\beta_k\bigl(-2\alpha_{k-1} + 2\beta_{k-1}\bigr) + 3\alpha_k\right] x_k^3 + O(x_k^4).$$

We, thus, define the sequence $\gamma_k := [-\beta_k(-2\alpha_{k-1} + 2\beta_{k-1}) + \alpha_k]$ for $k \in \mathbb{Z}$ and the trivial solutions of (6.15), (6.16) and (6.18) are

- unstable, if $\liminf_{k\to\infty} \gamma_k > -\frac{1}{3}$,
- asymptotically stable, if $\limsup_{k\to\infty} \gamma_k < -\frac{1}{3}$.

For a bifurcation analysis of the trivial solution to (6.15) we augment the planar system (6.16) with the trivial equation $\lambda_{k+1} = \lambda_k$ and introduce the new variables $z_k := (x_k, \lambda_k) \in \mathbb{R}^3$ to obtain

$$z_{k+1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} z_k + \begin{pmatrix} F_k^1(z_k) \\ z_k^2 z_k^3 + F_k^2(z_k) \\ 0 \end{pmatrix}. \tag{6.19}$$

Due to Thm. 6.1 this difference equation has a 2-dimensional center fiber bundle $\mathcal{C} \subseteq \mathbb{Z} \times \mathbb{R}^3$ being graph of mappings c_k, which allow the ansatz

$$c_k(\zeta_2, \zeta_3) = \sum_{i=0}^{2} \zeta_2^{2-i}\zeta_3^i c_k^{2-i,i} + O\left(\sqrt{\zeta_2^2 + \zeta_3^2}^{\,3}\right)$$

in order to determine the desired coefficient sequences $c_k^{2-i,i}$, $k \in \mathbb{Z}$, for $i \in \{0, 1, 2\}$. This yields the homological equations

$$c_{k+1}^{2,0} = \tfrac{1}{2} + \alpha_k + \beta_k, \qquad c_{k+1}^{1,1} = 0, \qquad c_{k+1}^{0,2} = 0$$

and the bifurcation equation (i.e. (6.19) reduced to the center fiber bundle $\mathcal{C}$) becomes

$$\begin{aligned} y_{k+1} &= (1 + \lambda)y_k + F_k^2(c_k(y_k, \lambda), y_k, \lambda) \\ &= (1 + \lambda)y_k - (\lambda + 1)\left[2\alpha_k(\lambda + 1) + 2\beta_k + (\lambda + 1)^2\right] y_k^2 + r_k(y_k, \lambda) \end{aligned} \tag{6.20}$$

with a remainder r_k satisfying $r_k(y, \lambda) = O(y^3)$ uniformly in $k \in \mathbb{Z}$. We now restrict to the generic situation where the sequence $\delta_k(\lambda) := 2\alpha_k(\lambda + 1) + 2\beta_k + (\lambda + 1)^2$ does not identically vanish. It is clear that (6.20) has the trivial solution for all parameters $\lambda > -1$. As suggested in Rem. 4.3(2) we can globally extend the bifurcation eqn. (6.20) outside a uniform neighborhood of the origin, such that it becomes globally invertible. Moreover, Hypothesis 4.1 is satisfied for (6.20) with the increasing functions $\gamma_-(\lambda) = \gamma_+(\lambda) = 1 + \lambda$. We can therefore apply Thm. 4.1 yielding a transcritical attractor bifurcation of the trivial solution at the critical parameter value $\lambda^* = 0$, provided one has

$$\liminf_{\lambda \to 0} \inf_{k \in \mathbb{Z}} (-\delta_k(\lambda)) \leq \limsup_{\lambda \to 0} \sup_{k \in \mathbb{Z}} (-\delta_k(\lambda)) < 0.$$

Remarks

By construction, the reduced difference eqn. (6.8) has a critical linear part. Hence, if it is not scalar, corresponding stability investigations require subtle techniques.

Our Thm. 6.3 can be used to obtain local approximations of invariant fiber bundles based on Taylor coefficients. This is sufficient for a local stability and bifurcation analysis. On the other hand, a numerical scheme to compute more global approximations was derived in [79].

7 Concluding remarks and perspectives

In our bifurcation analysis we explored two essentially different approaches, namely a dynamical one (*attractor bifurcation*) in Sect. 4 and a functional analytical one in Sect. 5. Both lead to independent scenarios indicating the fact that nonhyperbolicity is a significantly wider concept in our nonautonomous setting. This necessitates to investigate the fine structure of the dichotomy spectrum in order to classify nonautonomous bifurcations on a linear level already (cf. [73]).

Nevertheless, we penally neglected various other approaches to a nonautonomous bifurcation theory — in part, to keep this survey short, in part since the corresponding results deal with differential equations only. Historically first, bifurcation results for almost periodic solutions to such differential equations can be traced back to the monograph [49], while corresponding results in discrete time are due to [33].

For instance, [59] contains a phenomenological approach to bifurcation phenomena in nonautonomous ODEs. The references [62, 63] understand bifurcations as changes in (pullback) stability notions for ODEs. Topological methods have been employed in [17] to describe bifurcations in control systems. Using a skew-product language, the contribution [65] gives elegant nonautonomous counterparts to the classical bifurcation patterns for scalar differential equations; we refer to [89] or [3] for related discrete time results.

The references [45, 46] investigate Hopf bifurcations along nonperiodic solutions and [44, 29] apply averaging techniques to deduce nonautonomous counterparts of transcritical and saddle-node bifurcations. A two-step bifurcation scenario significantly different from [75] was investigated in [43].

Eventually, we did not deal with bifurcations for random dynamical systems in this survey, but refer to [19] or [5] for corresponding results.

The reduction to center fiber bundles yields a dynamical way to understand the behavior of critical nonautonomous difference equations via dimension reduction. An algebraic approach to simplify difference equations are normal forms. The corresponding nonautonomous theory has been established in [83] containing a nice and natural formulation of the nonresonance conditions in terms of the dichotomy spectrum.

The monograph [64, pp. 114ff, Sect. 5.2] contains an approach to attractor bifurcations for autonomous differential equations. A generalization to nonautonomous equations, or suitable discrete time versions, would be interesting. The contribution [53] might be helpful for a connection between solution and attractor bifurcation. Here, it is shown that every compact forward (or backward) invariant set contains a strictly invariant nonautonomous set. Since attractors consist of bounded entire solutions (cf. [70, p. 17, Cor. 1.3.4]), attractor bifurcation will lead to solution bifurcation.

All the (nonlinear) results mentioned in this survey are of local nature. This is due to that fact that global continuation and bifurcation results often rely on degree theory. However, for instance, the Leray-Schauder degree (cf., e.g., [50, pp. 178ff, Sect. II.2]) requires the considered equations to be compact perturbations of the identity. In large function spaces as ℓ^∞, where we look for solutions, such compactness requires very restrictive assumptions on the right-hand side of (Δ_λ). The Fredholm degree used in [28, 27] might be a suitable alternative.

A Global extension

Let K denote a (nonempty) set, X be a C^m-Banach space and Y be a Banach space. Here, being a *C^m-Banach space* means that the norm on X is of class C^m away from 0; we refer to [70, pp. 364–371, Sect. C.2] for a survey.

Proposition A.1 (global extension of local diffeomorphisms). *Let $m \in \mathbb{N}$ and U be an open neighborhood of $x_0 \in X$. If $f_k : U \to Y$, $k \in K$, is a C^m-mapping satisfying*

(i) $Df_k(x_0) \in GL(X, Y)$ *with* $\sup_{k\in K} \left\|Df_k(x_0)^{-1}\right\| < \infty$,

(ii) $\lim_{x\to x_0} \|Df_k(x) - Df_k(x_0)\| = 0$ *uniformly in* $k \in K$,

then there exists a $\rho > 0$ and a C^m-diffeomorphism $F_k : X \to Y$ with $F_k(x) \equiv f_k(x)$ on $\bar{B}_\rho(0)$ for all $k \in K$.

Proof. Above all, we define the C^m-mapping $\tilde{f}_k(x) := f_k(x + x_0) - f_k(x_0)$, $k \in K$ on the open neighborhood $U_0 := \{x \in X : x + x_0 \in U\}$ of 0 and choose $\rho_1 > 0$ so small that $B_{\rho_1}(0) \subseteq U_0$. Since X is assumed to be a C^m-Banach space we obtain from [70, p. 369, Prop. C.2.16] that for every $\rho > 0$ there exists a C^m-function $\chi_\rho : X \to [0, 1]$ such that

$$\chi_\rho(x) \equiv 1 \quad \text{on } \bar{B}_\rho(0), \qquad \chi_\rho(x) \equiv 0 \quad \text{on } X \setminus B_{2\rho}(0), \qquad \|D\chi_\rho(x)\| \le \tfrac{2}{\rho} \quad \text{for all } x \in X.$$

Thanks to the assumption (i) we can choose a $c > 0$ such that $c \sup_{k\in\mathbb{Z}} \left\|D\tilde{f}_k(0)^{-1}\right\| < 1$ and assumption (ii) ensures that there exists a $\rho \in (0, \rho_1)$ with

$$\left\|D\tilde{f}_k(x) - D\tilde{f}_k(0)\right\| < \tfrac{c}{6} \quad \text{for all } x \in B_{2\rho}(0),\ k \in K. \tag{A.1}$$

We introduce the C^m-mappings $\hat{f}_k, g_k : X \to Y$ given by $\hat{f}_k(x) := D\tilde{f}_k(0)x + g_k(x)$ and

$$g_k(x) := \begin{cases} \chi_\rho(x)\left[\tilde{f}_k(x) - D\tilde{f}_k(0)x\right], & \|x\| < 2\rho, \\ 0, & \|x\| \ge 2\rho. \end{cases}$$

Thanks to the mean value theorem the derivative of g_k is given by

$$\begin{aligned} Dg_k(x) &= D\chi_\rho(x)\left[\tilde{f}_k(x) - D\tilde{f}_k(0)x\right] + \chi_\rho(x)\left[D\tilde{f}_k(x) - D\tilde{f}_k(0)\right] \\ &= D\chi_\rho(x)\int_0^1 D\tilde{f}_k(hx) - D\tilde{f}_k(0)\,dh\,x + \chi_\rho(x)\left[D\tilde{f}_k(x) - D\tilde{f}_k(0)\right] \end{aligned}$$

and therefore satisfies

$$\|Dg_k(x)\| \overset{(A.1)}{\leq} 2\rho\,\|D\chi_\rho(x)\| \int_0^1 \frac{c}{6}\,dh + \|\chi_\rho(x)\|\,\frac{c}{6} \leq \frac{2c}{3} + \frac{c}{6} < c \quad \text{for all } \|x\| \leq 2\rho,\ k \in K.$$

Since the nonlinearity g_k vanishes identically outside of the ball $B_{2\rho}(0)$ we obtain from the mean value inequality that $c > 0$ is a global Lipschitz constant of g_k (uniformly in $k \in K$). Due to the choice of c, this in turn, ensures that both $\hat{f}_k$ and $D\hat{f}_k(x)$ are homeomorphisms (cf. [8, Cor. 6.2]) for all $x \in X$. Hence, $\hat{f}_k$ is proper (see [90, p. 173, Ex. 4.39]) and due to the local inverse function theorem (see [90, p. 172, Thm. 4.F]) also a local C^m-diffeomorphism. Given this information, [90, p. 174, Thm. 4.G] implies that $\hat{f}_k : X \to Y$ is a global C^m-diffeomorphism. Finally, the desired global extension $F_k : X \to Y$ of f_k reads as $F_k(x) := \hat{f}_k(x - x_0) + f(x_0)$ and satisfies our assertion. □

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References

[1] R.P. Agarwal, *Difference equations and inequalities*, Monographs and Textbooks in Pure and Applied Mathematics 228, Marcel Dekker, New York, 2000.

[2] A.I. Alonso, J. Hong & R. Obaya, *Exponential dichotomy and trichotomy for difference equations*, Comput. Math. Appl. **38** (1999), no. 1, 41–49.

[3] V. Anagnostopoulou & T. Jäger, *Nonautonomous saddle-node bifurcations: Random and deterministic forcing*, Preprint, http://www.math.tu-dresden.de/ anagnost/na-saddlenode05.pdf, 2011.

[4] B. Aulbach & J. Kalkbrenner, *Exponential forward splitting for noninvertible difference equations*, Comput. Math. Appl. **42** (2001), 743–754.

[5] L. Arnold, *Random dynamical systems*, Monographs in Mathematics, Springer, Berlin etc., 1998.

[6] B. Aulbach & S. Siegmund, *The dichotomy spectrum for noninvertible systems of linear difference equations*, J. Difference Equ. Appl. **7** (2001), no. 6, 895–913.

[7] ———, *A spectral theory for nonautonomous difference equations*, Proceedings of the 5th Intern. Conference of Difference Eqns. and Application (Temuco, Chile, 2000) (J. López-Fenner, et al., ed.), Taylor & Francis, London, 2002, pp. 45–55.

[8] B. Aulbach, *The fundamental existence theorem on invariant fiber bundles*, J. Difference Equ. Appl. **3** (1998), 501–537.

[9] B. Aulbach & N. Van Minh, *The concept of spectral dichotomy for linear difference equations II*, J. Difference Equ. Appl. **2** (1996), 251–262.

[10] B. Aulbach, N. Van Minh & P.P. Zabreiko, *The concept of spectral dichotomy for linear difference equations*, J. Math. Anal. Appl. **185** (1994), 275–287.

[11] ______, *Structural stability of linear difference equations in Hilbert space*, Comput. Math. Appl. **36** (1998), no. 10–12, 71–76.

[12] A. Ben-Artzi & I. Gohberg, *Dichotomy, discrete Bohl exponents, and spectrum of block weighted shifts*, Integral Equations Oper. Theory **14** (1991), no. 5, 613–677.

[13] ______, *Dichotomies of perturbed time varying systems and the power method*, Indiana Univ. Math. J. **42** (1993), no. 3, 699–720.

[14] A.G. Baskakov, *Invertibility and the Fredholm property of difference operators*, Mathematical Notes **67** (2000), no. 6, 690–698.

[15] W.-J. Beyn & J.-M. Kleinkauf, *The numerical computation of homoclinic orbits for maps*, SIAM J. Numer. Anal. **34** (1997), no. 3, 1209–1236.

[16] K.L. Cooke, D.F. Calef & E.V. Level, *Stability or chaos in discrete epidemic models*, Nonlinear Systems and Applications (New York) (V. Lakshmikantham, ed.), Academic Press, 1977, pp. 73–93.

[17] F. Colonius, R. Fabbri, R.A. Johnson & M. Spadini, *Bifurcation phenomena in control flows*, Topological Methods in Nonlinear Analysis **30** (2007), no. 1, 87–111.

[18] S.-N. Chow & J.K. Hale, *Methods of bifurcation theory*, Grundlehren der mathematischen Wissenschaften 251, Springer, Berlin etc., 1996.

[19] H. Crauel, P. Imkeller & M. Steinkamp, *Bifurcations of one-dimensional stochastic differential equations*, Stochastic Dynamics (H. Crauel, ed.), Springer, New York, 1999, pp. 27–47.

[20] D.N. Cheban, P.E. Kloeden & B. Schmalfuß, *Pullback attractors in dissipative nonautonomous differential equations under discretization*, J. Dyn. Differ. Equations **13** (2000), no. 1, 185–213.

[21] ______, *The relationship between pullback, forward and global attractors of nonautonomous dynamical systems*, Nonlin. Dynam. Syst. Theory **2** (2002), 9–28.

[22] S.L. Campbell & C.D. Meyer, *Generalized inverses of linear transformations*, Surveys and reference works in mathematics 4, Pitman, San Francisco, 1979.

[23] J.D. Crawford, *Introduction to bifurcation theory*, Rev. Mod. Phys. **63** (1991), 991–1037.

[24] C.V. Coffman & J.J. Schäffer, *Dichotomies for linear difference equations*, Math. Ann. **172** (1967), 139–166.

[25] K. Deimling, *Nonlinear functional analysis*, Springer, Berlin etc., 1985.

[26] S. Elaydi, *An introduction to difference equation*, Undergraduate Texts in Mathematics, Springer, New York, 2005.

[27] G. Evéquoz, *Erratum: Global continuation for first order systems over the half-line involving parameters*, Math. Nachr. **283** (2010), no. 4, 630–631.

[28] ______, *Global continuation for first order systems over the half-line involving parameters*, Math. Nachr. **282** (2010), no. 8, 1036–1078.

[29] R. Fabbri, R.A. Johnson & F. Mantellini, *A nonautonomous saddle-node bifurcation pattern*, Stoch. Dyn. **4** (2004), no. 3, 335–350.

[30] J. Guckenheimer & P.J. Holmes, *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, Applied Mathematical Sciences 42, Springer, Berlin etc., 1983.

[31] W. Govaerts, Y.A. Kuznetsow, A. Yu & B. Sijnava, *Bifurcation of maps in the software package CONTENT*, Proceedings of the 2nd Workshop on Computer Algebra in Scientific Computing. CASC '99 (V.G. Ganzha, et al., ed.), Springer, 1999, pp. 191–206.

[32] P. Giesl & M. Rasmussen, *A note on almost periodic variational equations*, Commun. Pure Appl. Anal. **10** (2011), no. 3, 983–994.

[33] Y. Hamaya, *Bifurcation of almost periodic solutions in difference equations*, J. Difference Equ. Appl. **10** (2004), no. 3, 257–297.

[34] D. Henry, *Geometric theory of semilinear parabolic equations*, Lect. Notes Math. 840, Springer, Berlin etc., 1981.

[35] J.K. Hale & H. Koçak, *Dynamics and bifurcations*, Texts in Applied Mathematics 3, Springer, Berlin etc., 1991.

[36] J. Hong & C. Núñez, *The almost periodic type difference equation*, Math. Comput. Modelling **28** (1998), no. 12, 21–31.

[37] J.M. Holtzman, *Explicit ϵ and δ for the implicit function theorem*, SIAM Review **12** (1970), no. 2, 284–286.

[38] M.W. Hirsch & S. Smale, *Differential equations, dynamical systems, and linear algebra*, Academic Press, Boston-New York-San Diego, 1974.

[39] T. Hüls, *A model function for non-autonomous bifurcations of maps*, Discrete Contin. Dyn. Syst. (Series B) **7** (2007), no. 2, 351–363.

[40] ______, *Numerical computation of dichotomy rates and projectors in discrete time*, Discrete Contin. Dyn. Syst. (Series B) **12** (2009), no. 1, 109–131.

[41] ______, *Computing Sacker-Sell spectra in discrete time dynamical systems*, SIAM Journal on Numerical Analysis **48** (2010), no. 6, 2043–2064.

[42] G. Iooss, *Bifurcation of maps and applications*, Mathematics Studies 36, North-Holland, Amsterdam etc., 1979.

[43] R.A. Johnson, P.E. Kloeden & R. Pavani, *Two-step transitions in nonautonomous bifurcations: An explanation*, Stoch. Dyn. **2** (2002), no. 1, 67–92.

[44] R.A. Johnson & F. Mantellini, *A nonautonomous transcritical bifurcation problem with an application to quasi-periodic bubbles*, Discrete Contin. Dyn. Syst. **9** (2003), no. 1, 209–224.

[45] R.A. Johnson, *Hopf bifurcation from nonperiodic solutions of differential equations. I. Linear theory*, J. Dyn. Differ. Equations **1** (1989), no. 2, 179–198.

[46] R.A. Johnson & Y. Yi, *Hopf bifurcation from nonperiodic solutions of differential equations. II*, J. Differ. Equations **107** (1994), no. 2, 310–340.

[47] J. Kalkbrenner, *Exponentielle Dichotomie und chaotische Dynamik nichtinvertierbarer Differenzengleichungen*, Ph.D. thesis, Universität Augsburg, 1994.

[48] T. Kato, *Perturbation theory for linear operators*, corrected 2nd ed., Grundlehren der mathematischen Wissenschaften 132, Springer, Berlin etc., 1980.

[49] M.A. Krasnosel'skij, V.Sh. Burd & Yu.S. Kolesov, *Nonlinear almost periodic oscillations*, A Halsted Press Book, John Wiley & Sons, Jerusalem - London, 1973.

[50] H. Kielhöfer, *Bifurcation theory: An introduction with applications to PDEs*, Applied Mathematical Sciences 156, Springer, New York, etc., 2004.

[51] P.E. Kloeden & V. Kozyakin, *Single parameter dissipativity and attractors in discrete time asynchronous systems*, J. Difference Equ. Appl. **7** (2001), no. 6, 873–894.

[52] P.E. Kloeden, *Pullback attractors in nonautonomous difference equations*, J. Difference Equ. Appl. **6** (2000), no. 1, 33–52.

[53] P.E. Kloeden & P. Marín-Rubio, *Negatively invariant sets and entire solutions*, J. Dyn. Differ. Equations **23** (2011), no. 3, 437–450.

[54] J. Kurzweil & G. Papaschinopoulos, *Structural stability of linear discrete systems via the exponential dichotomy*, Czech. Math. J. **38** (1988), 280–284.

[55] ———, *Topological equivalence and structural stability for linear difference equations*, J. Differ. Equations **89** (1991), 89–94.

[56] P.E. Kloeden, C. Pötzsche & M. Rasmussen, *Discrete-time nonautonomous dynamical systems*, Manuscript, 2011.

[57] ———, *Limitations of pullback attractors for processes*, **to appear** in J. Difference Equ. Appl., DOI: 10.1080/10236198.2011.578070 (2011).

[58] P.E. Kloeden & M. Rasmussen, *Nonautonomous dynamical systems*, Mathematical Surveys and Monographs 176, AMS, Providence, RI, 2011.

[59] P.E. Kloeden & S. Siegmund, *Bifurcations and continuous transitions of attractors in autonomous and nonautonomous systems*, International Journal of Bifurcation and Chaos **5** (2005), no. 2, 1–21.

[60] Y.A. Kuznetsow, *Elements of applied bifurcation theory*, 3rd ed., Applied Mathematical Sciences 112, Springer, Berlin etc., 2004.

[61] Ta Li, *Die Stabilitätsfrage bei Differenzengleichungen*, Acta Math. **63** (1934), 99–141.

[62] J.A. Langa, J.C. Robinson & A. Suárez, *Stability, instability, and bifurcation phenomena in non-autonomous differential equations*, Nonlinearity **15** (2002), 887–903.

[63] ______, *Bifurcations in non-autonomous scalar equations*, J. Differ. Equations **221** (2006), 1–35.

[64] T. Ma & S. Wang, *Bifurcation theory and applications*, Series on Nonlinear Sciences 53, World Scientific, Hackensack, NJ, 2005.

[65] C. Núñez & R. Obaya, *A non-autonomous bifurcation theory for deterministic scalar differential equations*, Discrete Contin. Dyn. Syst. (Series B) **9** (2008), no. 3–4, 701–730.

[66] K.J. Palmer, *Exponential dichotomies, the shadowing lemma and transversal homoclinic points*, Dynamics Reported 1 (U. Kirchgraber & H.-O. Walther, eds.), B.G. Teubner/John Wiley & Sons, Stuttgart/Chichester etc., 1988, pp. 265–306.

[67] G. Papaschinopoulos, *Exponential dichotomy for almost periodic linear difference equations*, Ann. Soc. Sci. Bruxelles, Sér. I **102** (1988), no. 1–2, 19–28.

[68] C. Pötzsche, *A note on the dichotomy spectrum*, J. Difference Equ. Appl. **15** (2009), no. 10, 1021–1025.

[69] ______, *Robustness of hyperbolic solutions under parametric perturbations*, J. Difference Equ. Appl. **15** (2009), no. 8–9, 803–819.

[70] ______, *Geometric theory of discrete nonautonomous dynamical systems*, Lect. Notes Math. 2002, Springer, Berlin etc., 2010.

[71] ______, *Nonautonomous bifurcation of bounded solutions I: A Lyapunov-Schmidt approach*, Discrete Contin. Dyn. Syst. (Series B) **14** (2010), no. 2, 739–776.

[72] ______, *Corrigendum to "A note on the dichotomy spectrum"*, **to appear** in J. Difference Equ. Appl., DOI: 10.1080/10236198.2011.573790 (2011).

[73] ______, *Fine structure of the dichotomy spectrum*, submitted (2011).

[74] ______, *Nonautonomous bifurcation of bounded solutions: Crossing curve situations*, **to appear** in Stoch. Dyn., DOI: 10.1142/S021949371200364X (2011).

[75] ______, *Nonautonomous bifurcation of bounded solutions II: A shovel bifurcation pattern*, Discrete Contin. Dyn. Syst. (Series A) **31** (2011), no. 1, 941–973.

[76] ———, *Nonautonomous continuation of bounded solutions*, Commun. Pure Appl. Anal. **10** (2011), no. 3, 937–961.

[77] ———, *Persistence and imperfection of nonautonomous bifurcation patterns*, J. Differ. Equations **250** (2011), 3874–3906.

[78] C. Pötzsche & M. Rasmussen, *Taylor approximation of invariant fiber bundles for nonautonomous difference equations*, Nonlin. Analysis (TMA) **60** (2005), no. 7, 1303–1330.

[79] ———, *Computation of nonautonomous invariant and inertial manifolds*, Numer. Math. **112** (2009), 449–483.

[80] M. Rasmussen, *Towards a bifurcation theory for nonautonomous difference equation*, J. Difference Equ. Appl. **12** (2006), no. 3–4, 297–312.

[81] ———, *Attractivity and Bifurcation for Nonautonomous Dynamical Systems*, Lect. Notes Math. 1907, Springer, Berlin etc., 2007.

[82] ———, *An alternative approach to Sacker-Sell spectral theory*, J. Difference Equ. Appl. **16** (2010), no. 2–3, 227–242.

[83] S. Siegmund, *Normal forms for nonautonomous difference equations*, Comput. Math. Appl. **45** (2003), no. 6–9, 1059–1073.

[84] V.E. Slyusarchuk, *Exponential dichotomies for solutions of discrete systems*, Ukr. Math. J. **35** (1983), no. 1, 98–103.

[85] R.J. Sacker & G.R. Sell, *A spectral theory for linear differential systems*, J. Differ. Equations **27** (1978), 320–358.

[86] V.I. Tkachenko, *On the exponential dichotomy of linear difference equations*, Ukr. Math. J. **48** (1996), no. 10, 1600–1608.

[87] T. Wanner, *Linearization of random dynamical systems*, Dynamics reported 4. New Series (C.K.R.T. Jones, et al., ed.), Springer, Berlin, 1995, pp. 203–269.

[88] S. Wiggins, *Introduction to applied nonlinear dynamical systems and chaos*, Texts in Applied Mathematics 2, Springer, Berlin etc., 1990.

[89] N.T. Yet, T.S. Doan, T. Jäger & S. Siegmund, *Nonautonomous saddle-node bifurcations in the quasiperiodically forced logistic map*, International Journal of Bifurcation and Chaos **21** (2011), no. 5, 1427–1438.

[90] E. Zeidler, *Nonlinear functional analysis and its applications I (Fixed-points theorems)*, Springer, Berlin etc., 1993.