Global Asymptotic Stability of a Generalization of the Pielou Difference Equation

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Abstract. The Pielou equation is a well-known discrete-time population model in which the per capita growth rate depends on the population size, but the density dependence operates with a delay of d generations. Thus, the between-year dynamics are governed by a difference equation of order $d + 1$. The main result in this paper establishes the global stability of the unique positive equilibrium for a generalization of the two-dimensional Pielou equation. Our proof is based on a rather natural combination of two techniques which could be, in principle, applicable to obtain global asymptotic stability in other problems: some dominance conditions and the determination of a first integral for a related equation, which turns out to be a quasi-Lyanupov function for the generalized Pielou equation. We provide additional results on the global dynamics of the generalized Pielou equation for dimensions higher than two, and discuss its relationship with other families of difference equations traditionally employed for modelling population dynamics.

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1. Introduction

Population growth of many species is modelled by difference equations in such a way that the population size x_n after n generations is described by a production function f depending on the previous generations:

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-d}), \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where the initial conditions $x_{-d}, \ldots, x_{-1}, x_0$ are nonnegative real numbers with $x_0 > 0$.

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A well-known example is given by the Levin–May model [16]:

$$x_{n+1} = x_n F(x_{n-d}), \quad n = 0, 1, 2, \ldots, \tag{1.2}$$

where $F : [0, \infty) \to (0, \infty)$ is continuous and decreasing. For some specific forms of F, see Table I in [16]. In particular, for $F(x) = \beta/(1 + \delta x)$, $\beta > 0$, $\delta > 0$, (1.2) gives the Pielou equation:

$$x_{n+1} = \frac{\beta x_n}{1 + \delta x_{n-d}}, \quad n = 0, 1, 2, \ldots, \tag{1.3}$$

introduced by Pielou in [23].

Another example of (1.1) is the Clark model [6], given by

$$x_{n+1} = \alpha x_n + F(x_{n-d}), \quad n = 0, 1, 2, \ldots, \tag{1.4}$$

where $0 < \alpha < 1$, and $F : [0, \infty) \to [0, \infty)$ is continuous.

Both Eqs. (1.2) and (1.4) are simple models to take into account some age structure in a population, and x_n denotes the adult population. Equation (1.2) assumes that the per capita growth rate depends on the population size, but the density dependence operates with a delay of d generations; Eq. (1.4) assumes that a proportion αx_n of the adult population survives to the next generation, and the newborns need a maturation delay d to become adults.

This paper is focused on the following generalization of the Pielou Eq. (1.3):

$$x_{n+1} = \frac{\beta x_n^\gamma}{1 + \delta x_{n-d}}, \quad n = 0, 1, 2, \ldots, \tag{1.5}$$

where β, γ and δ are positive real numbers and d is a nonnegative integer. In the one-dimensional case $d = 0$, Eq. (1.5) has been recently studied in [18], and it belongs to a family of first-order difference equations given by

$$x_{n+1} = x_n^\gamma F(x_n), \quad n = 0, 1, 2, \ldots, \tag{1.6}$$

which includes other models with applications in population dynamics, economics, and social science (see [17,19] and their references).

In this way, our study of Eq. (1.5) can be considered as a step to understand some qualitative properties of the family of delayed difference equations

$$x_{n+1} = x_n^\gamma F(x_{n-d}), \quad n = 0, 1, 2, \ldots, \tag{1.7}$$

where $\gamma > 0$ and d is a positive integer. In particular, Eq. (1.7) has been recently considered in [3] with $\gamma \geq 1$ and $F(x) = e^{r(1-x)}$, $r > 0$, to investigate Allee effects in a Ricker-type population model with delay.

Note that (1.7) with $\gamma = 1$ gives the Levin–May model (1.2), and a change of variables transforms (1.7) into the Clark model (1.4). We provide more details in Sect. 2. These two facts make the study of Eq. (1.7) of interest from a theoretical point of view. From the perspective of its applications, assuming that F is decreasing, Eq. (1.7) can be viewed as a population model where the per capita growth rate x_{n+1}/x_n is the product of two density-dependent factors: the delayed factor of intraspecific competition $F(x)$ already present in the Levin–May equation, and a new factor $x^{\gamma-1}$, which can be a cooperation factor ($\gamma > 1$) or an interference factor ($0 < \gamma < 1$). We emphasize that for $\gamma > 1$, the cooperation factor induces a strong Allee effect,

in such a way that populations below a threshold are doomed to extinction [3,17]. For $0 < \gamma \leq 1$, (1.7) has a unique positive equilibrium (the additional assumption $F(0) > 1$ is necessary when $\gamma = 1$), and the parameter γ is used to gain flexibility to fit population data; for details and more references, see [17,18,24].

Our main results concern local and global stability properties for the unique positive equilibrium of (1.5) in the case $0 < \gamma < 1$. In particular, we prove that this equilibrium is globally asymptotically stable in the two-dimensional case ($d = 1$), thus extending the result for $\gamma = 1$ (Pielou equation), proved by Kuruklis and Ladas [15, Theorem 4]. Our proof is based on a rather natural combination of two techniques which could be, in principle, applicable to obtain global asymptotic stability in other problems: the dominance conditions introduced in [7,11], and the determination of a first integral for a related equation, which turns out to be a quasi-Lyanupov function for (1.5) in a certain parameter regime, where we can show global stability with the aid of a Lyapunov-type result (Lemma 3.11).

For other values of d, we get some global stability results that work for the more general Eq. (1.7) with $\gamma \in (0, 1)$. In this case, we apply some previous results of Tkachenko and Trofimchuk [25], using the relationship between (1.7) and Clark Eq. (1.4).

The paper is organized as follows: in Sect. 2, we provide some results on the global stability of the positive equilibrium of (1.7). Section 3 is devoted to Eq. (1.5). In the subsequent subsections, we address the local stability of the equilibrium (Sect. 3.1), the global stability in the two-dimensional case (Sect. 3.2), and some partial global stability results for dimensions higher than two (Sects. 3.3, 3.4). Finally, Sect. 4 contains some remarks and open problems.

2. Preliminary Results

As we have stated before, a change of variables transforms (1.7) into the Clark model (1.4). Since this fact will be useful in some of the subsequent results, in this section we give the explicit form of this transformation and we use it to derive some results.

We consider (1.7) with $0 < \gamma < 1$ and $F \colon [0, \infty) \to (0, \infty)$ continuous and decreasing. If p is the unique equilibrium of (1.7), then the change of variables $y_n = -\log(x_n/p)$ transforms (1.7) into

$$y_{n+1} = \gamma y_n + g(y_{n-d}), \quad n = 0, 1, 2, \dots, \tag{2.1}$$

where $g(y) = -\log\left(p^{\gamma-1}F(pe^{-y})\right)$.

The following global stability result is a straightforward consequence of Theorem 7 in the paper [25] by Tkachenko and Trofimchuk.

Theorem 2.1. *Assume that $0 < \gamma < 1$, and let $A = -p^{-\gamma}(1-\gamma)/F'(p)$. The positive equilibrium p of* (1.7) *is globally asymptotically stable if the following conditions hold:*

(H1) *Either $A \geq 1$, or $A < 1$ and*

$$\gamma^{d+1} > \frac{1}{A} \log \left(\frac{1+A}{1+A^2} \right). \tag{2.2}$$

(H2) *F is $\mathcal{C}^3$-differentiable, $F'(x) < 0$ for all $x > 0$, and*

$$x^2 \left(2\frac{F'''(x)}{F'(x)} - 3\left(\frac{F''(x)}{F'(x)}\right)^2 + \left(\frac{F'(x)}{F(x)}\right)^2 \right) < 1, \quad \forall\, x > 0. \tag{2.3}$$

For $\gamma = 1$, Eq. (1.2) has also been considered by Tkachenko and Trofimchuk, and the following global stability result can be derived from [26, Theorem 1.3].

Proposition 2.2. *If **(H2)** holds, then the positive equilibrium p of (1.2) is globally asymptotically stable if the following inequality is satisfied:*

$$-pF'(p) < \frac{3}{2(d+1)} + \frac{1}{2(d+1)^2}. \tag{2.4}$$

Remark 2.3. The technical condition (2.3) is equivalent to say that the Schwarzian derivative of the transformed map $g(y) = -\log\left(p^{\gamma-1}F(pe^{-y})\right)$ is negative (see [19, Proposition 6]).

3. Results for the Gamma–Pielou Equation

In this section, we consider Eq. (1.5), which we refer to as the Gamma–Pielou equation, in analogy with the one-dimensional gamma model (1.6) introduced in [19]. Since we are mainly interested in global stability results, we only consider the case $0 < \gamma \leq 1$. As we mentioned in Sect. 1, in this case (1.5) has a unique positive equilibrium p. Our first remark is that we can normalize the equilibrium to 1, thus dropping one of the parameters in (1.5); indeed, since $\beta p^{\gamma-1} = 1 + \delta p$, the change of variables $x_n \mapsto x_n/p$ transforms (1.5) into

$$x_{n+1} = \frac{(\alpha+1)x_n^\gamma}{\alpha + x_{n-d}}, \quad n = 0, 1, 2, \ldots \tag{3.1}$$

with $\alpha = 1/(\delta p) > 0$.

Note also that $x_n > 0$ holds for all $n \geq 1$ under any admissible initial conditions, i.e. provided $x_{-d}, \ldots, x_0$ are all nonnegative, and $x_0 > 0$, so it is not restrictive to assume that our state space is the open set $\mathbb{R}_+^{d+1}$.

3.1. Local Stability

We begin with the study of the local asymptotic stability of the equilibrium 1 in (3.1). For completeness, we consider the general case $\gamma > 0$.

The linearized equation about the positive equilibrium 1 of (3.1) is

$$x_{n+1} = \gamma x_n - q x_{n-d}, \quad q = \frac{1}{\alpha+1} < 1. \tag{3.2}$$

Applying the Schur–Cohn criterion (see, e.g. [13, Theorem 1.3.4]), we get the following result for $d = 1$:

Proposition 3.1. *The equilibrium $p = 1$ of*

$$x_{n+1} = \frac{(\alpha + 1)x_n^\gamma}{\alpha + x_{n-1}}, \quad n = 0, 1, 2, \dots$$

is locally asymptotically stable if and only if $0 < \gamma < (\alpha + 2)/(\alpha + 1)$.

Proof. The Schur–Cohn conditions are equivalent to

$$\gamma < 1 + q < 2 \Longleftrightarrow \gamma < \frac{\alpha + 2}{\alpha + 1} < 2.$$

The second inequality trivially holds for all $\alpha > 0$. □

Corollary 3.2. *If $\gamma \in (0, 1]$, then the equilibrium $p = 1$ of*

$$x_{n+1} = \frac{(\alpha + 1)x_n^\gamma}{\alpha + x_{n-1}}, \quad n = 0, 1, 2, \dots$$

is locally asymptotically stable for all $\alpha > 0$.

For $d > 1$, we use Corollary 4.2 in [5], which provides the following result:

Theorem 3.3. *The equilibrium $p = 1$ of* (3.1) *is locally asymptotically stable if and only if either $\gamma \leq \alpha/(1 + \alpha)$, or*

$$\frac{\alpha}{\alpha + 1} < \gamma < \frac{\alpha + 2}{\alpha + 1} \tag{3.3}$$

and

$$d < \frac{\arccos\left(\frac{(\gamma^2 - 1)(1 + \alpha)^2 + 1}{2\gamma(1 + \alpha)}\right)}{\arccos\left(\frac{(\gamma^2 + 1)(1 + \alpha)^2 - 1}{2\gamma(1 + \alpha)^2}\right)}. \tag{3.4}$$

Corollary 3.4. *If $\gamma \geq 2$, then the equilibrium $p = 1$ of* (3.1) *is unstable for all positive integer d and all real number $\alpha > 0$.*

If we denote by $G(\gamma, \alpha)$ the function on the right-hand side of (3.4), it is clear that $\lim_{\alpha \to 0^+} G(\gamma, \alpha) = 1$ for every fixed $\gamma > 0$. Thus, if $d > 1$, for every $\gamma > 0$, there is a value of $\alpha > 0$ such that the equilibrium $p = 1$ of (3.1) is unstable. This means that we cannot extend the result of Corollary 3.2 to larger values of the delay.

Analogously, Theorem 3.3 establishes a sufficient condition $\gamma \leq \alpha/(1 + \alpha)$ for the asymptotic stability of the equilibrium independent of the delay d (absolute local stability). This condition is also necessary because if $\gamma > \alpha/(1 + \alpha)$, then there is a value of d for which (3.4) does not hold. The region R_d of absolute local stability is represented in Fig. 1 in the parameter plane (α, γ).
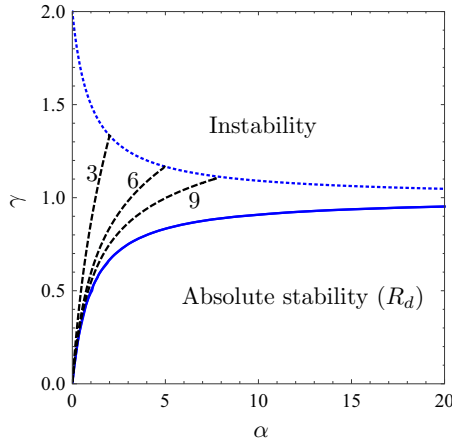
Figure 1. Stability diagram of the positive equilibrium $p = 1$ of (3.1) in the parameter plane (α, γ). The equilibrium is asymptotically stable for every d in the region R_d below the blue solid line, and unstable above the dotted blue line. For particular values of d, the stability region is the one below the dotted blue line and the corresponding black dashed line (examples for $d = 3, 6, 9$) (color figure online)

3.2. Main Results: Global Stability in the Two-Dimensional Case

This subsection is devoted to show that the conclusion of Corollary 3.2 has a global character; that is, the positive equilibrium $p = 1$ of (3.1) is globally asymptotically stable if $d = 1$ and $0 < \gamma \leq 1$.

In the case $\gamma = 1$ (Pielou equation), this result was proved by Kuruklis and Ladas [15, Theorem 4]. We notice that in the one-dimensional case $d = 0$, the equilibrium is also globally asymptotically stable if $0 < \gamma \leq 1$. See [15, Theorem 3] for $\gamma = 1$ and [18, Theorem 3.1] for $0 < \gamma < 1$.

For convenience, we state the form of (3.1) for $d = 1$:

$$x_{n+1} = \frac{(\alpha + 1)x_n^\gamma}{\alpha + x_{n-1}}, \quad n = 0, 1, 2, \ldots, \tag{3.5}$$

with positive initial conditions x_{-1}, x_0.

Theorem 3.5. *The equilibrium $p = 1$ is globally asymptotically stable for* (3.5) *if $\alpha > 0$ and $0 < \gamma \leq 1$.*

First, we list some notation. If $x, y > 0$, we introduce

$$z = \frac{(\alpha + 1)y^\gamma}{\alpha + x}, \quad a = \frac{(\alpha + 1)z^\gamma}{\alpha + y} = y^{\gamma^2} \frac{(\alpha + 1)^{\gamma+1}}{(\alpha + x)^\gamma (\alpha + y)},$$

and we denote

$$\rho_1 = \gamma(1 - \gamma), \quad \rho_2 = 1 + \gamma(1 - \gamma),$$
$$\rho_3 = \gamma(1 + \gamma) - 1, \quad \rho_4 = \frac{(1 - \gamma)(2 + \gamma)}{\gamma(1 + \gamma) - 1}.$$

Also, we consider the maps

$$h(t) = \left(\frac{\alpha+1}{\alpha+t}\right)^{\mu+1} (\mu > 0), \quad f(t) = t^{\gamma^2}\left(\frac{\alpha+1}{\alpha+t}\right)^{\gamma-\mu} \quad (\mu > 0),$$

$$u(t) = \frac{\alpha+t}{\alpha+1}, \quad v(t) = t^{\rho_3}$$

for $t > 0$, and the one-dimensional equation

$$t_{n+1} = h(t_n), \quad n = 0, 1, 2, \ldots \tag{3.6}$$

with positive initial condition t_0. Finally, we consider the scalar map

$$V(x, y) = \log^2 x - \gamma \log x \log y + \log^2 y$$

for $x, y > 0$. Note that $V(1, 1) = 0$ and $V(x, y) > 0$ otherwise. In fact, this map is a first integral (that is, it is invariant for the orbits) of (3.5) in the limit case $\alpha = 0$; one can easily find V by taking logarithms in this limit equation and integrating the ensuing linear equation.

Lemma 3.6. *If $\alpha \geq \mu$, then 1 is globally asymptotically stable for* (3.6)*. If $\alpha < \mu$, then there is a 2-periodic orbit attracting all orbits of* (3.6) *(except the equilibrium 1).*

Proof. This is an easy consequence of three facts: h is decreasing, the Schwarzian derivative of h is negative, and $h'(-1) = -(1+\mu)/(1+\alpha)$ (see, e.g. [20, Theorem 1]). □

     The following notion of dominance is necessary for stating our next lemma, and it will also play an essential role in our arguments.

Definition 3.7 ([11, p. 106]). Let $I \subseteq \mathbb{R}$ be an open interval, and let $G\colon I^{k+1} \to I$ and $H\colon I \to I$ be continuous maps, and consider the difference equation

$$x_{n+1} = G(x_n, \ldots, x_{n-k}), \quad n \geq 0, \quad (x_0, \ldots, x_{-k}) \in I^{k+1}. \tag{3.7}$$

Then G is said to be dominated by H if the following condition is fulfilled:

(DC) There exists $l \geq 0$ such that for every semi-orbit $(x_n)_{n=-k}^{l+1}$ of (3.7), the following holds: if $x_{l+1} \geq \max\{x_l, \ldots, x_{-k}\}$ (respectively, $x_{l+1} \leq \min\{x_l, \ldots, x_{-k}\}$), then there exists some $x \in \text{conv}\{x_l, \ldots, x_{-k}\}$, for which inequality $h(x) \geq x_{l+1}$ (respectively, $h(x) \leq x_{l+1}$) holds, where $\text{conv } A$ denotes the smallest connected set containing the set A.

Lemma 3.8. *Assume $\mu \geq \rho_1$. Then h dominates* (3.5) *in the sense of Definition* 3.7*, with $l = 1$.*

Proof. Note first that, due to the condition on μ, the map f is increasing.
     We must show that if $a \geq \max\{x, y, z\}$ (respectively, $a \leq \min\{x, y, z\}$), then there is $t \in \text{conv}\{x, y, z\}$ such that $h(t) \geq a$ (respectively, $h(t) \leq a$). We just consider the cases (i) $1 \leq x \leq y$, (ii) $1 \leq y \leq x$ and (iii) $y < 1 < x$, because the argument in the other cases is analogous. Of course, we can discard the trivial possibility $x = y = 1$; hence, in cases (i) and (ii), we also assume, respectively, $y > 1$ and $x > 1$.

Case (i): $1 \le x \le y$. If $z > 1$, then $a < z$, and if $z \le 1$, then $a < y$. Thus, in either case, $a < \max\{x, y, z\}$, so our statement follows after proving $h(y) \le a$, that is,

$$\left(\frac{\alpha+1}{\alpha+y}\right)^{\mu+1} \le y^{\gamma^2} \frac{(\alpha+1)^{\gamma+1}}{(\alpha+x)^{\gamma}(\alpha+y)}.$$

If fact, since $x \le y$, it suffices to show

$$\left(\frac{\alpha+1}{\alpha+y}\right)^{\mu+1} \le y^{\gamma^2} \left(\frac{\alpha+1}{\alpha+y}\right)^{\gamma+1},$$

that is, $f(y) \ge 1$, which is true because f is increasing and $y > 1$.

Case (ii): $1 \le y \le x$. As in Case (i), if $z > 1$, then $a < z$, and if $z \le 1$, then $a < x$. Thus, it suffices to show now that $h(x) \le a$, that is,

$$\left(\frac{\alpha+1}{\alpha+x}\right)^{\mu+1} \le y^{\gamma^2} \frac{(\alpha+1)^{\gamma+1}}{(\alpha+x)^{\gamma}(\alpha+y)},$$

or, equivalently,

$$\frac{(\alpha+1)^{\mu+1}}{(\alpha+x)^{\mu+1-\gamma}} \le y^{\gamma^2} \frac{(\alpha+1)^{\gamma+1}}{\alpha+y},$$

Since $y \le x$, it suffices to show

$$\frac{(\alpha+1)^{\mu+1}}{(\alpha+y)^{\mu+1-\gamma}} \le y^{\gamma^2} \frac{(\alpha+1)^{\gamma+1}}{\alpha+y},$$

which, again, amounts to $f(y) \ge 1$.

Case (iii): $y < 1 < x$. Now $z < 1$, so $a > z^{\gamma} > z$. Then it suffices to show that $h(y) \ge a$, that is,

$$\left(\frac{\alpha+1}{\alpha+y}\right)^{\mu+1} \ge y^{\gamma^2} \frac{(\alpha+1)^{\gamma+1}}{(\alpha+x)^{\gamma}(\alpha+y)}.$$

Since $x > y$, we just need to show

$$\left(\frac{\alpha+1}{\alpha+y}\right)^{\mu+1} \ge y^{\gamma^2} \left(\frac{\alpha+1}{\alpha+y}\right)^{\gamma+1},$$

that is, $f(y) \le 1$, which follows, once again, because f is increasing and now $y < 1$. □

Lemma 3.9. *If $\alpha \ge \rho_1$, then 1 is globally asymptotically stable for (3.5). If $\alpha < \rho_1$, then (3.5) is permanent and the ω-limit set of every orbit of (3.5) is included in the interval $[r(\alpha), s(\alpha)]$ whose endpoints are the points of the 2-periodic orbit of (3.6) for $\mu = \rho_1$.*

Proof. This follows from Lemmas 3.6 and 3.8 after applying [11, Theorem B] to h with $\mu = \rho_1$. (We emphasize that, although strictly speaking [11, Theorem B] just states permanence in the case $\mu = \rho_1$, its proof guarantees that the ω-limit sets of the orbits of (3.5) are included in any compact invariant (for h) interval M whose interior contains all ω-limit sets of the orbits of (3.6). In our present situation, if $x = r(\alpha) - \epsilon$ for an arbitrarily small number

$\epsilon > 0$, then the interval $M = [x, h(x)]$ has this property. As a conclusion, all ω-limit sets of (3.5) are, in fact, included in $[r(\alpha), s(\alpha)]$.) $\qquad\square$

Remark 3.10. Observe that, in particular, we have just shown global asymptotical stability for the classical Pielou equation ($\gamma = 1$), first proved, as previously mentioned, by Kuruklis and Ladas in [15]. While elementary as well, their proof relies on very specific properties of the equation (to begin with, the equilibrium is relocated in the origin after a linear change of variables, which would be rather unpractical in the generalized case $0 < \gamma < 1$), allowing them to estimate the relative distance to the origin of consecutive blocks of positive and negative points of any given orbit. In contrast to this, both our choice of the dominance map and the checking of the dominance condition (**DC**) in Lemma 3.8 are completely natural, and the argument works (partially) as well when $\gamma < 1$.

The rest of the proof bases on the following Lyapunov-type result.

Lemma 3.11. *Consider the discrete-time semi-dynamical system $S: \mathbb{N}_0 \times X \to X$, $(n, \xi) \mapsto S(n, \xi)$ on an arbitrary metric space X, which possesses a global attractor. Assume that $\xi^* \in X$ is a locally attractive fixed point and that $V: X \to [0, \infty)$ is a quasi-Lyapunov function in the following sense: V is continuous on X, $V(\xi^*) = 0$, whereas for all $\xi^* \neq \xi \in X$, $V(\xi) > 0$ holds, and there exists $n = n(\xi) \geq 1$ such that $V(S(n, \xi)) < V(\xi)$.*

Then ξ^* is the global attractor of the semi-dynamical system.

The main idea of quasi-Lyapunov technique has been already used as an effective tool to prove global stability in rational difference equations; this technique appeared in the paper of Merino [22] to prove global stability in another generalization of the Pielou equation (solving the so-called Y2K problem suggested by G. Ladas). Motivated by the approach of Merino, Bastien and Rogalski [4] introduced the concept of quasi-Lyapunov functions for discrete dynamical systems of arbitrary order k, and used them in the context of Lyness type difference equations. A similar idea already appeared in a paper by Kruse and Nesemann [14], which was later successfully applied by Hogan and Zeilberger [10] to obtain an algorithmic approach for proving global asymptotic stability for rational difference equations.

In spite of the similarities in the definition, in the statement and in the technique of the proof, there are also differences in each part, so we give our own proof of the lemma below.

Proof. Since ξ^* is locally attractive, it is sufficient to show that for all $\xi \in X$, $\xi^* \in \omega(\xi)$ holds. Note that the existence of the global attractor implies that $\omega(\xi)$ is nonempty, compact and invariant for all $\xi \in X$ (see, e.g. [9, Corollary 2.2.4]).

Now, seeking for contradiction, suppose that there exists $\xi \in X$ such that $\xi^* \notin \omega(\xi)$. Let $m := \min_{\eta \in \omega(\xi)} V(\eta)$ and $\eta_0 \in \omega(\xi)$ such that $V(\eta_0) = m$. Our assumptions on V yield on the one hand that $m > 0$, and on the other hand, there exists $n = n(\eta_0)$ such that $V(S(n, \eta_0)) < V(\eta_0) = m$. This, together with invariance of $\omega(\xi)$, contradicts the minimality property of η_0. $\qquad\square$

In the following lemma, we show that $V(x, y)$ is a quasi-Lyapunov function for (3.5) if γ or α are not too large. In the following lemmas, we assume $(x, y) \neq (1, 1)$.

Lemma 3.12. *Assume $\rho_3 \leq 0$. Then either $V(y, z) < V(x, y)$ or $V(z, a) < V(x, y)$. As a consequence, 1 is globally asymptotically stable for (3.5).*

Proof. Concerning the first statement, observe that

$$\Delta V(x, y) := V(x, y) - V(y, z) = \log\left(\frac{x(\alpha + x)}{(\alpha + 1)y^\gamma}\right) \log\left(\frac{x(\alpha + 1)}{\alpha + x}\right).$$

Therefore, $\Delta V(x, y) > 0$ trivially holds if $x \leq 1 \leq y$ or $y \leq 1 \leq x$, so we assume in what follows that either $x, y \leq 1$ or $x, y \geq 1$. In fact, $\Delta V(x, y) \leq 0$ if and only if

$$(x, y) \in \left\{(x, y) \in \mathbb{R}_+^2 : x \geq 1 \quad \text{and} \quad \left(\frac{x(\alpha + x)}{\alpha + 1}\right)^{1/\gamma} \leq y\right\}$$

or

$$(x, y) \in \left\{(x, y) \in \mathbb{R}_+^2 : x \leq 1 \quad \text{and} \quad \left(\frac{x(\alpha + x)}{\alpha + 1}\right)^{1/\gamma} \geq y\right\},$$

so we assume that we are in this case. This, in particular, implies $y < x$ (in the case $x, y \leq 1$) and $y > x$ (in the case $x, y \geq 1$). Proving $V(z, a) < V(x, y)$ is equivalent to showing $-\Delta V(x, y) < \Delta V(y, z)$, that is,

$$\log\left(\frac{(\alpha + 1)y^\gamma}{x(\alpha + x)}\right) \log\left(\frac{x(\alpha + 1)}{\alpha + x}\right) < \log\left(\frac{y(\alpha + y)}{(\alpha + 1)z^\gamma}\right) \log\left(\frac{y(\alpha + 1)}{\alpha + y}\right).$$

In fact, it is easy to see—by eliminating z from the right-hand side, and comparing terms on both sides—that it suffices to show

$$\log(y^\gamma) \log\left(\frac{x(\alpha + 1)}{\alpha + x}\right) < \log\left(\frac{y^{1-\gamma^2}(\alpha + y)}{\alpha + 1}\right) \log\left(\frac{y(\alpha + 1)}{\alpha + y}\right)$$

or just

$$|\log(y^\gamma)| < \left|\log\left(\frac{y^{1-\gamma^2}(\alpha + y)}{\alpha + 1}\right)\right|,$$

which in turn amounts to check that $u(y) < v(y)$ (respectively, $u(y) > v(y)$) whenever $y < 1$ (respectively, $y > 1$). This is obvious because u is increasing and, due to the hypothesis on γ, v is decreasing, and $v(1) = u(1)$.

To prove the last statement, we simply need to show that Lemma 3.11 can be applied to the semi-dynamical system corresponding to (3.5) and generated by the two-dimensional mapping:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ \frac{(\alpha+1)y^\gamma}{\alpha+x} \end{pmatrix} \tag{3.8}$$

with the state space $X := \mathbb{R}_+^2$.

In the first statement of the lemma, we showed that V has the quasi-Lyapunov property (with $\xi^* = (1, 1)$). Furthermore, Corollary 3.2 implies

that $(1,1)$ is locally asymptotically stable; hence, it is only left to verify that there exists a global attractor, which is ensured by permanence (see Lemma 3.9).

Therefore, application of Lemma 3.11 yields global asymptotic stability of the equilibrium 1 of (3.5). $\qquad\square$

Lemma 3.13. *Assume $\rho_3 > 0$ and $\alpha < \rho_1$. Then there is exactly one point $0 < q < 1$ satisfying $u(q) = v(q)$, and $u(t) < v(t)$ (respectively, $u(t) > v(t)$) whenever $q < t < 1$ (respectively, $t > 1$). Moreover, $q < r(\alpha)$.*

Proof. The first statement follows easily from $u(1) = v(1)$, $v(0) < u(0)$, the concavity of v, and from $\alpha < \rho_1 < \rho_4$, which implies

$$v'(1) = \rho_3 < \frac{1}{\alpha + 1} = u'(1).$$

Proving $q < r(\alpha)$ amounts to show $h^2(q) > q$ (when $\mu = \rho_1$), that is,

$$\left(\frac{\alpha+1}{\alpha+q}\right)^{\rho_2} q^{1/\rho_2} < 1 + \alpha(1 - q^{1/\rho_2}).$$

In fact, the stronger statement

$$\left(\frac{\alpha+1}{\alpha+q}\right)^{\rho_2} q^{1/\rho_2} < 1,$$

that is,

$$q^{1/\rho_2 - \rho_2\rho_3} < 1,$$

holds because

$$1 - \rho_2^2\rho_3 = (1 - \gamma)((3\gamma + 1)(1 - \gamma^2) + 1 - \gamma^3 + \gamma^5) > 0.$$

$\qquad\square$

The following lemma is probably true as well without any restrictions on α or y, but this is enough for our purposes.

Lemma 3.14. *Assume $\rho_3 > 0$ and $\alpha < \rho_1$ and set q as defined in Lemma 3.13. If $y > q$, then either $V(y, z) < V(x, y)$ or $V(z, a) < V(x, y)$. As a consequence, if $\rho_3 > 0$ and $\alpha < \rho_1$, then 1 is globally asymptotically stable for (3.5).*

Proof. The statement concerning V can be proved, due to the properties of u and v to the right of q (Lemma 3.13), with the same argument as that of Lemma 3.12. To deduce asymptotic stability recall first that, by Lemma 3.13, $q < r(\alpha) < s(\alpha) < h(q)$ and $h^2(q) > q$. This last inequality implies, because h is decreasing, that h maps $(q, h(q))$ into itself. Then, as a consequence of the dominance condition, the two-dimensional mapping (3.8) maps the square $X = (q, h(q))^2$ into itself; hence, the corresponding semi-dynamical system $S : \mathbb{N} \cup \{0\} \times X \to X$ is well defined. Now $q < r(\alpha) < s(\alpha) < h(q)$ and Lemma 3.9 guarantees the existence of a global attractor for S, and we can apply Lemma 3.11 to get asymptotic stability for this "restricted" system. But, again by Lemma 3.9, every orbit of (3.5) eventually lies between q and $h(q)$. This finishes the proof. $\qquad\square$

Proof of Theorem 3.5. The   proof   follows   from   Lemmas   3.9,   3.12
and 3.14. ☐

3.3. Global Stability for $d > 1$

In this subsection, we address the global stability of the equilibrium $p = 1$ of
(3.1) for $\gamma \in (0, 1]$ and $d > 1$.

From Theorem 2.1, we get the following result:

Corollary 3.15. *Assume that $\alpha > 0$, $0 < \gamma < 1$, and let $A = (\alpha + 1)(1 - \gamma)$.
The equilibrium $p = 1$ of (3.1) is globally asymptotically stable if either $A \geq 1$,
or $A < 1$ and*

$$\gamma^{d+1} > \frac{1}{A} \log \left(\frac{1 + A}{1 + A^2} \right). \tag{3.9}$$

Proof. In this case, $F(x) = (\alpha + 1)/(\alpha + x)$, and $F'(1) = -1/(\alpha + 1)$. Hence,
the value of A in the statement of Theorem 2.1 is

$$A = \frac{-p^{-\gamma}(1 - \gamma)}{F'(p)} = \frac{-(1 - \gamma)}{F'(1)} = (\alpha + 1)(1 - \gamma).$$

On the other hand, inequality (2.3) reads

$$\frac{x^2}{\alpha + x^2} < 1, \quad \forall\, x > 0,$$

which obviously holds for all $\alpha > 0$. ☐

It is worth clarifying that in the special case $d = 1$, Theorem 3.5 is more
general than Corollary 3.15.

For the Pielou equation ($\gamma = 1$), we can apply Proposition 2.2 to get
the following result:

Corollary 3.16. *Assume that $\alpha > 0$ and $\gamma = 1$. The equilibrium $p = 1$ of (3.1)
is globally asymptotically stable if the following inequality holds:*

$$\alpha > \frac{2d^2 + d - 2}{3d + 4}. \tag{3.10}$$

Since condition $A \geq 1$ is equivalent to the absolute stability condition
$\gamma \leq \alpha/(1 + \alpha)$, we have the following consequence of Corollary 3.15:

Corollary 3.17. *The equilibrium $p = 1$ of (3.1) is globally asymptotically sta-
ble for all $d \geq 0$ if and only if $\gamma \leq \alpha/(1 + \alpha)$.*

In Fig. 2, we plot the stability diagram of the positive equilibrium $p = 1$
of (3.1) in the parameter plane (γ, d) for $\alpha = 1$.

Remark 3.18. Dominance allows to prove Corollary 3.17 in a very simple,
direct way. Actually, we prove that condition

$$\alpha \geq \gamma + \cdots + \gamma^d = \frac{\gamma - \gamma^{d+1}}{1 - \gamma} \tag{3.11}$$

is sufficient for the global stability in (3.1). Thus, for every d, 1 is GAS if
$\alpha \geq \gamma/(1 - \gamma)$, which is equivalent to $\gamma \leq \alpha/(1 + \alpha)$.

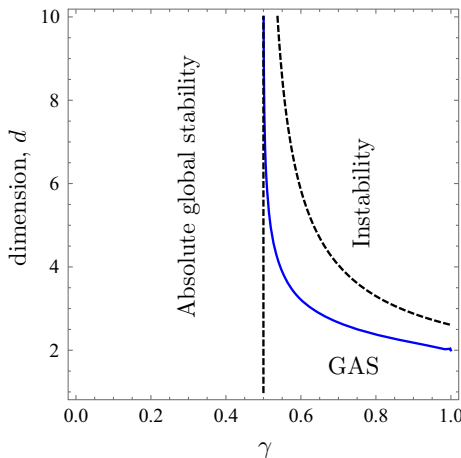Figure 2. Stability diagram of the positive equilibrium $p = 1$ of (3.1) in the parameter plane (γ, d) for $\alpha = 1$. By Corollary 3.17, the equilibrium is asymptotically stable for every d if $\gamma \leq 1/2$. The blue solid line corresponds to the global stability condition (3.9), and the black dashed line is the boundary of local asymptotic stability given by (3.4). Notice that the relevant values on the d axes are the positive integers (color figure online)

Indeed, in view of Lemma 3.6, the statement follows after proving that $h(t)$ dominates (3.1) when $\mu = \gamma + \cdots + \gamma^d$ and $l = d$.

We must show that if $(x_n)_{n=-d}^{d+1}$ is a semi-orbit of (3.1) and $x_{d+1} \geq \max\{x_d, \ldots, x_{-d}\}$ (respectively, $x_{d+1} \leq \min\{x_d, \ldots, x_{-d}\}$), then there is $t \in \mathrm{conv}\{x_d, \ldots, x_{-d}\}$ such that $h(t) \geq x_{d+1}$ (respectively, $h(t) \leq x_{d+1}$). Say, for instance, $x_{d+1} \geq \max\{x_d, \ldots, x_{-d}\}$ (the other case is similar). Then either $x_d \leq 1$ or $x_0 \leq 1$, and also $t := \min\{x_d, \ldots, x_{-d}\} \leq 1$. If $x_d \leq 1$, then

$$x_{d+1} = \frac{(\alpha + 1)x_d^\gamma}{\alpha + x_0} \leq \frac{\alpha + 1}{\alpha + t} \leq h(t).$$

If $x_0 \leq 1$, then

$$x_{d+1} = \frac{(\alpha + 1)^{1+\gamma+\cdots+\gamma^d} x_0^{\gamma^{d+1}}}{(\alpha + x_0)(\alpha + x_{-1})^\gamma \cdots (\alpha + x_{-d})^{\gamma^d}} \leq \frac{(\alpha + 1)^{1+\gamma+\cdots+\gamma^d}}{(\alpha + t)^{1+\gamma+\cdots+\gamma^d}} = h(t).$$

This finishes the proof.

For parameters $\gamma \in (0, 1)$ for which global stability is not granted, we can show permanence:

Proposition 3.19. *Equation* (3.1) *is permanent: if* $\alpha < \gamma + \cdots + \gamma^d$, *then the* ω-*limit set of every orbit of* (3.1) *is included in the interval* $[C, D]$ *whose endpoints are the points of the 2-periodic orbit of* (3.6) *for* $\mu = \gamma + \cdots + \gamma^d$.

Proof. For $\alpha \geq \gamma + \cdots + \gamma^d$, the 1 equilibrium of Eq. (3.1) is GAS (see Remark 3.18); therefore, (3.1) is also permanent. Thus, we can focus on parameters with $\alpha < \gamma + \cdots + \gamma^d$. First, we note that permanence was already proved in [1, Theorem 1].

We have seen in Remark 3.18 that the map h with $\mu = \gamma + \cdots + \gamma^d$ dominates g in the sense of Definition 3.7. The rest of the proof coincides with that of Lemma 3.9 (with (3.5) replaced by (3.1)). □

Note that monotonicity of h immediately implies the uniform ultimate upper bound

$$D \leq D_{\alpha,\gamma} := \left(\frac{\alpha+1}{\alpha}\right)^{(1-\gamma^{d+1})/(1-\gamma)}. \tag{3.12}$$

3.4. A Complementary Result for Global Stability

Making use of the uniform ultimate upper bound (3.12) and the main idea of Remark 3.18 we can give another criteria for global stability of (3.1). This extends somewhat the GAS regime (in the (α, γ) parameter plane) in the $d = 2, 3$ cases (see Fig. 3 for $d = 2$).

For convenience, let us use the notation $c\colon [0,\infty) \to [0,\infty)$,

$$c(x) = \frac{x^{\gamma^{d+1}}}{\alpha + x}.$$

Theorem 3.20. *Assume that $\alpha > 0$ and $\gamma \in (0,1)$. Then the equilibrium 1 of* (3.1) *is GAS provided the following three assumptions are fulfilled:*

(i) $\gamma(1 - \gamma^d) > 1 - \gamma$,
(ii) $c(D_{\alpha,\gamma}) \geq \frac{1}{\alpha+1}$,
(iii) $(1+\alpha)(1-\gamma) \geq \gamma(1-\gamma^d)$.
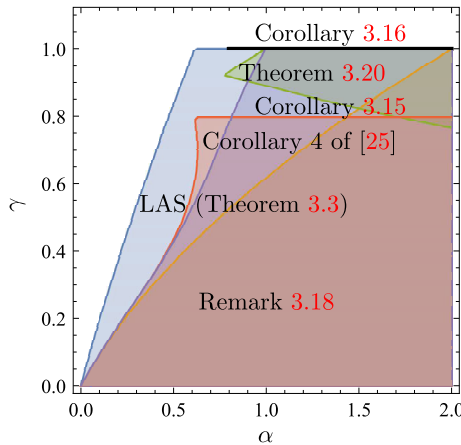

Figure 3. Comparison of the regions of LAS and GAS provided by various criteria for $d = 2$

Proof. Proposition 3.19 allows us to concentrate only on solutions for which $x_n \leq D_{\alpha,\gamma}$ holds for all $n \geq 0$.

We show that h dominates (3.1) under the above assumptions with $l = d$ and $\mu = \frac{\gamma(1-\gamma^d)}{1-\gamma} - 1 = \gamma + \cdots + \gamma^d - 1$. Then (i) guarantees that $\mu > 0$, while (iii) ensures that $\alpha \geq \mu$ holds and, therefore, by virtue of Lemma 3.6, (3.1) is GAS.

To show dominance of h, we need to consider two cases.

Case 1 $x_{d+1} \geq \max\{x_d, \ldots, x_{-d}\}$. Then $x_d \leq 1$ or $x_0 \leq 1$ must hold. Let $t = \min\{x_d, \ldots, x_{-d}\} \leq 1$.

If $x_d \leq 1$, then

$$x_{d+1} = \frac{(\alpha+1)x_d^\gamma}{\alpha+x_0} \leq \frac{\alpha+1}{\alpha+t} \leq \left(\frac{\alpha+1}{\alpha+t}\right)^{\gamma+\cdots+\gamma^d} = h(t)$$

holds, where the last inequality follows from assumption (i) and $t \leq 1$.

If $x_0 \leq 1$, then consider

$$
\begin{aligned}
x_{d+1} &= \frac{(\alpha+1)^{1+\gamma+\cdots+\gamma^d} x_0^{\gamma^{d+1}}}{(\alpha+x_0)(\alpha+x_{-1})^\gamma \cdots (\alpha+x_{-d})^{\gamma^d}} \\
&= \frac{(\alpha+1)^{1+\gamma+\cdots+\gamma^d} c(x_0)}{(\alpha+x_{-1})^\gamma \cdots (\alpha+x_{-d})^{\gamma^d}}.
\end{aligned}
\tag{3.13}
$$

Note that $c(0) = 0$, $c(1) = \frac{1}{\alpha+1}$ and, moreover, c is strictly increasing on $(0, \xi)$ and strictly decreasing on (ξ, ∞), where $\xi = \frac{\alpha\gamma^{d+1}}{1-\gamma^{d+1}}$. These together with $x_0 \leq 1 < D_{\alpha,\gamma}$ and assumption (ii) infer that $c(x_0) \leq \frac{1}{\alpha+1}$ and, hence,

$$x_{d+1} \leq \frac{(\alpha+1)^{\gamma+\cdots+\gamma^d}}{(\alpha+x_{-1})^\gamma \cdots (\alpha+x_{-d})^{\gamma^d}} \leq \frac{(\alpha+1)^{\gamma+\cdots+\gamma^d}}{(\alpha+t)^{\gamma+\cdots+\gamma^d}} = h(t).$$

Case 2 $x_{d+1} \leq \min\{x_d, \ldots, x_{-d}\}$. Then, similar to Case 1, $x_d \geq 1$ or $x_0 \geq 1$ must hold. Let $t = \max\{x_d, \ldots, x_{-d}\} \geq 1$.

The case $x_d \geq 1$ is completely analogous to the $x_d \leq 1$ part of Case 1 and, therefore, omitted.

It remains the case if $x_0 \geq 1$. Then using the properties of c and (ii) ensures that $c(x_0) \geq \frac{1}{\alpha+1}$ and thus from formula (3.13) we obtain that

$$x_{d+1} \geq \frac{(\alpha+1)^{\gamma+\cdots+\gamma^d}}{(\alpha+t)^{\gamma+\cdots+\gamma^d}} = h(t).$$

This completes the proof. $\qquad\square$

4. Final Remarks

In this section, we provide some remarks about some related papers and suggest some open problems.

- Agarwal et al. [1] considered a more general equation

$$x_{n+1} = x_n^\gamma f(x_{n-k_1}, x_{n-k_2}, \ldots, x_{n-k_r}), \quad n = 0, 1, \ldots, \tag{4.1}$$

where $\gamma > 0$, $k_1 < k_2 < \cdots < k_r$ are nonnegative integers, and the continuous map $f \colon [0, \infty)^r \to (0, \infty)$ is nonincreasing in each of its arguments. For $\gamma < 1$, their Theorem 2 establishes that the unique positive equilibrium p of (4.1) is a global attractor of all positive solutions if p is a global attractor for the related one-dimensional map

$$h(x) = p^{\gamma^d}(f(x, x, \ldots x))^{(1-\gamma^d)/(1-\gamma)},$$

where $d = k_r$. However, there is a mistake in the proof of their theorem, and the map h should be defined as

$$h(x) = p^{\gamma^{d+1}}(f(x, x, \ldots x))^{(1-\gamma^{d+1})/(1-\gamma)}.$$

We can apply the corrected version of this result to equation (3.1), using Lemma 3.6. The sufficient condition for the global attractivity of the equilibrium 1 provided by that result is exactly the dominance condition (3.11).

- The global stability results given in this paper are particular examples of global stability results for Clark Eq. (1.4). Actually, Theorem 3.5 provides an additional example for which the conjecture 'LAS implies GAS' in Clark's equation holds for $d = 1$. For related results in this direction, see [2,12]. Our conjecture is that, at least for $d = 2$, the equilibrium 1 of (3.1) is a global attractor when it is locally asymptotically stable, that is, in the conditions of Theorem 3.3 (with $\gamma < 1$).
- The paper [25] by Tkachenko and Trofimchuk gives additional results that allow to sharpen the global stability region in some cases. For example, for $d = 2$, Corollary 4 in [25] establishes the global stability of 1 in (3.1) if $\gamma \leq 0.796$ and

$$\gamma^3 \geq \frac{1-A}{1+A},$$

where $A = (\alpha + 1)(1 - \gamma)$ was defined in Corollary 3.15 (see Fig. 3).
- In case $\gamma < 1$, one may get some information about the global dynamics even if the positive equilibrium of (1.5) is not globally attractive: the interested reader may easily check that the results of [8] can be applied for equation (1.5) to obtain a Morse decomposition of the global attractor, which is based on a discrete valued Lyapunov functional.

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References

[1] Agarwal, R.P., Li, W.T., Pang, P.Y.H.: Asymptotic behavior of a class of nonlinear delay difference equations. J. Differ. Equ. Appl. **8**, 719–728 (2002)

[2] Bartha, F.A., Garab, Á.: Necessary and sufficient condition for the global stability of a delayed discrete-time neuron model. J. Comput. Dyn. **1**, 213–232 (2014)

[3] Bashkirtseva, I., Ryashko, L.: Stochastic sensitivity analysis of noise-induced extinction in the Ricker model with delay and Allee effect. Bull. Math. Biol. **80**, 1596–1614 (2018)

[4] Bastien, G., Rogalski, M.: Results and problems about solutions of perturbed Lyness' type order k difference equations in $\mathbb{R}_*^+$ $u_{n+k}(u_n + \lambda) = f(u_{n+k-1}, \ldots, u_{n+1})$, with examples, and test of the efficiency of a quasi-Lyapunov function method. J. Differ. Equ. Appl. **19**, 1331–1352 (2013)

[5] Čermak, J., Jánský, J., Kundrát, P.: On necessary and sufficient conditions for the asymptotic stability of higher order linear difference equations. J. Differ. Equ. Appl. **18**, 1781–1800 (2012)

[6] Clark, C.W.: A delayed recruitment model of population dynamics with an application to Baleen Whale populations. J. Math. Biol. **3**, 381–391 (1976)

[7] El-Morshedy, H.A., Jiménez López, V.: Global attractors for difference equations dominated by one-dimensional maps. J. Differ. Equ. Appl. **14**, 391–410 (2008)

[8] Garab, Á., Pötzsche, C.: Morse decompositions for delay-difference equations. J. Dyn. Differ. Eq. https://doi.org/10.1007/s10884-018-9685-8

[9] Hale, J.: Asymptotic Behavior of Dissipative Systems. Mathematical Surveys and Monographs, vol. 25. American Mathematical Society, Providence (1988)

[10] Hogan, E., Zeilberger, D.: A new algorithm for proving global asymptotic stability of rational difference equations. J. Differ. Equ. Appl. **18**(11), 1853–1873 (2012)

[11] Jiménez López, V.: The Y2K problem revisited. J. Differ. Equ. Appl. **16**, 105–119 (2010)

[12] Jiménez López, V., Parreño, E.: LAS and negative Schwarzian derivative do not imply GAS in Clark's equation. J. Dyn. Differ. Eq. **28**, 339–374 (2016)

[13] Kocić, V.L., Ladas, G.: Global Behavior of Nonlinear Difference Equations of Higher Order with Applications. Mathematics and its Applications, vol. 256. Kluwer Academic Publishers Group, Dordrecht (1993)

[14] Kruse, N., Nesemann, T.: Global asymptotic stability in some discrete dynamical systems. J. Math. Anal. Appl. **235**, 151–158 (1999)

[15] Kuruklis, S.A., Ladas, G.: Oscillations and global attractivity in a discrete delay logistic model. Q. Appl. Math. **50**, 227–233 (1992)

[16] Levin, S.A., May, R.M.: A note on difference delay equations. Theor. Pop. Biol. **9**, 178–187 (1976)

[17] Liz, E.: A global picture of the gamma-Ricker map: a flexible discrete-time model with factors of positive and negative density dependence. Bull. Math. Biol. **80**, 417–434 (2018)

[18] Liz, E.: A new flexible discrete-time model for stable populations. Discr. Contin. Dyn. Syst. B **23**, 2487–2498 (2018)

[19] Liz, E., Buedo-Fernández, S.: A new formula to get sharp global stability criteria for one-dimensional discrete-time models. Q. Theory Dyn. Syst. https://doi.org/10.1007/s12346-018-00314-4

[20] Liz, E., Röst, G.: Dichotomy results for delay differential equations with negative Schwarzian derivative. Nonlinear Anal. Real World Appl. **11**, 1422–1430 (2010)

[21] Mallet-Paret, J., Sell, G.R.: Differential systems with feedback: time discretizations and Lyapunov functions. J. Dyn. Differ. Equ. **15**, 659–698 (2003)

[22] Merino, O.: Global attractivity of the equilibrium of a difference equation: an elementary proof assisted by computer algebra system. J. Differ. Equ. Appl. **17**, 33–41 (2011)

[23] Pielou, E.C.: Population and Community Ecology. Gordon and Breach, New York (1974)

[24] Quinn, T.J., Deriso, R.B.: Quantitative Fish Dynamics. Oxford University Press, New York (1999)

[25] Tkachenko, V., Trofimchuk, S.: Global stability in difference equations satisfying the generalized Yorke condition. J. Math. Anal. Appl. **303**, 173–187 (2005)

[26] Tkachenko, V., Trofimchuk, S.: A global attractivity criterion for nonlinear nonautonomous difference equations. J. Math. Anal. Appl. **322**, 901–912 (2006)

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