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Some results on periodicity of difference equations

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Abstract

We will deal with the topic of the periodicity for difference equations with real values. In the autonomous case, $x_{n+k} = f(x_{n+k-1}, \dots, x_n)$, we will survey two topics: the forcing relations of periods and the global periodicity. In the non-autonomous case, $x_{n+k} = f(n, x_{n+k-1}, \dots, x_n)$ we focus our attention on periodic non-autonomous difference equations given by $x_{n+1} = f_{n \bmod p}(x_n)$, with $n \geq 1$. For these equations, we present a Sharkovsky type result characterizing their periodic structure. In both the autonomous and the non-autonomous case, we pose some open problems.

1 Introduction

In general, if X is a set, $f : \Omega \subseteq X^k \to X$ is a map defined on some subset of a finite Cartesian product of X, and $x_0, \dots, x_{k-1}$ are elements of X (so-called *initial conditions*), we say that

$$x_{n+k} = f(x_{n+k-1}, \dots, x_{n+1}, x_n), \quad n \geq 0, \tag{1}$$

is an (autonomous) *difference equation of order* k. Notice that, once we have introduced the initial conditions, the recurrence constructed by f gives a unique *solution* $\{x_n\}_{n=0}^{\infty}$ if that the map f is well defined for any element $(x_{n+k-1}, \dots, x_{n+1}, x_n)$.

If $f : \Omega \times \left(\mathbb{Z}^+ \cup \{0\}\right) \subseteq X^k \times \left(\mathbb{Z}^+ \cup \{0\}\right) \to X$ and

$$x_{n+k} = f(x_{n+k-1}, \dots, x_{n+1}, x_n, n), \quad n \geq 0, \tag{2}$$

we have a non-autonomous system.

When we obtain a constant solution $\{\overline{x}, \overline{x}, \dots\}$ we say that $\overline{x}$ is an *equilibrium point* of the difference equation. Notice that in the autonomous case $\overline{x}$ satisfies the equation $\overline{x} = f(\overline{x}, \dots, \overline{x})$.

The analysis of the asymptotic behavior of the solutions $\{x_n\}_{n=0}^{\infty}$ as n tends to infinity is a question of paramount importance. In applied mathematics, we find a lot of biological, economical, chemical, physical, ... models following laws described by recurrences or difference equations (the reader is referred to the textbooks [1, 30, 36, 38, 52, 58], in order to get examples illustrating the importance of difference equations as applied models).

In this paper we will concentrate on the special type of periodic behavior. Recall that $\{x_n\}_{n=0}^{\infty}$ is a *periodic solution* if $x_{n+m} = x_n$ for all $n \geq 0$ and some positive integer m. The smallest of such values is called the *period* of $\{x_n\}_{n=0}^{\infty}$. If, additionally, all the solutions are periodic and p is the least common multiple of their periods, we say that the difference equation is a $p-$*cycle* or is *globally periodic*. In the next two sections we will deal with the topic of the periodic structure of discrete dynamical systems and in the last section we will comment some aspects of the global periodicity. Most of the part of the following notes is extracted from [44].

Before starting our theoretical development, let us present some historical notes relative to the abundance of the difference equations. In fact, they appear during all the periods of *the History of Mathematics*, although, obviously, the mathematical formulation for difference equations was nonexistent. In their appearances the common idea is to give a recurrence method for making some arithmetic or geometric construction.

- There are archaeological signs (for instance the clay tablet YBC 7289) which allow us to know an arithmetic procedure in the Babylonian civilization for approximating $\sqrt{2}$ as the arithmetic mean of lower and upper bounds of this root. In our notation, $x_{n+1} = \frac{1}{2}(x_n+\frac{2}{x_n})$. To obtain more information about the mentioned tablet, the reader is referred to [32] and references therein.

- In the Egyptian culture, the geometrical sequences are included in the Rhind's papyrus (or Ahmes' papyrus) as the 79th problem, an amusing exercise whose statement says something like this (see [49]): "there are seven houses; in each house there are seven cats; each cat eat seven mice; each mouse eats seven ears of spelt; each ear of spelt produces seven hekat of grain. Find the total number of items involved".

- In the Ancient Greek culture, triangular numbers were an important and intriguing class of integer numbers for the Pythagorean school. Each number represented the amount of pebbles necessary to do the corresponding figure with form of triangle. They are defined by the non-autonomous difference equation $t_{n+1} = t_n + n + 1, t_1 = 1$. In spite of their importance, they are not mentioned in the thirteen books of Euclid's Elements, although in the seventh book a variety of integer numbers is presented: odd, even, multiple, perfect, ... It is interesting to mention that in the ninth book Euclid proves the formula to calculate the sum of a geometrical progression, but surprisingly nothing about arithmetical progressions is said.

- Theon of Smyrna (first-second century A.D.) was a neoplatonist philosopher who approximates $\sqrt{2}$ by using a process based on a linear system of difference equations:

  $$x_{n+1} = x_n + y_n, \qquad y_{n+1} = 2x_n + y_n.$$

  Theon observed that the successive differences between the areas of the squares of lengths y_n and δ_n, with δ_n being the diagonal of the square having length x_n, were always ± 1, that is, $y_n^2 - 2x_n^2 = (-1)^n$ and consequently $\frac{y_n}{x_n} = \sqrt{2 + \frac{(-1)^n}{x_n^2}}$. If we take $x_1 = y_1 = 1$, the above successive quotients give us a good approximation of $\sqrt{2}$.

- Fibonacci (1180-1250) is famous, above all, for the ubiquitous sequence receiving his name, $\{1, 1, 2, 3, 5, 8, 13, \dots\}$. It is the solution of the difference equation of second order $F_{n+2} = F_{n+1} + F_n$, with initial conditions $F_0 = F_1 = 1$, and appears in Chapter 12 of the "Liber abaci" [61] as the solution of a (somewhat artificial) growth problem of the rabbits population. The first time that Fibonacci difference equation appears in modern notation

$F_{n+2} = F_{n+1} + F_n$ is in *Les Oeuvres mathématiques de Simon Stevin augmentées par Albert Girard*, from Albert Girard, Leyde, 1634. In this work he also stated that the ratios of terms of the Fibonacci sequence tend to the golden ratio, that is, $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \Phi := \frac{1+\sqrt{5}}{2}$.

- Under suitable conditions, the Newton-Raphson method for approximating the roots of an equation $f(x) = 0$ is given by the difference equation $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. The first time we may recognize this method is inside the Newton's tract *De analysi per aequationes numero terminorum infinitas,* around 1669. It is interesting to mention that Vieta was a forerunner of this formula because in his *De numerosa potestatum,* Paris, 1600, he presented a numerical method similar to the above-mentioned one. To obtain more information about the history of Newton-Raphson method, see [68].

- At the XVIII century, and according to Jordan's book [35], "the origin of the Calculus of Finite Differences may be ascribed to Brook Taylor's *Methodus Incrementorum* (London, 1717), but the real founder of the theory was Jacob Stirling, who in his *Methodus Differentialis* (London, 1730) solved very advanced questions, and gave useful methods, introducing the famous Stirling numbers; these, though hitherto neglected, will form the backbone of the Calculus of Finite Differences." Jordan continues to say *"The first treatise on this Calculus is contained in Leonhardo Eulero Institutiones Calculus Differentialis* (Academiae Imperialis Scientarum Petropolitanae, 1755. See also *Opera Omnia*, Series I. Vol. X. 1913) in which he was the first to introduce the symbol Δ for the differences, which is universally used now. From the early works in this subject the interesting article *Différence* in the Encyclopédie Méthodique (Paris, 1784), written by l'Abbé Charles Bossut, should be mentioned, and also F.S. Lacroix's *Traité des différences et séries*, Paris, 1800."

- During the centuries XVIII-XIX we observe the creation of the fundamentals of the modern theory of difference equations, especially in the linear case. As an example of a famous mathematician working in this subject, we can cite, for instance, H. Poincaré [55]. Even in first half of the XX century the researchers continued constructing new methods for solving difference equations, as the case of U. Broggi [10] or L.M. Milne-Thomson [53].

- Difference equations appear in the setting of numerical methods of differential equations while approximating solutions through finite difference schemes both for ordinary differential equations -let us mention Euler's method, Runge-Kutta's methods, multistep methods- and partial difference equations as well (consult [12, 43, 50, 65, 66]).

2 Structure of the set of periods

In this section we briefly survey Sharkovsky's Theorem and solve the problem of determining the set of periods of any power f^n of a continuous interval map f by establishing the corresponding initial segment of Sharkovsky's ordering. Let us recall that we consider a discrete dynamical system as a pair (X, f), where f is a continuous map defined from a topological space X into itself (in short, $f \in C(X, X)$). In this setting, by f^n is meant the composition $f \circ f^{n-1}$, $n \geq 1$, with $f^0 = \mathrm{Id}|_X$ the identity map. A periodic point $x \in X$ satisfies $f^m(x) = x$ for some $m \geq 1$, and its *period* or *order*, denoted by $\mathrm{ord}_f(x)$, is the smallest of such values m. We use $\mathrm{Per}(f)$ to denote the set of periods of f.

2.1 Sharkovsky's Theorem

In our opinion, Sharkovsy's Theorem is the most famous theorem of combinatorial nature dealing with discrete dynamical systems. It is easy to state, even secondary students can understand it because they are only required to know the concept of continuity of a real map and the intermediate value property. Let $f : I \to I$ be a continuous map, with $I = [0, 1]$. Reorder the natural numbers as follows

$$\begin{aligned} &3 \gtrdot_s 5 \gtrdot_s 7 \gtrdot_s \ldots \gtrdot_s 2\cdot 3 \gtrdot_s 2\cdot 5 \gtrdot_s 2\cdot 7 \gtrdot_s \ldots \\ &2^n\cdot 3 \gtrdot_s 2^n\cdot 5 \gtrdot_s 2^n\cdot 7 \gtrdot_s \ldots \gtrdot_s 2^{n+1} \gtrdot_s 2^n \gtrdot_s \ldots \gtrdot_s 2 \gtrdot_s 1. \end{aligned}$$

For $n \in \mathbb{N} \cup \{2^\infty\}$ define

$$\mathcal{S}(n) = \{m \in \mathbb{N} : n \gtrdot_s m\} \cup \{n\} \quad \text{and} \quad \mathcal{S}(2^\infty) = \{2^n : n \in \mathbb{N} \cup \{0\}\}.$$

Then the *direct part* of Sharkovsky's theorem states that if f has a periodic orbit of period n, then $\mathcal{S}(n) \subseteq \mathrm{Per}(f)$, and the *converse statement* asserts that for any $n \in \mathbb{N} \cup \{2^\infty\}$ there is an $f_n \in C(I, I)$ such that $\mathrm{Per}(f_n) = \mathcal{S}(n)$ (see [59] or its English translation [60]).

There are different extensions of Sharkovsky's Theorem to other spaces and other particular classes of continuous maps: continuous circle maps $f : \mathbb{S}^1 \to \mathbb{S}^1$ (here by $\mathbb{S}^1$ is meant the circle $\{z \in \mathbb{C} : |z| = 1\}$); continuous triangular maps $f : I^k \to I^k$,

$$f(x_1, x_2, \ldots, x_k) = \big(f_1(x_1), f_2(x_1, x_2), \ldots, f_k(x_1, x_2, \ldots, x_k)\big);$$

continuous permutation maps $F : I^k \to I^k$ or $F : \mathbb{T}^k \to \mathbb{T}^k$,

$$F(x_1, x_2, ..., x_k) = \left(f_{\sigma(1)}(x_{\sigma(1)}), f_{\sigma(2)}(x_{\sigma(2)}), ..., f_{\sigma(k)}(x_{\sigma(k)})\right),$$

where $\mathbb{T}^k = \mathbb{S}^1 \times \overset{k}{\ldots} \times \mathbb{S}^1$ is the k-dimensional torus and σ is a cyclic permutation; etcetera. For more information, the reader is referred to [2] and [44].

2.2 Computing periods of powers

A curious problem is to find the set of periods of f^p for any $p \geq 2$ if we know the set of periods of $f \in C(I, I)$. This is a question of combinatorial nature and in order to solve it we have in mind the following general facts (X is a general topological space):

- *Fact 1: For $k \in \mathbb{Z}^+$, $\varphi \in C(X, X)$ and $x \in X$, it holds:*

$$\varphi^k(x) = x \Longleftrightarrow \mathrm{ord}_\varphi(x)|k,$$

where $m|k$ means that m divides to k, and $\mathrm{ord}_\varphi(x)$ is the period of the sequence $\{\varphi^n(x)\}_n$.

- *Fact 2:*

$$\mathrm{Per}(f^p) = \Big\{\frac{k}{\gcd(k, p)} : k \in \mathrm{Per}(f)\Big\},$$

where $\gcd(k, p)$ is the greatest common divisor of k and p.

So, the latter fact answers our question. But we can say even more, we can describe the sets of periods in terms of the initial segments of the Sharkovsky's ordering. Consider $2 \cdot \mathbb{N} + 1$, the set of odd numbers greater than 1, and define for any pair p, q of positive integers

$$t(p,q) := \min\{m \in 2 \cdot \mathbb{N} + 1 : q \leq m \cdot p\}.$$

For instance, $t(7,13) = 19$, $t(5,5) = 3$ and $t(7,67) = 11$. Then we can state the sets of periods of a power in terms of initial segments:

Theorem 2.1 ([16]). *Let $f \in C(I,I)$ be such that $\mathrm{Per}(f) = \mathcal{S}(q \cdot 2^r)$, with $q \geq 1$ odd and $r \in \mathbb{N} \cup \{0, \infty\}$. Let $p \in \mathbb{N}$, $p \neq 1$.*

(a) If $p \geq 3$ is odd:

(a1) If $q = 1$, $\mathrm{Per}(f^p) = \mathcal{S}(2^r)$.

(a2) If $q > 1$, $\mathrm{Per}(f^p) = \mathcal{S}(t(p,q) \cdot 2^r)$.

(b) If $p = l \cdot 2^k$, $l \geq 1$ odd, $k \geq 1$:

(b1) If $q = 1$,

$$\mathrm{Per}(f^p) = \left\{ \begin{array}{ll} \mathcal{S}(1) & \text{if } k \geq r, \\ \mathcal{S}(2^{r-k}) & \text{if } k < r. \end{array} \right\}$$

(here if $2^k = 2^\infty$, we consider that $2^{k-r} = 2^\infty$).

(b2) If $q > 1$,

$$\mathrm{Per}(f^p) = \left\{ \begin{array}{ll} \mathcal{S}(3) & \text{if } k > r, \\ \mathcal{S}(t(l,q)) & \text{if } k = r, \\ \mathcal{S}(t(l,q) \cdot 2^{r-k}) & \text{if } k < r. \end{array} \right\}$$

Let us present some examples illustrating the above theorem.

Example 2.1. *Let $f \in C(I,I)$ with $\mathrm{Per}(f) = \mathcal{S}(2^7 \cdot 61)$. Since $t(3,61) = 21$, $t(9,61) = 7$, $t(313,61) = 3$, we have*

$$\mathrm{Per}(f^3) = \mathcal{S}(2^7 \cdot 21), \quad \mathrm{Per}(f^9) = \mathcal{S}(2^7 \cdot 7), \quad \mathrm{Per}(f^{313}) = \mathcal{S}(2^7 \cdot 3).$$

On the other hand, since $t(1,61) = 61$, $t(5,61) = 13$, $t(7,61) = 9$, we have

$$\begin{array}{lll} \mathrm{Per}(f^2) = \mathcal{S}(2^6 \cdot 61), & \mathrm{Per}(f^{2^2 \cdot 5}) = \mathcal{S}(2^5 \cdot 13), & \mathrm{Per}(f^{2^7}) = \mathcal{S}(61), \\ \mathrm{Per}(f^{2^7 \cdot 7}) = \mathcal{S}(9), & \mathrm{Per}(f^{2^8}) = \mathcal{S}(3), & \mathrm{Per}(f^{2^9 \cdot 11}) = \mathcal{S}(3). \end{array}$$

Example 2.2. *Let $g \in C(I,I)$ with $\mathrm{Per}(g) = \mathcal{S}(2^{15})$. Then $\mathrm{Per}(g^{2m+1}) = \mathcal{S}(2^{15})$ for all $m \geq 1$. For some even values of p we have:*

$$\mathrm{Per}(g^2) = \mathcal{S}(2^{14}), \quad \mathrm{Per}(g^{2^3 \cdot 7}) = \mathcal{S}(2^{12}), \quad \mathrm{Per}(g^{2^{15}}) = \mathcal{S}(1), \quad \mathrm{Per}(g^{2^{29} \cdot 101}) = \mathcal{S}(1).$$

There exists a similar result for circle maps. Since it is rather long to be described here, the reader interested in its description can consult [16], where it can be also found the proof for the case of interval maps.

Once we have studied the set of periods for a power of a continuous (real or circle) map, an obvious question concerning the periods of powers is the following one:

Question: If we know the periodic structure of certain class $\mathcal{M}$ of maps (interval maps, circle maps, σ-permutation maps, tree maps, ...), given a map $f \in \mathcal{M}$ try to describe the new set of periods of f^p in terms of the sets of periods appearing in the corresponding result which characterizes the periodic structure of the class $\mathcal{M}$.

2.3 Some isolated results

Notice that the above results on periodic structure are obtained for difference equations of the form $X_{n+1} = \Phi(X_n)$, in a suitable space (interval, circle, n-dimensional interval, ...) So, it is natural to look for the results describing the periods of a difference equation of order equal to or bigger than 2. We can say that in the literature we only find some isolated results obtained for some particular difference equations, that is, the authors give the set of periods for a particular difference equation, but no forcing relationship between the periods is attained. Hence, to obtain new general results containing some type of forcing relationship between the periods of new classes of difference equations of order bigger than 1 could be an interesting line of research.

As an example, we show some periodicity results concerning the Lyness max equation [31]

$$x_{n+1} = \frac{\max(x_n^k, A)}{x_n^l x_{n-1}}, k, l \in \mathbb{R}, A > 0.$$

- For $k = 1$ and $l = 0$, we get

$$x_{n+1} = \frac{\max(x_n, A)}{x_{n-1}}, \quad A > 0, \tag{MAX-5}$$

  the so-called "max-5" equation. In this case, all the periods of (MAX-5) are:

  (a) $A > 1 : 5r + 4s, r, s \in \mathbb{Z}^+$.

  (b) $A < 1 : 5r + 6s, r, s \in \mathbb{Z}^+$.

  (c) $A = 1$: It is a 5-cycle: all the sequences are periodic of period 5 or 1.

- For $k = 1$, $l = 0$, now

$$x_{n+1} = \frac{\max(x_n, A)}{x_n x_{n-1}}, \quad A > 0 \tag{MAX-7}$$

  and all the periods of (MAX-7) are:

  (a) $A > 1 : 7r + 3s, r, s \in \mathbb{Z}^+$.

  (b) $A < 1 : 7r + 4s, r, s \in \mathbb{Z}^+$.

  (c) $A = 1$: It is a 7-cycle: all the sequences are periodic having period 7 or 1.

Related to this class of difference equations, we could propose the following *open problems*: What happens in the case $k = 2, l = 1$ (called 9-max),

$$x_{n+1} = \frac{\max(x_n^2, A)}{x_n x_{n-1}}, k, l \in \mathbb{R}, A > 0,$$

what are their periods? Does it exist some type of forcing between the periods? Is it true the conjecture of Kulenovic-Ladas (see [41, Conjecture 5.4.8]) establishing that all positive solutions of this equation are bounded? Try to give some information about the periods in the general case, being k, l, arbitrarily taken.

3 Periodic structure of alternated maps

Another way to describe applied models (biological, economic, physic, ...) is to employ an iteration involving several functions according to different steps of the evolution of the system,

$$x, f(x), g(f(x)), h(g(f(x))), \ldots$$

(see, for instance, [11, 48, 64, 67]).

A particular case of this idea is to apply the functions in a periodic form,

$$f_1, f_2, ..., f_p, f_1, f_2, ..., f_p, f_1, f_2, ..., f_p, \ldots$$

Then we obtain a new (non-autonomous) dynamical system, denoted by $[f_1, f_2, f_3, \ldots, f_p]$, where the solutions are generated by the alternated use of p individual discrete systems $(f_i, X), i = 1, \ldots, p$, with each $f_i \in C(X, X)$:

$$x_1 \to f_1(x_1) := x_2 \to f_2(f_1(x_1)) := x_3 \to f_3(f_2(f_1(x_1))) := x_4 \ldots$$

We obtain a new dynamics from the individual dynamics of each $f_j, j = 1, \ldots, p$. If we consider $X = I := [0, 1]$, we propose to study the periodic sequences and their corresponding periods when we apply the strategy of alternating continuous interval maps. In this new frame, we use the notation $\mathrm{ord}_{[f_1,\ldots,f_p]}(x_n)$ or $\mathrm{ord}_{[f_1,\ldots,f_p]}(x_1)$ to represent the period of the sequence $(x_n)_n = \{x_1, x_2, \ldots\}$, and $\mathrm{Per}([f_1, f_2, \ldots, f_p])$ means the set of periods of the alternated system $[f_1, f_2, \ldots, f_p]$.

3.1 Initial results

We survey some contributions to this topic of the periodic structure of alternated systems defined from interval maps $f_j \in C(I, I)$. Notice that for $p \geq 2$, an alternated system is precisely a periodic non-autonomous difference equation, namely

$$x_{n+1} = f_n(x_n),$$

with $f_n \in C(I, I), f_{n+p} \equiv f_n, n \geq 1$.

Obviously, when $p = 1$ we have a discrete dynamical system and Sharkovsky's Theorem solves the problem of the description of the periodic structure of the system.

When $p \geq 2$ we mention [4] and [15] and pass to describe briefly their contributions.

In [4], for $q \in \mathbb{Z}^+$, the authors define

$$\mathcal{A}_q = \{n : \operatorname{lcm}(n, p) = q \cdot p\},$$

where lcm denotes the lowest common multiple of two natural numbers. Notice that $p \cdot q \in \mathcal{A}_q$.

Then if $\mathcal{A}_n \cap \operatorname{Per}([f_1, ..., f_p]) \neq \emptyset$, it holds $\mathcal{A}_m \cap \operatorname{Per}([f_1, ..., f_p]) \neq \emptyset$ for any $n >_{\mathrm{s}} m$ (remember that $\geqslant_{\mathrm{s}}$ is Sharkovsky's ordering). For instance, if $p = 3$ we have

$$\begin{aligned}
\mathcal{A}_3 = \{3 \cdot 3\} >_{\mathrm{s}} \mathcal{A}_5 = \left\{ \begin{array}{c} 5 \\ 3 \cdot 5 \end{array} \right\} >_{\mathrm{s}} \mathcal{A}_7 = \left\{ \begin{array}{c} 7 \\ 3 \cdot 7 \end{array} \right\} >_{\mathrm{s}} \mathcal{A}_9 = \{3 \cdot 9\} >_{\mathrm{s}} \dots \\
\mathcal{A}_{2\cdot 3} = \{2 \cdot 3 \cdot 3\} >_{\mathrm{s}} \mathcal{A}_{2\cdot 5} = \left\{ \begin{array}{c} 2 \cdot 5 \\ 3 \cdot 2 \cdot 5 \end{array} \right\} >_{\mathrm{s}} \mathcal{A}_{2\cdot 7} = \left\{ \begin{array}{c} 2 \cdot 7 \\ 3 \cdot 2 \cdot 7 \end{array} \right\} >_{\mathrm{s}} \mathcal{A}_{2\cdot 9} = \{3 \cdot 2 \cdot 9\} >_{\mathrm{s}} \dots \\
\mathcal{A}_{2^2\cdot 3} = \{2^2 \cdot 3 \cdot 3\} >_{\mathrm{s}} \mathcal{A}_{2^2\cdot 5} = \left\{ \begin{array}{c} 2^2 \cdot 5 \\ 3 \cdot 2^2 \cdot 5 \end{array} \right\} >_{\mathrm{s}} \mathcal{A}_{2^2\cdot 7} = \left\{ \begin{array}{c} 2^2 \cdot 7 \\ 3 \cdot 2^2 \cdot 7 \end{array} \right\} >_{\mathrm{s}} \mathcal{A}_{2^2\cdot 9} = \{3 \cdot 2^2 \cdot 9\} >_{\mathrm{s}} \dots \\
\mathcal{A}_{2^k\cdot 3} = \{2^k \cdot 3 \cdot 3\} >_{\mathrm{s}} \mathcal{A}_{2^k\cdot 5} = \left\{ \begin{array}{c} 2^k \cdot 5 \\ 3 \cdot 2^k \cdot 5 \end{array} \right\} >_{\mathrm{s}} \mathcal{A}_{2^k\cdot 7} = \left\{ \begin{array}{c} 2^k \cdot 7 \\ 3 \cdot 2^k \cdot 7 \end{array} \right\} >_{\mathrm{s}} \mathcal{A}_{2^k\cdot 9} = \{3 \cdot 2^k \cdot 9\} >_{\mathrm{s}} \dots \\
\dots >_{\mathrm{s}} \mathcal{A}_{2^m} = \left\{ \begin{array}{c} 2^m \\ 3 \cdot 2^m \end{array} \right\} >_{\mathrm{s}} \dots >_{\mathrm{s}} \mathcal{A}_2 = \left\{ \begin{array}{c} 2 \\ 3 \cdot 2 \end{array} \right\} >_{\mathrm{s}} \mathcal{A}_1 = \left\{ \begin{array}{c} 1 \\ 3 \end{array} \right\}.
\end{aligned}$$

However, the above result does not give information on the forcing between the elements of two classes $\mathcal{A}_r$ and $\mathcal{A}_q$, all we know is that the presence of a period in the block $\mathcal{A}_r$ implies the existence of other period in $\mathcal{A}_q$, whenever $r \geqslant_{\mathrm{s}} q$. But the result does not provide a detailed frame of forcing between the periods of two classes.

Independently of the work of AlSharawi et al., in [15] it was possible to characterize completely the structure of the set of periods $\operatorname{Per}([f_1, f_2])$, including a detailed study of the forcing between elements of two different blocks $\mathcal{A}_r$ and $\mathcal{A}_q$. If we put $\mathbb{N}^* = \{4, 6, 8, 10, \dots\}$, the main result of [15] is:

Theorem 3.1 ([15]). *Let $f_1, f_2 \in C(I, I)$. Then*

(a) If $[f_1, f_2]$ has a periodic orbit of period $n \in \mathbb{N}^* \cup \{2^\infty\}$, then $\mathcal{S}(n) \backslash \{1, 2\} \subset \operatorname{Per}[f_1, f_2]$.

(b) If $2n + 1 \in \operatorname{Per}[f_1, f_2]$, $n \geq 1$, then $\mathcal{S}(2 \cdot 3) \backslash \{1\} \subset \operatorname{Per}[f_1, f_2]$.

(c) There are $f_1, f_2 \in C(I, I)$ such that $\operatorname{Per}([f_1, f_2])$ is $\{1\}$, $\{2\}$ or $\{1, 2\}$.

(d) For any $n \in \mathbb{N}^* \cup \{2^\infty\}$:

d.1. There are $f_1, f_2 \in C(I, I)$ such that $\operatorname{Per}([f_1, f_2]) = \mathcal{S}(n)$.

d.2. There are $f_1, f_2 \in C(I, I)$ such that $\operatorname{Per}([f_1, f_2]) = \mathcal{S}(n) \backslash \{1\}$.

d.3. There are $f_1, f_2 \in C(I, I)$ such that $\operatorname{Per}([f_1, f_2]) = \mathcal{S}(n) \backslash \{2\}$.

(e) Let $\operatorname{Imp} \subseteq \{2n + 1 : n \in \mathbb{N}\}$. Then

e.1. For any subset of odd numbers Imp there are $f_1, f_2 \in C(I, I)$ such that $\operatorname{Per}([f_1, f_2]) = \operatorname{Imp} \cup (\mathcal{S}(2 \cdot 3) \backslash \{1\})$.

e.2. For any subset of odd numbers Imp there are $f_1, f_2 \in C(I, I)$ such that $\operatorname{Per}([f_1, f_2]) = \operatorname{Imp} \cup \mathcal{S}(2 \cdot 3)$.

Notice that the case $\mathrm{Per}([f_1, f_2]) = \mathrm{Imp} \cup (\mathcal{S}(2 \cdot 3)\backslash\{2\})$ is not allowed, that is, if $2n + 1 \in \mathrm{Per}([f, f_2])$ for some $n \in \mathbb{N}$, then automatically $2 \in \mathrm{Per}([f_1, f_2])$. In addition, for $n \in \mathbb{N}^* \cup \{2^\infty\}$, $n \neq 2 \cdot 3$, there are not continuous maps $f_1, f_2 \in C(I)$ such that $\mathrm{Per}([f_1, f_2]) = \mathrm{Imp} \cup (\mathcal{S}(n)\backslash\{1\})$ or $\mathrm{Per}([f_1, f_2]) = \mathrm{Imp} \cup (\mathcal{S}(n)\backslash\{2\})$ or $\mathrm{Per}([f_1, f_2]) = \mathrm{Imp} \cup \mathcal{S}(n)$. The result of Cánovas-Linero can be summarized in the following frame of forcing (where $n >_2 m$ means that the presence of a period n in the alternated system forces the existence of periodic sequences having order m):

$$
\begin{aligned}
&\{2 \cdot n + 1 : n \in \mathbb{N}\} >_2 2 \cdot 3 >_2 2 \cdot 5 >_2 2 \cdot 7 >_2 ... \\
&2^n \cdot 3 >_2 2^n \cdot 5 >_2 2^n \cdot 7 >_2 ... >_2 2^n >_2 ... >_2 2^2 >_2 (1 \text{ or/and } 2).
\end{aligned}
$$

Other interesting papers dealing with the subject of finding periodicities in alternated systems are [3], [5] and [6].

3.2 Some advances

It is an open problem to determine all the sets of periods for an alternated system when $p \geq 3$. The main result of [4] gives only a partial answer because it ensures that the existence of periods in certain block $\mathcal{A}_q$ only guarantees the existence of periods in blocks $\mathcal{A}_r$ for which $r <_{\mathrm{s}} q$, but it does not establish what periods of the block are attained in $\mathrm{Per}[f_1, \dots, f_p]$.

In this sense, if we want to give a step forward in relation with the knowledge of the periodic structure of alternated systems in the interval, then the following results will be useful for the task of determining what elements of a block $\mathcal{A}_q$ can belong to $\mathrm{Per}[f_1, \dots, f_p]$. The first one is of combinatorial nature and characterize the periods coprime with the dimension p, and is a generalization of the corresponding result of [15].

Lemma 3.2. *Let $f_1, ..., f_p \in C(I, I)$, with p a positive integer. Fix an integer $m \geq 1$, with $\gcd(m, p) = 1$; then $m \in \mathrm{Per}[f_1, \cdots, f_p]$ if and only if $f_1, ..., f_p$ share a common orbit $(x_1, ..., x_m)$ of period m. Moreover, in this case $f_i(x_j) = x_{(j+1)\mathrm{mod} m}$, $i = 1, \dots, p$, $j = 1, \dots, m$.*

According to this result, we see that a period m coprime with the dimension p does not force necessarily the existence of other periods coprime with p. However, the following result shows us that the presence of a period coprime m forces certain smaller periods in Sharkovsky's ordering. The proof is based upon the use of techniques of digraphs in the interval, [2].

Lemma 3.3. *Let $q \in \mathrm{Per}([f_1, ..., f_p])$, with $\gcd(q, p) = 1$. Then*

$$p \cdot m \in \mathrm{Per}([f_1, ..., f_p]) \textit{ for all } m \in \mathbb{N},\ m \geq 1, \textit{ such that } q \succcurlyeq_{\mathrm{s}} p \cdot m.$$

It is interesting to mention that the above results appear in [17], inside of a more ambitious article answering adequately the problem of finding all the forcing relations of an alternated system. They are based mainly on techniques of discrete dynamical systems.

Some interesting *open problems* related with the periodic structure of alternated systems could be:

- *To study the periodic structure in the case of alternated continuous circle maps.*
- *Once we know the periodic structure of a general class of continuous maps, to analyze the periodic structure of the corresponding alternated systems.*

- *How to extend the study of the periodicity of alternated systems when we consider a family of infinite continuous maps?*
- *To study other dynamical aspects (entropy, ω-limit sets, ...) of alternated systems, even general non-autonomous systems. In this direction, we mention [13, 14, 28, 39, 40].*

4 Global periodicity

In this section we concentrate our attention on p–cycles. Recall that a difference equation is a p–cycle when all its possible solutions are periodic sequences and p is the least common multiple of their periods. We survey some results concerning the existence of families of p-cycles, the classification of p–cycles by conjugation and the existence of equilibrium points. In the results presented below, we assume mostly that $X = \mathbb{R}$ or $X = (0, \infty)$.

We summarize the aspects we will deal with:

- Most popular $p-$cycles in the literature, including the Lyness equation and rational cycles.
- Some results on particular families of $p-$cycles having a specific typology.
- Abundance of $p-$cycles, and its possible classification by topological conjugation.
- Open problems related with the search for new families of $p-$cycles, with the classification of cycles and with the existence of equilibria points.

Before starting the development of this section, let us notice that for the study of the global periodicity, the following approaches have been employed: sometimes it is necessary to solve functional equations (for instance, [9, 20]); in other cases it is appropriate to use techniques of discrete dynamical systems (as in [19, 22] and [24]); and, finally, some direct arguments of real analysis also work in some cases (for example, see [69]).

4.1 A historical digression

Probably, the most popular $p-$cycle is the so-called *Lyness cycle*

$$x_{n+2} = \frac{1 + x_{n+1}}{x_n},$$

a second order difference equation for which all its solutions $\{x_n\}_n$ verify $x_{n+5} = x_n$, therefore it is a 5-cycle.

In fact, Lyness cycle is a particular case -take $a = 1$- of the family of 5-cycles $x_{n+2}x_n - a^2 = ax_{n+1}$, $a \neq 0$. It is precisely under this aspect that Lyness cycle appeared: in fact, Gauss obtained it when working in the spherical geometry of the *pentagrama mirificum,* a spherical pentagram formed by five successively orthogonal great-circle arcs. To see its construction and the relation with the 5-cycle, the reader can consult [26]. According to this paper: "This $5-$cycle seems to have been transmitted in the form of mathematical gossip for a long time". The 5-cycle receives the name of Lyness cycle because R.C. Lyness accounted for it in a series of papers dealing with the existence of cycles (see [45, 46, 47] and also [34]). Surprisingly, the interest of Lyness was associated neither to dynamical systems nor difference equations, he found the equation while investigating a problem related to the number theory: to obtain three integer numbers such that the sum or the difference of any different pair of them is a square. The first time that the equation is referred to as the "Lyness equation" occurred in 1961, in [57].

A variant of the Lyness cycle was presented in [26], namely, the difference equation of order m given in terms of continued fractions

$$c_{n+m} = 1 - \frac{c_{m+n-1}}{1 - \frac{c_{m+n-2}}{1-\cdots\frac{\cdots}{1-c_n}}}, \tag{3}$$

which is a $(m+3)-$cycle and for $m = 2$ coincides with the Lyness cycle.

Other rational $p-$cycles, different to Lyness equation $x_{n+2} = \frac{1+x_{n+1}}{x_n}$, are $x_{n+1} = \frac{1}{x_n}$ ($2-$cycle) and $x_{n+3} = \frac{1+x_{n+2}+x_{n+1}}{x_n}$ ($8-$cycle, called the Todd's one, because Lyness in [46, p. 233] commented that professor H. Todd, from Bristol University, discovered the $8-$cycle $u_{n+3}u_n = u_{n+2} + u_{n+1} + 1$). At the first glance the reader could think about a generalization and could conjecture that for $k \geq 4$ the equation

$$x_{n+k} = \frac{1 + x_{n+k-1} + x_{n+k-2} + \cdots + x_{n+1}}{x_n}$$

can be a p–cycle for some $p \in \mathbb{Z}^+$. However, in [69] it was established that this conjecture is false and, even more, the general difference equation

$$x_{n+1} = \frac{a + b_0 x_n + b_1 x_{n-1} + \cdots + b_{k-2} x_{n-k+2}}{x_{n-k+1}}, \quad n = 0, 1, ...$$

where $a, b_i \in (0, \infty)$, $i \in \{0, 1, \cdots, k-2\}$, and $x_{-k+1}, x_{-k+2}, \cdots, x_0$ are arbitrary positive numbers, is not a p–cycle.

4.2 Families of p–cycles

Now we give a quick survey of results about the existence of families of $p-$cycles. We will pay attention to: Rational cycles; first and second order difference equations; difference equations of order greater than 2; and potential cycles.

4.2.1 Rational cycles

Here we will give a brief mention to the results in [23] and [27]. In [27] the authors look for p–cycles of the form

$$x_n = \frac{a_0 + a_1 x_{n-1} + a_2 x_{n-2} + \cdots + a_k x_{n-k}}{x_{n-k-1}}$$

with $a_0, a_1, \cdots, a_k \in \mathbb{C}$, $a_1 \cdot a_2 \cdot \cdots \cdot a_k \neq 0$, $a_k = 1$. They obtain:

$$x_n = \frac{x_{n-1}}{x_{n-2}} : \ 6-\text{cycle}, \quad x_n = \frac{-1 - x_{n-1} + x_{n-2}}{x_{n-3}} : \ 8-\text{cycle}.$$

In [23], the authors consider rational equations

$$x_{n+k} = \frac{A_1 x_n + A_2 x_{n+1} + A_3 x_{n+2} + \cdots + A_k x_{n+k-1} + A_0}{B_1 x_n + B_2 x_{n+1} + B_3 x_{n+2} + \cdots + B_k x_{n+k-1} + B_0}, \tag{4}$$

where the initial conditions are positive numbers, and the coefficients verify $A_i \geq 0$, $B_i \geq 0$, $\sum_i A_i > 0$, $\sum_i B_i > 0$, and $A_1^2 + B_1^2 \neq 0$. They prove that any non trivial p–cycle of order $k = 2$ or $k = 4$ is equivalent to one of the following difference equations:

$$\begin{aligned} u_{n+2} &= \frac{u_{n+1}}{u_n} : \text{ 6−cycle}; \quad u_{n+2} = \frac{1+u_{n+1}}{u_n} : \text{ 5−cycle} \quad \text{(Lyness cycle)}; \\ u_{n+4} &= \frac{u_{n+2}}{u_n} : (2\times 6)\text{−cycle}; \quad u_{n+4} = \frac{1+u_{n+2}}{u_n} : (2\times 5)\text{−cycle}. \end{aligned}$$

and for odd orders $k \in \{1,3,5,7,9,11\}$ they prove that any non trivial p-cycle is equivalent to one of the following ones:

$$u_{n+3} = \frac{1+u_{n+2}+u_{n+1}}{u_n} : \text{ 8−cycle}; \quad u_{n+9} = \frac{1+u_{n+6}+u_{n+3}}{u_n} : (3\times 8)\text{−cycle}.$$

A natural question is to ask whether we can find other non trivial p–cycles of order k of the form (4), and not generated by one of the above mentioned cycles. Evidently, according to [23] the answer is negative for $k = 1,2,3,4,5,7,9,11$. The cases $k = 6,8,10$, and $k \geq 12$ are still open.

4.2.2 First and second order difference equations

Evidently, the unique 1−cycle generated by a difference equation of first order, $x_{n+1} = f(x_n)$, is the trivial one $x_{n+1} = x_n$. Moreover, if $f : (0,\infty) \to (0,\infty)$ is a continuous map, and f produces a p−cycle different to the trivial one, then in [18] it is proved that necessarily $p = 2$ and f must satisfy

$$f(x) = \begin{cases} f_0(x), \text{ if } x \in (0,x_0), \\ x_0, \text{ if } x = x_0, \\ f_0^{-1}(x), \text{ if } x \in (x_0,\infty), \end{cases}$$

where $x_0 > 0$ and $f_0 \in C((0,x_0),(x_0,\infty))$ is a decreasing map with $\lim_{x\to x_0} f_0(x) = x_0$.

Concerning the non-autonomous (of first order) case

$$x_{n+1} = f_n(x_n),\ n = 0,1,2,\dots ,$$

where each f_n is a continuous map from $(0,\infty)$ into itself, it is a remarkable fact that *there exist ∞-cycles* in the non-autonomous case, that is, we can construct examples of non-autonomous difference equations of first order such that all their solutions are periodic, but their set of periods is infinite (see [19]).

In relation with p–cycles giving by difference equations of second order, $x_{n+2} = f(x_{n+1}, x_n)$, firstly it is obligatory to mention [51]. In this paper the author considers difference equations having the form

$$x_{n+2} = \frac{f(x_{n+1})}{x_n}, \tag{5}$$

and he looks for p−cycles, with $p = 3,4,5$. This question is answered by solving appropriate functional equations appearing as a consequence of imposing $x_{n+p} = x_n$. For instance, for 5−cycles, the global periodicity problem amounts to solve the system of two functional equations

$$\frac{f\left(\frac{f(x)}{y}\right)}{x} = \frac{f\left(\frac{f(y)}{x}\right)}{y}, \qquad \frac{f(x)}{y}\frac{f(y)}{x} = f\left(\frac{f\left(\frac{f(y)}{x}\right)}{y}\right),$$

which gets $f(x) = A(A^\alpha + x^\alpha)^{\frac{1}{\alpha}}$, with $A, \alpha > 0$.

Open problem: For $p \geq 6$, obtain $p-$cycles having the form of Eq. (5).

We continue with p–cycles of second order and cite the paper [8]. Here, general $3-$cycles of the form $x_{n+2} = f(x_{n+1}, x_n)$ are searched. Now the functional equation $f \in C((0,\infty),(0,\infty))$ must obey is

$$f(u, f(w,u)) = w$$

for all $u, w > 0$. The study is focussed on the analysis of the fiber maps $f_z(\cdot) = f(z,\cdot)$ and $f^z(\cdot) = f(\cdot, z)$. They are strictly decreasing maps, with $f^z \circ f_z = f_z \circ f^z = \mathrm{Id}$. If, in addition, we assume that f separates the variables, $f(x,y) = \sigma(x)\cdot\rho(y)$, then it is proved that $f^z \equiv f_z$ (the difference equation is symmetric), and $f(x,y) = \frac{k}{xy}$ for some positive constant k. At the same time, in [8] it is proposed the problem of finding non-symmetric $3-$cycles of second order. This was done in [18], where it is constructed a continuous map $\widetilde{F}$ from $(0,\infty)^2$ into itself, with the form $\widetilde{F}(x,y) = (y, \widetilde{f}(y,x))$, that holds $\widetilde{F}^3 = \mathrm{Id}|_{(0,\infty)^2}$, and $\widetilde{f}$ is non-symmetric. The idea of the construction is to modify $F(x,y) = (y, \frac{1}{xy})$ and its iterates on an appropriate rectangle. Other example of non-symmetric $3-$cycle of second order can be found in [24].

Under the *involution hypothesis for fiber maps*, $f_z \circ f_z = \mathrm{Id}$, and assuming *separation of variables,* we can get new results on global periodicity of $x_{n+2} = f(x_{n+1}, x_n)$. Again, the main tool for achieving them consists of solving functional equations.

In [20] it is proved that $x_{n+2} = \sigma(x_{n+1})\rho(x_n)$ is a $4-$cycle if and only if $\sigma(x) = k$ is a constant map for some $k > 0$, and ρ is a bijective decreasing map satisfying $c\rho \circ c\rho = \mathrm{Id}|_{(0,\infty)}$. And if it is a $p-$cycle, with $p \geq 5$, then necessarily $\rho(x) = \frac{c}{x}$ for all $x > 0$ and some constant $c > 0$. This implies that all the $p-$cycles of order two, fibers acting as involutions, and separated variables, have to follow the pattern of the equations described in Mestel's paper [51].

Open problem: Analyze if the above characterizations of the different $p-$cycles of second order remain true if we remove the involution hypothesis.

4.2.3 Order greater than two

New families of $p-$cycles of third order $x_{n+3} = f(x_{n+2}, x_{n+1}, x_n)$ were found in [9]. To be more precise, the following equation is studied

$$x_{n+3} = x_i f(x_j, x_k) \tag{6}$$

where $i, j, k \in \{n, n+1, n+2\}$ are pairwise distinct and $f : (0,\infty)^2 \to (0,\infty)$ is continuous.

It is proved, with the help of functional equations, that the unique $3-$cycle of Eq. (6) is $x_{n+3} = x_n$; the unique $4-$cycle is given by $x_{n+3} = x_n\frac{x_{n+2}}{x_{n+1}}$; and there are two $5-$cycles, namely $x_{n+3} = x_n(\frac{x_{n+2}}{x_{n+1}})^\Phi$, and $x_{n+3} = x_n(\frac{x_{n+2}}{x_{n+1}})^\varphi$, where $\Phi = \frac{1+\sqrt{5}}{2}$ (golden number) and $\varphi = -\Phi^{-1}$.

In [20], two new classes of difference equations for positive real numbers are studied, namely, $x_{n+3} = x_n g(x_{n+2})/g(x_{n+1})$, and $x_{n+3} = \frac{f(x_{n+2},x_{n+1})}{x_n}$. The first class contains $6-$cycles if and only if $g(x) = cx^2$ for some constant $c > 0$, and the second one have neither $3-$cycles nor $5-$cycles, and presents $4-$cycles if and only if $f(x,y) = c/(xy)$, $c > 0$ is a constant.

Open problem: Find new families of p–cycles, $p \geq 6$, having the form $x_{n+3} = x_i f(x_j, x_k)$ and, even, find general $p-$cycles having the form $x_{n+3} = f_2(x_{n+2})f_1(x_{n+1})f_0(x_n)$.

We have seen that the equations $x_{n+2} = c/(x_{n+1}x_n)$, $c > 0$, are the unique $3-$cycles defined in $(0,\infty)$ if we separate the variables of the map f which defines the recurrence. This property can be generalized in the following way:

Theorem 4.1 ([21]). *Consider the difference equation of order $k \geq 3$*

$$x_{n+k} = F(x_{n+k-1}, \dots, x_{n+2}, x_{n+1}) f_0(x_n),$$

with $F \in C((0,\infty)^{k-1}, (0,\infty))$, $f_0 \in C((0,\infty),(0,\infty))$, and $x_i \in (0,\infty)$, $i = 0, 1, \dots, k-1$; then, assuming additionally that for each $x > 0$ the one-dimensional map $f_x(y) = F(x, \dots, x)f_0(y)$ is an involution

$$f_x(f_x(y)) = y \quad \textit{for all } y,$$

the equation is a $(k+1)$–cycle if and only if it has one of the following forms:

(i) k is even and

$$x_{n+k+1} = \frac{C}{x_{n+1}x_{n+2}\cdots x_{n+k}}, \qquad \textit{for some } \; C > 0; \tag{7}$$

(ii) k is odd and either $x_{n+k+1} = \frac{C}{x_{n+1}x_{n+2}\cdots x_{n+k}}$, for some $C > 0$ or

$$x_{n+k+1} = \frac{\prod_{j=1}^{(k+1)/2} x_{n+2j-1}}{\prod_{j=1}^{(k-1)/2} x_{n+2j}}. \tag{8}$$

Open problem: Study whether or not the property will remain true if we remove the condition on f_x to be an involution.

4.2.4 Potential cycles

At the moment, most of the showed $p-$cycles on $(0,\infty)$ have a "potential" form. That is, the p–cycle can be written as

$$x_{n+k} = c x_{n+k-1}^{\alpha_{k-1}} \cdots x_{n+1}^{\alpha_1} x_n^{\alpha_0}, \quad n = 0, 1, \dots, \tag{9}$$

where the constant c and the initial values $x_0, x_1, \dots, x_{k-1}$ are positive real numbers, and the exponents α_j, $j = 0, 1, \dots, k-1$, are real numbers.

The following result describes all the p–cycles of order k and potential form, that is, gives the values of the constants α_i and $c > 0$ for which the difference equation is a p–cycle, $p \geq k \geq 2$, where $x_i > 0, i = 0, 1, \dots k-1$.

Theorem 4.2 ([21]). *Equation (9), with $p \geq k \geq 2$, is a p–cycle of order k if and only if either $k = p$ and (9) is reduced to $x_{n+k} = x_n$, or $p > k$ and the real exponents $\alpha_j, j = 0, 1, \ldots, k-1$, are the coefficients of the characteristic polynomial $P(\lambda) = \lambda^k - \alpha_{k-1}\lambda^{k-1} - \ldots - \alpha_1\lambda - \alpha_0$, and according to the Cardano-Vieta's formulas we have*

$$-\alpha_j = (-1)^{k-j} \sum_{i_1 < \ldots < i_{k-j}} \lambda_{i_1} \cdots \lambda_{i_{k-j}} \tag{10}$$

where the eigenvalues λ_j of $P(\lambda)$ are simple pth roots of the unity verifying one of the following cases:

1. *Case 1: k is even, $\lambda_1 = 1$, $\lambda_2 = -1$ are eigenvalues of $P(\lambda)$ and the rest of the eigenvalues are complex numbers $\lambda_{j,\pm} = \cos\frac{2\pi l_j}{p} \pm i\sin\frac{2\pi l_j}{p}$, $j = 1, \ldots, \frac{k}{2} - 1$, for some integers $l_j < \frac{p}{2}$. Moreover, $c = 1$, $\alpha_0 = 1$, and p is even.*

2. *Case 2: k is even, and all the eigenvalues are complex numbers $\lambda_{j,\pm} = \cos\frac{2\pi l_j}{p} \pm i\sin\frac{2\pi l_j}{p}$, $j = 1, 2, \ldots, \frac{k}{2}$, for some integers $l_j < \frac{p}{2}$. Moreover, $c > 0$ and $\alpha_0 = -1$.*

3. *Case 3: k is odd, $\lambda_1 = 1$ is the unique real eigenvalue of $P(\lambda)$, and the rest are complex numbers $\lambda_{j,\pm} = \cos\frac{2\pi l_j}{p} \pm i\sin\frac{2\pi l_j}{p}$, $j = 1, 2, \ldots, \frac{k-1}{2}$, for some integers $l_j < \frac{p}{2}$. Moreover, $c = 1$, and $\alpha_0 = 1$.*

4. *Case 4: k is odd, $\lambda_1 = -1$ is the unique real eigenvalue of $P(\lambda)$, and the rest are complex numbers $\lambda_{j,\pm} = \cos\frac{2\pi l_j}{p} \pm i\sin\frac{2\pi l_j}{p}$, $j = 1, 2, \ldots, \frac{k-1}{2}$, for some integers $l_j < \frac{p}{2}$. Moreover, $c > 0$, $\alpha_0 = -1$, and p is even.*

In all the above four cases, l_j are integer numbers satisfying $\gcd(\{l_j\}, p) = 1$ and with $l_j \neq l_s$ if $j \neq s$.

In order to prove this theorem, we apply a change of variables and then we analyze the obtained linear difference equation. In fact, by taking $y_m = \ln x_m$ we linearize the potential equation, because Equation (9) is a p–cycle if and only if equation

$$\begin{aligned} y_{n+k} &= \ln c + \alpha_{k-1}y_{n+k-1} + \ldots + \alpha_1 y_{n+1} + \alpha_0 y_n, \\ & \quad y_i \in \mathbb{R}, \quad i = 0, 1, \ldots, k-1, \end{aligned}$$

is a p–cycle for $p \geq k \geq 2$. Moreover, we recall that if Eq. (9) is a p–cycle, then the zeros associated to its characteristic polynomial are simple pth roots of unity.

Remark 4.1. The above theorem allows us to give a counterexample to Conjecture 2.1 in [33]: *"the only 8-cycle of the form $x_{n+3} = \frac{f(x_{n+2}, x_{n+1})}{x_n}$ with $f \in C^1((0,\infty) \times (0,\infty), (0,\infty))$ is*

$$x_{n+3} = \frac{1 + x_{n+2} + x_{n+1}}{x_n}." \tag{11}$$

The conjecture is false since $x_{n+3} = \frac{c(x_{n+1}x_{n+2})^\beta}{x_n}$, with $\beta = -1 + \sqrt{2}$ or $\beta = -1 - \sqrt{2}$, is a 8-cycle different from (11).

Potential cycles appear again in the search of $(k+2)$-cycles of order k, when the difference equation has separated variables and certain symmetric property holds.

Theorem 4.3 ([21]). *Consider the difference equation of order $k \geq 2$*

$$x_{n+k} = f_{k-1}(x_{n+k-1}) \cdots f_2(x_{n+2}) f_1(x_{n+1}) f_0(x_n),$$

$n = 0, 1, \ldots$, *with initial conditions* $x_i \in (0, \infty)$, *and continuous maps* $f_i : (0, \infty) \to (0, \infty)$, $i = 0, \ldots, k-1$. *Assume that for each* $x > 0$ *the one-dimensional map* $f_x(y) := f_{k-1}(x) \cdots f_2(x) f_1(x) f_0(y)$ *is an involution* ($f_x(f_x(y)) = y$ *for all* $x, y > 0$). *Then, this equation is a* $(k+2)$*–cycle if and only if one of the following conditions occurs:*

1. $k = 2$, $f_1(x) = c$ *for all* $x > 0$ *and some constant* $c > 0$, *and* f_0 *is a decreasing and bijective map such that* $(cf_0 \circ cf_0) = \mathrm{Id}|_{(0,\infty)}$.
2. $k \geq 4$ *is even and* $x_{n+k} = \frac{c}{x_{n+k-2} x_{n+k-4} \cdots x_{n+2} x_n}$, *for any constant* $c > 0$.
3. $k \geq 3$ *and the difference equation has a potential form*

$$x_{n+k} = x_{n+k-1}^{\alpha_{k-1}} \cdots x_{n+2}^{\alpha_2} x_{n+1}^{\alpha_1} x_n,$$

where the constants α_j, $j = 1, \ldots, j-1$ *are given by Cases 1 and 3 of Theorem 4.2.*

Open problem: Prove that the above theorem remains valid independently of the condition of symmetry imposed to f_x.

4.3 Classification of p–cycles

In this section we recall the well known relationship between difference equations and discrete dynamical systems, and with the help of one dynamical concept, namely the topological conjugation, we try to classify all p–cycles.

4.3.1 Difference equations and discrete dynamical systems

Let X be a metric space, we can associate to the difference equation

$$x_{n+k} = f(x_{n+k-1}, \ldots, x_{n+1}, x_n), \quad n \geq 0, \tag{12}$$

the following map $F : X^k \to X^k$ given by

$$F(x_1, x_2, \ldots, x_k) = (x_2, \ldots, x_k, f(x_k, \ldots, x_2, x_1)). \tag{13}$$

The advantage of transforming a difference equation into a discrete dynamical system was already noticed in [42]. If (12) is a $p-$cycle and F is its associated dynamical system, then $F^p \equiv \mathrm{Id}|_{X^k}$, and, reciprocally, if F is given by (13) and p is the smallest positive integer satisfying $F^p \equiv \mathrm{Id}|_{X^k}$ then its associated difference equation is a $p-$cycle.

4.3.2 Topological conjugation

We have presented only some examples of $p-$cycles. We could think that global periodicity is a property only satisfied by a few equations, as Lyness one, or rational and potential equations. But one can realize that the topological conjugation of systems gives us the possibility of finding more and more $p-$cycles.

According to the equivalence between difference equations and discrete dynamical systems given by Eq. (12) and the map of (13), we can establish whether two difference equations are conjugate in terms of their correspondent dynamical systems. Remember that if X_1, X_2 are two metric spaces and $g_1 : X_1 \to X_1$, $g_2 : X_2 \to X_2$, are two continuous maps, we say that (X_1, g_1) and (X_2, g_2) are *topologically conjugate* if there is a homeomorphism $h : X_1 \to X_2$ such that $h \circ g_1 = g_2 \circ h$.

So, if $x_{n+k} = f(x_{n+k-1}, \ldots, x_n)$ is a p–cycle of order k, with $f \in C((0,\infty)^k, (0,\infty))$, and we consider an arbitrary homeomorphism α defined from $(0,\infty)$ into itself, then

$$F(x_1, x_2, \ldots, x_k) = (x_2, \ldots, x_k, f(x_k, \ldots, x_1))$$

is topologically conjugate to $G(x_1, x_2, \ldots, x_k) = (x_2, \ldots, x_k, \alpha^{-1}(f(\alpha(x_k), \ldots, \alpha(x_1))))$, therefore if $x_{n+k} = f(x_{n+k-1}, \ldots, x_n)$ is a $p-$cycle then

$$x_{n+k} = \alpha^{-1}(f(\alpha(x_{n+k-1}), \ldots, \alpha(x_n)))$$

so is.

For instance, if $x_{n+2} = \frac{1+x_{n+1}}{x_n}$ is the $5-$Lyness cycle and we take $\alpha(z) = z^2, \alpha^{-1}(z) = \sqrt{z}$, we obtain

$$z_{n+2} = \sqrt{\frac{1 + z_{n+1}^2}{z_n^2}}, \qquad \text{a new 5-cycle.}$$

4.3.3 Classification of p–cycles

The above notion of conjugation of difference equations allows us to *classify the set of different $p-$cycles.*

In the case $x_{n+1} = f(x_n)$, $f \in C((0,\infty))$, the problem of the *classification of cycles* is completely solved, the unique $1-$cycle is given by the identity map, and in [18, Theorem B] it is shown that there is only an equivalence class on the set of $2-$cycles, any $2-$cycle is topologically conjugate to $x_{n+1} = \frac{1}{x_n}$. Notice that when we apply the change $z_n = \ln x_n$ we obtain $z_{n+1} = -z_n$, a linear cycle.

When the order of the cycle is greater than or equal to 2 we have a few results about the classification of $p-$cycles. For instance, all the $p-$cycles of second order $x_{n+2} = f(x_{n+1}, x_n)$, with $p \geq 3$, are topologically conjugate to rotations of the plane (see [19]). Other general result classifying all the $(k+1)-$cycles of order k is (see also [24]):

Theorem 4.4 ([19, Theorem C]). *Let $f \in C((0,\infty)^k, (0,\infty))$ and let Eq. (12) be a $(k+1)-$cycle.*

(i) If k is even, the $(k+1)-$cycle is topologically conjugate to

$$x_{n+k} = \frac{1}{x_{n+k-1}x_{n+k-2}\cdots x_n}. \tag{14}$$

(ii) If k is odd, the $(k+1)-$cycle is topologically conjugate either to (14) or to

$$x_{n+k} = \frac{\prod_{j=1}^{(k+1)/2} x_{n+2j-2}}{\prod_{j=1}^{(k-1)/2} x_{n+2j-1}}. \tag{15}$$

The proof of the theorem is constructive, we explicitly define the homeomorphisms of conjugation, given by

$$h(x_1,\dots,x_k)=\left(\frac{f(x_k,x_{k-1},\dots,x_1)}{x_k},\frac{x_1}{f(x_k,x_{k-1},\dots,x_1)},\frac{x_2}{x_1},\dots,\frac{x_{k-1}}{x_{k-2}}\right)$$

and

$$h(x_1,\dots,x_k)=(x_1f(x_k,x_{k-1},\dots,x_1),x_2x_1,x_3x_2,\dots,x_kx_{k-1}).$$

Therefore, the number of different equivalence classes for $(k+1)-$cycles of order k defined for positive real numbers is either 1 or 2.

Notice that the canonical representatives are Eq. (14) and Eq. (15), two potential cycles. In fact, all potential $p-$cycles of order k are topologically conjugate to linear $p-$cycles via the homeomorphism $h((x_j)_{j=1}^k)=((\ln(x_j))_{j=1}^k)$. Next, we present an example illustrating how to construct linear cycles.

Example 4.1. *To construct a linear $5-$cycle of order 3, take three $5-$roots of the unity such that not all of them are $s-$roots of the unity, for $s<5$. For instance:*

$$\lambda_1=1,\lambda_2=\cos\left(\frac{2\pi}{5}\right)+i\sin\left(\frac{2\pi}{5}\right),\ \textit{and}\ \lambda_3=\cos\left(\frac{2\pi}{5}\right)-i\sin\left(\frac{2\pi}{5}\right).$$

Then its characteristic polynomial will be $P(\lambda)=\lambda^3-\left(1+\cos\left(\frac{2\pi}{5}\right)\right)\lambda^2+\left(1+\cos\left(\frac{2\pi}{5}\right)\right)\lambda-1$, and

$$y_{n+3}=\left(1+\cos\left(\frac{2\pi}{5}\right)\right)y_{n+2}-\left(1+\cos\left(\frac{2\pi}{5}\right)\right)y_{n+1}+y_n$$

is a linear $5-$cycle since all the solutions are linear combinations of the linearly independent solutions

$$\{1^n\}_n,\quad \left\{\cos\left(\frac{2\pi n}{5}\right)\right\}_n,\ \textit{and}\ \left\{\sin\left(\frac{2\pi n}{5}\right)\right\}_n.$$

In potential form, if we make the change of variables $y_j=\ln(x_j)$ we find the $5-$cycle

$$x_{n+3}=x_n\left(\frac{x_{n+2}}{x_{n+1}}\right)^{1+2\cos\left(\frac{2\pi}{5}\right)}.$$

Consequently, in the set of $p-$cycles of order k there exist classes whose canonical representative is a potential cycle (if we move into $(0,\infty)$) or a linear cycle (if we consider $\mathbb{R}$ as the ambient space).

Open problem: Determine whether or not any $p-$cycle of order k is linearizable, or equivalently, is conjugate to a potential cycle. As a first step it would be interesting to focus attention on the set of $(k+2)-$cycles of order k.

Open problem: If a p–cycle separates variables, is it conjugate to a potential cycle? And if the cycle does not separate variables?

4.4 Existence of equilibrium points

We have seen that the difference equation (12) is a $p-$cycle if and only if the map given by (13) is a periodic homeomorphism, that is, $F^p = \mathrm{Id}|_X$.

This functional equation receives the name of *Babbage's equation.* From this point of view, recently some authors concentrate their attention in some classes of rational maps (see [7, 56]) and try to search discrete rational systems F satisfying $F^p = \mathrm{Id}|_X$. Notice that these systems do not come necessarily from a difference equation, that is, the map F does not have necessarily the form (13)

Realize that it is impossible to find a periodic homeomorphism F from a Euclidean space into itself such that all the points of the space are periodic for F and the map contains an infinite set of periods: indeed, according to [54] if all the points of the space are periodic for some homeomorphism F, then necessarily $F^k = \mathrm{Id}$ for some positive integer k, so the set of periods is bounded and, consequently, there are no "$\infty-$cycles" when we consider autonomous difference equations taking real values.

All the $p-$cycles showed in this talk have equilibrium points, or equivalently all the associated homeomorphisms present fixed points. But *does any periodic transformation of a Euclidean space, E^n, always admit a fixed point?*

This question was posed by P.A. Smith in 1946 (see [29]). Smith knew that the answer is positive if the period p of the transformation is a prime number [62] or a power of a prime number [63]. In view of the precedents, Smith conjectured that the periodic transformations of a Euclidean space have a fixed point.

However, in 1961, by using a slight modification of the Conner-Floyd's example [25], it was possible in [37] to show examples of periodic homeomorphisms on E^n without fixed points.

But, it is still an *open problem* to determine whether the periodic transformation given by Eq. (13) has or not equilibrium points, or equivalently, whether the corresponding p–cycle given by Eq. (12) *has or not equilibrium points.* If we look for a counterexample, a possible strategy would be to analyze the Conner-Floyd's and Kister's examples and to realize if these transformations could be written in the form $T(x_1, x_2, \ldots, x_k) = (x_2, \ldots, x_k, \varphi(x_1, \ldots, x_k))$.

If we were able to find a counterexample of a $p-$cycle without equilibria points, at the same time we would find a new class of $p-$cycles not conjugate to potential or linear cycles, and consequently we would deduce that the classification by conjugation of p–cycles cannot be reduced to the class of potential or linear cycles.

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References

[1] R.P. Agarwal, *Difference Equations and Inequalities. Theory, Methods and Applications*, second edition, Monographs and Textbooks in Pure and Applied Mathematics **228**, Marcel Dekker Inc., New York, 2000.

[2] Ll. Alsedà, J. Llibre, M. Misiurewicz, *Combinatorial dynamics and entropy in dimension one*, second edition, Adv. Ser. in Nonlinear Dynamics **5**, World Scientific Publishing, River Edge, 2000.

[3] Z. AlSharawi, *Periodic orbits in periodic discrete dynamics*, Comput. Math. Appl. **56** (2008), 1966–1974.

[4] Z. AlSharawi, J. Angelos, S. Elaydi, L. Rakesh, *An extension of Sharkovsky's theorem to periodic difference equations*, J. Math. Anal. Appl. **316** (2006), 128–141.

[5] J.F. Alves, *Odd periods of* $2-$*periodic nonautonomous dynamical systems*, Grazer Math. Ber. **351** (2007), 21–26.

[6] J.F. Alves, *What we need to find out the periods of a periodic difference equation*, J. Difference Equ. Appl. **15** (2009), 833–847.

[7] I. Bajo, E. Liz, *Periodicity on discrete dynamical systems generated by a class of rational mappings*, J. Difference Equ. Appl. **12** (2006), 1201–1212.

[8] F. Balibrea, A. Linero, *On global periodicity of* $x_{n+2} = f(x_{n+1}, x_n)$, Elaydi, S. (Ed.) et al., Difference Equations, special functions and orthogonal polynomials. Proceedings of the International Conference, Munich, Germany, July 25–30, 2005. Hackensack, NJ: World Scientific. 41–50 (2007).

[9] F. Balibrea, A. Linero, *On the global periodicity of some difference equations of third order*, J. Difference Equ. Appl. **13** (2007), 1011–1027.

[10] U. Broggi, *Serie di fattoriali ed equazioni alle differenze*, Atti Accad. Naz. Lincei, Rend., **VI**. Ser. **15**, (1932), 707–712 (in Italian).

[11] J. Buceta, C. Escudero, F.J. de la Rubia, K. Lindenberg, *Outbreaks of Hantavirus induced by seasonality*, Phys. Rev. E **69** (2004), 021906.

[12] J.C. Butcher, *The numerical analysis of ordinary differential equations. Runge-Kutta and general linear methods*, John Wiley & Sons, Ltd., Chichester, 1987.

[13] J.S. Cánovas, *On* ω*-limit sets of non-autonomous discrete systems*, J. Difference Equ. Appl. **12** (2006), 95–100.

[14] J.S. Cánovas, *Li-Yorke chaos in a class of nonautonomous discrete systems*, J. Difference Equ. Appl. **17** (2011), 479–486.

[15] J.S. Cánovas, A. Linero, *Periodic structure of alternating continuous interval maps*, J. Difference Equ. Appl. **12** (2006), 847–858.

[16] J.S. Cánovas, A. Linero, *Computing periods of powers*, Int. J. Pure Appl. Math., in press, 2011.

[17] J.S. Cánovas, A. Linero, *Periodic structure of alternated systems. The general case*, preprint.

[18] J.S. Cánovas, A. Linero Bas, G. Soler López, *On global periodicity of difference equations*, Taiwanese J. Math. **13** (2009), 1963–1983.

[19] J.S. Cánovas, A. Linero Bas, G. Soler López, *A characterization of $k-$cycles,* Nonlinear Anal. **72** (2010), 364–372.

[20] A. Caro, A. Linero, *Existence and uniqueness of $p-$cycles of second and third order,* J. Difference Equ. Appl. **15** (2009), 489–500.

[21] A. Caro, A. Linero, *General cycles of potential form,* Internat. J. Bifur. Chaos Appl. Sci. Engrg. **20** (2010), 2735–2749.

[22] A. Cima, A. Gasull, V. Mañosa, *Global periodicity and complete integrability of discrete dynamical systems,* J. Difference Equ. Appl. **12** (2006), 696–716.

[23] A. Cima, A. Gasull, F. Mañosas, *On periodic rational difference equations of order k,* J. Difference Equ. Appl. **10** (2004), 549–559.

[24] A. Cima, A. Gasull, F. Mañosas, *Global linearization of periodic difference equations*, preprint, 2010.

[25] P.E. Conner, E.E. Floyd, *On the construction of periodic maps without fixed points,* Proc. Amer. Math. Soc. **10** (1959), 354–360.

[26] H.S.M. Coxeter, *Frieze patterns,* Acta Arith. **18** (1971), 297–310.

[27] M. Csörnyei, M. Laczkovich, *Some periodic and non-periodic recursions*, Monatsh. Math. **132** (2001), 215–236.

[28] E. D'Aniello, H. Oliveira, *Pitchfork bifurcation for non-autonomous interval maps,* J. Difference Equ. Appl. **15** (2009), 291–302.

[29] S. Eilenberg, *On the problems of Topology,* Ann. Math. **50** (1949), 247–260.

[30] S. Elaydi, *An introduction to difference equations,* third edition, Undergraduate Texts in Mathematics, Springer, New York, 2005.

[31] J. Feuer, *Periodic solutions of the Lyness max equation,* J. Math. Anal. Appl. **288** (2003), 147–160.

[32] D. Fowler, E. Robson, *Square root approximations in Old Babylonian Mathematics: YBC 7289 in context,* Historia Math. **25** (1998), 366–378.

[33] E.A. Grove, G. Ladas, *Periodicities in nonlinear difference equations,* Advances in Discrete Mathematics and Applications **4**, Chapman & Hall/CRC, Boca Raton, FL, 2005.

[34] W. Howard Joint, *Note 1767. Cycles,* Math. Gaz. **28** (1944), 196–197.

[35] C. Jordan, *Calculus of Finite Differences*, third edition reprint, Chelsea, New York, 1979.

[36] W. Kelley, A. Peterson, *Difference Equations: An Introduction with Applications*, Academic Press, New York, 2001.

[37] J.M. Kister, *Examples of periodic maps on Euclidean spaces without fixed points,* Bull. Amer. Math. Soc. **67** (1961), 471–474.

[38] V.L. Kocić, G. Ladas, *Global behavior of Nonlinear Difference Equations of Higher Order with Applications,* Mathematics and Its Applications **256**, Kluwer Academic Publishers, Netherlands, 1993.

[39] S. Kolyada, M. Misiurewicz, L'. Snoha, *Topological entropy of nonautonomous piecewise monotone dynamical systems on the interval,* Fund. Math. **160** (1999), 161–181.

[40] S. Kolyada, L'. Snoha, *Topological entropy of nonautonomous dynamical systems,* Random Comput. Dynam. **4** (1996), 205–233.

[41] M.R.S. Kulenović, G. Ladas, *Dynamics of second order rational difference equations. With open problems and conjectures,* Chapman & Hall/CRC, Boca Raton, FL, 2002.

[42] R.P. Kurshan, B. Gopinath, *Recursively generated periodic sequences,* Can. J. Math. **26** (1974), 1356–1371.

[43] J.D. Lambert, *Numerical Methods for Ordinary Differential Systems. The Initial Value Problem,* John Wiley & Sons, Ltd., Chichester, 1991.

[44] A. Linero Bas, *Advances in Discrete Dynamics. Chapter 1: Periodic structure of discrete dynamical systems and global periodicity,* J. S. Cánovas Ed., Nova Publishers, 2011.

[45] R.C. Lyness, *Note 1581. Cycles,* Math. Gaz. **26** (1942), 62.

[46] R.C. Lyness, *Note 1847. Cycles,* Math. Gaz. **29** (1945), 231–233.

[47] R.C. Lyness, *Note 2952. Cycles,* Math. Gaz. **45** (1961), 207–209.

[48] M.P.S. Maier, E. Peacock-López, *Switching induced oscillations in the logistic map,* Physics Letters A **374** (2010), 1028–1032.

[49] E. Maor, *Trigonometric Delights,* Princeton University Press, Princeton, NJ, 1998.

[50] D.F. Mayers, K.W. Morton, *Numerical Solution of Partial Differential Equations. An Introduction,* second edition, Cambridge University Press, Cambridge, 2005.

[51] B.D. Mestel, *On globally periodic solutions of the difference equation* $x_{n+1} = f(x_n)/x_{n-1}$, J. Difference Equ. Appl. **9** (2003), 201–209.

[52] R.E. Mickens, *Difference Equations. Theory and Applications,* second edition, Van Nostrand Reinhold Co., New York, 1990.

[53] L.M. Milne-Thomson, *On the operational solution of the homogeneous linear equation of finite differences by generalised continued fractions,* Proc. R. Soc. Edinb. **51**, (1931), 91-96 (1931).

[54] D. Montgomery, *Pointwise periodic homeomorphisms,* Amer. J. Math. **59** (1937), 118–120.

[55] H. Poincaré, *Sur les Équations Linéaires aux Différentielles ordinaires et aux Différences finies,* Amer. J. Math. **7** (1885), 203–258 (in French).

[56] J. Rubió-Massegú, *On the global periodicity of discrete dynamical systems and application to rational difference equations,* J. Math. Anal. Appl. **343** (2008), 182–189.

[57] W.W. Sawyer, *Note 2951. Lyness' periodic sequence*, Math. Gaz. **45** (1961), 207.

[58] H. Sedaghat, *Nonlinear Difference equations. Theory with Applications to Social Science Models*, Mathematical Modelling: Theory and Applications **15**, Kluwer Academic Publishers, Dordrecht, 2003.

[59] A.N. Sharkovsky, *Coexistence of the cycles of a continuous mapping of the line into itself*, Ukrain. Math. Zh. **16** (1964), 61–71 (in Russian).

[60] A.N. Sharkovsky, *Coexistence of the cycles of a continuous mapping of the line into itself,* Internat. J. Bifur. Chaos Appl. Sci. Engrg. **5** (1995), 1263–1273.

[61] L.E. Sigler (translator), *Fibonacci's Liber Abaci: Leonardo Pisano's Book of Calculation,* Springer-Verlag, New York, 2003.

[62] P.A. Smith, *A theorem on fixed points for periodic transformations,* Ann. Math. **35** (1934), 572–578.

[63] P.A. Smith, *Fixed-point theorems for periodic transformations,* Amer. J. Math. **63** (1941), 1–8.

[64] R. Spurgin and M. Tamarkin, *Switching investments can be a bad idea when Parrondo's paradox applies,* J. Behavioral Finance **6** (2005), 15–18.

[65] J. Stoer, R. Bulirsch, *Introduction to numerical analysis*, third edition, Texts in Applied Mathematics **12**, Springer-Verlag, New York, 2002.

[66] J.W. Thomas, *Numerical Partial Differential Equations: Finite Difference Methods*, Texts in Applied Mathematics **22**, Springer-Verlag, New York, 1995.

[67] D.M. Wolf, V.V. Vazirani, A.P. Arkin, *Diversity in times of adversity: probabilistic strategies in microbial survival games,* J. Theor. Biol. **234** (2005), 227–253.

[68] T.J. Ypma, *Historical development of the Newton-Raphson Method,* SIAM Review **34** (1995), 531–551.

[69] Y. Zheng, *On periodic cycles of the Lyness equations,* Differential Equations Dynam. Systems **6** (1998), 319–324.