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A counterexample on global attractivity for Clark's equation

Víctor Jiménez López

Universidad de Murcia, Spain

vjimenez@um.es

Abstract

It has been conjectured for Clark's equation $x_{n+1} = \alpha x_n + (1-\alpha)h(x_{n-k})$ that a locally attracting fixed point is also globally attracting whenever h is a unimodal or decreasing map with negative Schwarzian derivative. In this note we present a counterexample to the conjecture when $k \geq 3$.

Proofs will appear in a forthcoming paper.

1 Introduction

Clark's equation is an important and very interesting higher order difference equation. It is given by

$$x_{n+1} = \alpha x_n + (1-\alpha)h(x_{n-k}), \quad n \geq 0, \quad (x_0, x_{-1}, \dots, x_{-k}) \in I^{k+1}, \tag{1}$$

where $0 < \alpha < 1$ is a parameter, $I = (a, b)$ is an open interval (with $-\infty \leq a < b \leq \infty$), $h : I \to \mathrm{Cl}\, I$ is a continuous function ($\mathrm{Cl}\, I$ denoting the closure of I) and k is a positive integer. Note that, although h may take values in the endpoints of I, (1) is well defined. In applications, we usually have $I = (0, \infty)$.

This equation was first studied by Clark in a 1976 paper [3], having been proposed by K. R. Allen in 1963 to model whale populations. For instance, in a report presented to the International Whaling Commission in 1980, Beddington and May suggested

$$h(x) = x\left[1 + q\left(1 - \left(\frac{x}{p}\right)^z\right)\right]_+, \tag{2}$$

with $p, q, z \in (0, +\infty)$, $[x]_+ = \max\{x, 0\}$, for the baleen whale [1]. Also used in the context of Clark's equation are, among other maps, Ricker's function

$$h(x) = pxe^{-qx}, \tag{3}$$

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and Shepherd's function

$$h(x) = \frac{px}{1 + x^q}, \tag{4}$$

$p, q > 0$. For these maps, equation (1) turns out to be the discretized version of two famous delay differential equations, the Nicholson blowflies equation [7] and the Mackey-Glass equation [13], respectively. A recent and very complete survey on (1) is [12].

Clark's equation is relevant not only because it arises from some more or less realistic biological models, but also because it provides a very convenient framework for studying a key issue in dynamics, namely, when the global dynamical behaviour is determined by the local one.

In this paper we are only concerned with the simplest dynamics at all. Generally speaking, we say that a point u is the *global attractor* of the equation

$$x_{n+1} = g(x_n, \dots, x_{n-k}), \quad n \geq 0, \quad (x_0, \dots, x_{-k}) \in I^{k+1}, \tag{5}$$

with $g : I^{k+1} \to I$ a continuous map, if all *orbits* $(x_n)_{n=-k}^{\infty}$ of (5) converge to u. A simple continuity argument shows that u is then a *fixed point* of (5), that is $u = g(u, \dots, u)$. Note that in the case of Clark's equation, u is a fixed point if and only if it is a fixed point of h. We say that the fixed point u is a *local attractor* of (5) if orbits with *initial conditions* $x_0, \dots, x_{-k}$ close enough to u converge to u. We say that u is *stable* for (5) if for any $\epsilon > 0$ there is $\delta > 0$ such that $|x_n - u| < \delta$ for any $n \in \{-k, \dots, 0\}$ implies $|x_n - u| < \epsilon$ for all n. If u is not stable then it is called *unstable*. Global (respectively, local) stable attractors are often called in the literature *globally* (respectively, *locally*) *asymptotically stable*, or, shortly, *G.A.S.* (respectively, *L.A.S.*).

The problem we are interested in is whether L.A.S. implies G.A.S. for Clark's equation. For a general equation like (5) this is a very difficult question to handle with, except when the *order* $k + 1$ of the equation is 1, that is, when we dealt with the equation $x_{n+1} = g(x_n)$. To begin with, in this case a global attractor is always stable, see for instance [15]. Moreover:

Definition 1.1. Let J be an interval and let $f : J \to \mathbb{R}$ a C^3 map. If $x \in J$ and $f'(x) \neq 0$, then the *Schwarzian derivative* of f in x, $Sf(x)$, is given by

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

It turns out that, under some mild additional assumptions, satisfied in practical cases quite often, L.A.S. and negative Schwarzian derivative imply G.A.S. This follows essentially from [16]:

Definition 1.2. We say that $g : I \to I$ *belongs to the class* S if it has the following properties:

(S1) g is a C^3 map and g' vanishes at most at one point c (which is a relative extremum of g);

(S2) $Sg(x) < 0$ for any $x \in I$ (except possibly at c);

(S3) there is $u \in I$ such that $g(x) > x$ (respectively, $g(x) < x$) for any $x < u$ (respectively, $x > u$).

Theorem 1.1. *If g belongs to the class S and $|g'(u)| \leq 1$, then u is the global attractor of $x_{n+1} = g(x_n)$.*

Note that condition (S3), usually called in the literature the *negative-feedback condition*, ensures that u is the only fixed point of g. Also, observe that just "vanishes at most at one point c" in condition (S1) is not enough: the map $g:\mathbb{R}\to\mathbb{R}$ given by $g(x)=-x^3$ has negative Schwarzian derivative except at the fixed and critical point $u=0$. However, u is not globally attracting because the points $-1,1$ form a two-cycle. Finally, we emphasize that $-1\le g'(u)<0$ is by far the most interesting case in Theorem 1.1. Indeed, it is quite simple to prove:

Theorem 1.2. *If a continuous map g satisfies (S3), there is $c\in I$ such that both restrictions $g|_{(a,c)}, g|_{(c,b)}$ are monotone, and one of the restrictions $g|_{(a,u)}, g|_{(u,b)}$ is increasing, then u is a global attractor of $x_{n+1}=g(x_n)$.*

Let us return to Clark's equation (1). A well-known and very nice property of this equation is that if u is the global attractor of $x_{n+1}=h(x_n)$, then it is also the global (stable) attractor of (1). This seems to suggest that, in a certain sense, dynamics of (1) is "at most" as complicated as that of h. Indeed, to get global attraction in (1) it suffices that u is a global attractor of $x_{n+1}=H(x_n)$, with $H(x)=\alpha^{k+1}u+(1-\alpha^{k+1})h(x)$ [4]. Note that if h belongs to the class S, then so does H. Now, applying Theorem 1.1, we get that

$$-1/(1-\alpha^{k+1})\le h'(u)<0 \tag{6}$$

is enough to guarantee global attraction for (1). This is not the best available result on global attraction for type of maps, see [17], but it illustrates the point we want to emphasize here: if h belongs to the class S, then local dynamics at u may hopefully govern the entire dynamics of (1).

Since in view of the above discussion and Theorems 1.1 and 1.2 global attraction is ensured if $h'(u)\ge -1$, we concentrate in what follows in the case $r=h'(u)<-1$. Now, after linearizing at u, we get that u is a local stable attractor of (1) if all roots of the characteristic polynomial

$$\lambda^{k+1}-\alpha\lambda^k-(1-\alpha)r \tag{7}$$

have modulus less than 1, while u is unstable if some of the roots have modulus greater than 1. It turns out that there is a number $\alpha_k=\alpha_k(r)$, $0<\alpha_k<1$, such that all roots of (7) have modulus less than 1 if and only if $\alpha>\alpha_k$. The curve $\alpha_k(r)$ is decreasing, with $\alpha_k(r)\to 1$ as $r\to-\infty$ and $\alpha_k(r)\to 0$ as $r\to -1$. It is given, in parametric form, by

$$r=\frac{\sin(\theta)}{\sin((k+1)\theta)-\sin(k\theta)},\quad \alpha=\frac{\sin((k+1)\theta)}{\sin(k\theta)},$$

$\theta\in(\pi/(2k+1),\pi/(k+1))$. In the particular case $k=1$ this amounts to $\alpha_1(r)=1+1/r$ or, alternatively, $r=-1/(1-\alpha_1(r))$ (compare to (6)). See [10, 14] for the details.

In this way we naturally arrive at the following conjecture, see [8, 6] and also [18, 12] for some numerical experiments supporting it:

Assume that h belongs to the class S and $r=h'(u)<-1$. Then u is the global attractor of (1) *if and only if $\alpha\ge\alpha_k(r)$. As a consequence, L.A.S. implies G.A.S. for* (1) *whenever h belongs to the class S.*

The aim of this note is to present a counterexample to this conjecture when $k\ge 3$.

2 Main results

Recall that we are assuming $r = h'(u) < -1$.

Looking for a possible counterexample to the conjecture, a very natural way to proceed is as follows. It can be shown that, at the bifurcation parameter $\alpha = \alpha_k(r)$, (7) has two complex (conjugated) roots $\mu_k(r), \overline{\mu_k(r)}$ with modulus 1 while all other roots have modulus less than 1. In such a case, under appropriate smoothness assumptions for h (C^4 is enough) a so-called *Neimark-Sacker bifurcation* generally occurs involving the appearance of an invariant curve near the fixed point [11]. ("Curve" refers here to a set $C \subset I^{k+1}$ homeomorphic to the circle, and "invariant" means that whenever the vector of initial conditions $(x_0, \ldots, x_{-k})$ of an orbit (x_n) belongs to C, then all vectors $(x_n, \ldots, x_{n-k})$ belong to C as well.) More precisely, if the parameter α is close enough to $\alpha_k(r)$, then for our equation (1) we have that either there is an invariant curve whenever $\alpha < \alpha_k(r)$ (the *supercritical* case) or there is an invariant curve whenever $\alpha > \alpha_k(r)$ (the *subcritical* case). Thus, if a subcritical Neimark-Sacker bifurcation occurs, the locally attracting fixed point coexists with an invariant curve so the point cannot be globally attracting.

Our first theorem shows that, when h belongs to the class S, there is "almost no chance" for a subcritical bifurcation to occur. Observe that $Sh(u) < 0$ is equivalent to

$$\Sigma h(u) := \frac{h'''(u)h'(u)}{(h''(u))^2} < \frac{3}{2}.$$

(If $h''(u) = 0$, then we mean $\Sigma h(u) = \infty > 3/2$ or $\Sigma h(u) = -\infty < 3/2$ according to, respectively, $h'''(u) \leq 0$ or $h'''(u) > 0$.)

Theorem 2.1. *Assume that h is a C^4 map and one of the following conditions holds:*

(a) $k \leq 2$ *and* $\Sigma h(u) < 3/2$*;*

(b) $r < -1.18$ *and* $\Sigma h(u) < 3/2$*;*

(c) $\Sigma h(u) < 1.49$.

Then (1) *exhibits a supercritical Neimark-Sacker bifurcation at* $\alpha = \alpha_k(r)$.

In particular, local negative Schwarzian derivative at u precludes the appearance of subcritical Neimark-Sacker bifurcations in the cases $k = 1$ and $k = 2$, or, regardless k, if r is "not too close" to -1. Of course, the existence of a supercritical bifurcation does not guarantee that the local attractor is global, it just rules out the existence of invariant curves near the fixed point for a small range of values of the parameter α close to the bifurcation parameter $\alpha_k(r)$. Yet, in combination with results as that indicated by (6), taking care of parameter values "not too close" to $\alpha_k(r)$, this helps to understand why, in random numerical experiments, local and global attraction are most probably to coalesce. For instance, it is easy to check that $\Sigma h(u) < 1$ for the Ricker function (3), so the bifurcation is always supercritical. It is worth emphasizing that the paper [18] is devoted to study the Neimark-Sacker bifurcation for (1) in this particular case. A theorem characterizing when the bifurcation is either supercritical o subcritical is given (Theorem 3 there), but the proof is wrong because the coefficient g_{21} is not correctly calculated (compare to [11, p. 187]). The paper includes just one numerical example where, in accordance with our results, the bifurcation is supercritical.

Nevertheless, we have:

Theorem 2.2. *Let h_ϵ, $0<\epsilon<\epsilon_0$, be C^4 maps. Assume that for any ϵ there is $u_\epsilon \in I$ such that the following conditions are satisfied:*

(i) $h_\epsilon(u_\epsilon)=u_\epsilon$, $h'_\epsilon(u_\epsilon)<-1$*;*

(ii) $\lim_{\epsilon\to 0} h'_\epsilon(u_\epsilon)=-1$*;*

(iii) the map $T(\epsilon):=\Sigma h_\epsilon(u_\epsilon)$ *is differentiable and* $\lim_{\epsilon\to 0} T'(\epsilon)=0$*;*

(iv) $\lim_{\epsilon\to 0} T(\epsilon)=3/2$.

Then, if $k\geq 3$, $\epsilon>0$ is small enough and we put $h=h_\epsilon$, $u=u_\epsilon$, (1) *exhibits a subcritical Neimark-Sacker bifurcation at $\alpha=\alpha_k(r)$, $r=h'(u)$. In particular, if $\alpha>\alpha_k(r)$ is close enough to $\alpha_k(r)$, then u is a local, but not global, attractor of* (1).

Remark 2.1. Theorem 2.2 admits an alternative version replacing $\lim_{\epsilon\to 0} T'(\epsilon)=0$ by the weaker assumption $\lim_{\epsilon\to 0} T'(\epsilon)<\pi/8$ in (iii). Then the statement holds true for all k *large enough*.

A remarkable example for which Theorem 2.2 holds is Shepherd's function (4) when the parameters $p,q>0$ are appropriately chosen. In fact, it is easily seen that if $1/p+2/q<1$, then h belongs to the class S and $h'(u)<-1$; here, $u=(p-1)^{1/q}$. Also,

$$\Sigma h(u)=\frac{(p(-1+q)-q)(6q^2-6pq^2+p^2(-1+q^2))}{(-1+p)(p(-1+q)-2q)^2 q}.$$

For any $\epsilon>0$ we define $h_\epsilon(x)=h(x)$ by taking $q=2+\epsilon^2$ and $p=\epsilon+q/(q-2)=1+\epsilon+2/\epsilon^2$. Thus we get

$$u_\epsilon=\left(\frac{2}{\epsilon^2}+\epsilon\right)^{1/(2+\epsilon^2)},$$

$$h'_\epsilon(u_\epsilon)=-\frac{2+\epsilon^2+\epsilon^3+\epsilon^5}{2+\epsilon^2+\epsilon^3}$$

and

$$T(\epsilon)=\frac{\left(2+\epsilon^2+\epsilon^3+\epsilon^5\right)\left(12-20\epsilon^2+12\epsilon^3-25\epsilon^4-2\epsilon^5-\epsilon^6-12\epsilon^7+5\epsilon^8-4\epsilon^9+\epsilon^{10}\right)}{\left(2+\epsilon^2\right)\left(2+\epsilon^3\right)\left(2-\epsilon^2+\epsilon^3-\epsilon^4+\epsilon^5\right)^2},$$

$$T'(\epsilon)=\frac{p(\epsilon)}{\left(2+\epsilon^2\right)^2\left(2+\epsilon^3\right)^2\left(2-\epsilon^2+\epsilon^3-\epsilon^4+\epsilon^5\right)^3},$$

with

$$\begin{aligned} p(\epsilon) \;=\;& \epsilon(-256-1408\epsilon^2-400\epsilon^3-2560\epsilon^4-1472\epsilon^5-2472\epsilon^6-1856\epsilon^7-1436\epsilon^8\\ &-1264\epsilon^9-74\epsilon^{10}-519\epsilon^{11}+545\epsilon^{12}+132\epsilon^{13}+336\epsilon^{14}+344\epsilon^{15}+102\epsilon^{16}\\ &+168\epsilon^{17}+44\epsilon^{18}+27\epsilon^{19}+17\epsilon^{20}+2\epsilon^{22}). \end{aligned}$$

Clearly, all hypotheses of Theorem 2.2 are satisfied.

Shepherd's function has a turning point. A simpler example, also fulfilling all hypotheses of Theorem 2.2, is given by

$$h_\epsilon(x)=\frac{1}{(1-2\epsilon)(\epsilon+(1-\epsilon)x)+2\epsilon(\epsilon+(1-\epsilon)x)^2}, \tag{8}$$

$0 < \epsilon < 1/2$. Then h_ϵ is a (bounded) decreasing function such that $Sh_\epsilon(x) < 0$ for any $x \in (0, \infty)$. Its only fixed point is $u_\epsilon = u = 1$ and $h'_\epsilon(u) = -1 - \epsilon + 2\epsilon^2 < -1$. Moreover,

$$T(\epsilon) = \frac{3(1+2\epsilon)^2\left(1+4\epsilon^2\right)}{2\left(1+2\epsilon+4\epsilon^2\right)^2},$$

$$T'(\epsilon) = \frac{48\epsilon^3 - 12\epsilon}{\left(1+2\epsilon+4\epsilon^2\right)^3}.$$

Figure 1 illustrates Theorems 2.2 for the map (8). Here we take $k = 3$, $\epsilon = 0.00167086$ and $\alpha = \alpha_3(r) + 0.0001 = 0.00573994$, and depict pairs (x_{n+1}, x_n) for orbits $(x_n)_{n=-3}^{\infty}$ starting at several initial conditions $(x_0, x_{-1}, x_{-2}, x_{-3})$. The closed curve in the picture (the thickest one, painted in light grey) is the first two coordinates projection of the invariant curve in $(0, \infty)^4$ whose existence is predicted by Theorem 2.2 and contains the orbit (approximately) starting at

$$(1.898919, 1.570831, 0.995705, 0.638023).$$

"Arcs" inside the curve (intermediate thickness, painted in dark grey) consist of pairs from the orbit starting at

$$(1.8, 1.570831, 0.995705, 0.638023).$$

Among them, those closest to the curve arise after dotting the first 400 iterates of this point, while closer arcs to $(1,1)$ correspond to iterations from 100000 to 100400. Outer arcs (the thinnest ones, black colour) are similarly obtained, this time starting at the point

$$(2, 1.570831, 0.995705, 0.638023).$$

Big black dots at $(1.8, 1.570831)$, $(1.898919, 1.570831)$, $(2, 1.570831)$ and $(1,1)$ indicate the different initial conditions and the equilibrium. It is worth emphasizing that the dynamics near the invariant curve is almost "eight-periodic". The reason is that, after rewriting (1) in vectorial form and linearizing at the fixed point, it is possible to decompose $\mathbb{R}^{k+1}$ (when α is the bifurcation parameter $\alpha_k(r)$) as a direct sum of the invariant subspaces L_s and L_c, with L_s being the $(k-1)$-dimensional eigenspace corresponding to eigenvalues with modulus less than 1, while L_c is the 2-dimensional eigenspace attached to eigenvalues $\mu_k(r), \overline{\mu_k(r)}$. Moreover, modulo an additional linear transformation, we get at L_c a rotation of angle $\theta_k(r)$, where we mean $\mu_k(r) = e^{i\theta_k(r)}$. The Neimark-Sacker theory now shows that, when the bifurcation is subcritical and α is slightly larger than $\alpha_k(r)$, there is still an invariant 2-dimensional manifold for (1), but here one repelling invariant curve (the curve whose projection is depicted in Figure 1) coexists with the attracting fixed point; dynamics inside this curve is nearly that of a rotation of angle $\theta_k(r)$. In our concrete example, we have that $\theta_3(r)$ is approximately $\pi/4$, which accounts for the "eight-almost periodicity" we see in the picture. Finally, observe that although initial conditions are different, some subsequent pairs are almost identical (which is why the "endpoints" of some grey and black arcs practically coincide). This is because α is very small, so the equation almost reads $x_{n+1} = h_\epsilon(x_{n-3})$. For instance, after two iterations we get in all three cases approximately the same pair $(h_\epsilon(0.995705), h_\epsilon(0.638023)) \approx (1.007566, 1.56965)$.

Pairs near the axes belong to the orbit starting at

$$(1.219971, 0.0768226, 0.00488285, 0.0308514),$$

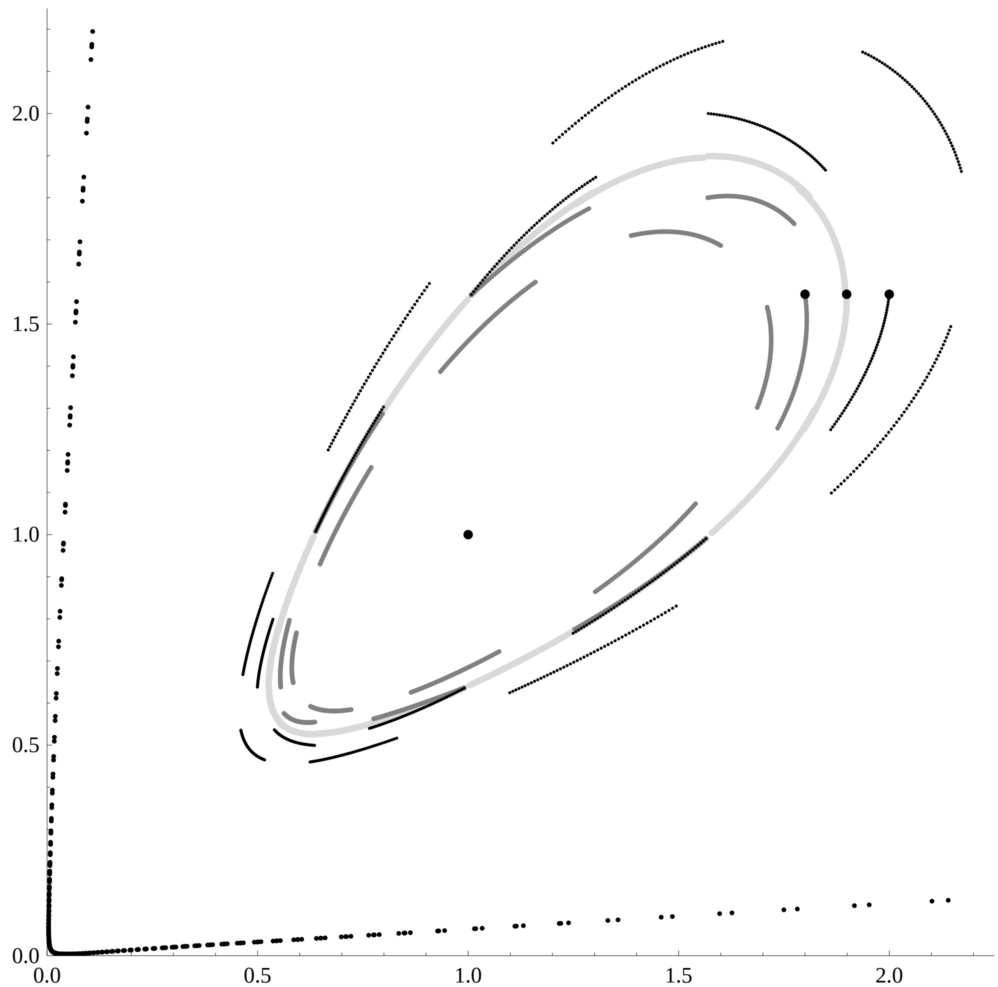

Figure 1: A local, but not global attractor, for Clark's equation.

which apparently is contained in another periodic orbit and seems to attract "most" orbits not attracted by the fixed point. We have no rigorous proof of these last two facts, but they are quite in accordance with a well-known result: if $I \neq \mathbb{R}$, then, regardless the map h and the parameter α, (1) is *permanent*, that is, there are numbers $a < c \leq d < b$ such that

$$c \leq \liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n \leq d$$

[4]. Note that it is necessary to exclude the case $I = \mathbb{R}$: for instance, the orbits of equation $x_{n+1} = x_n/2 + 3x_{n-1}/2$ are explicitly given by

$$x_n = \frac{3(x_0 + x_{-1})}{5}\left(\frac{3}{2}\right)^n + \frac{2x_0 - 3x_{-1}}{5}(-1)^n,$$

so there is no permanence here.

Theorems 2.1 and 2.2 are consequences of a more general result fully characterizing when a supercritical or a subcritical Neimark-Sacker bifurcation occurs in Clark's equation. Its statement is somewhat complicated and will not be given here. In the simplest case $k = 1$ it amounts to the following:

Theorem 2.3. *Assume that h is a C^4 map and $k = 1$. If $\Sigma h(u) < (1 - 2r)/(1 - r)$ (respectively, $\Sigma h(u) > (1 - 2r)/(1 - r)$), then* (1) *exhibits a supercritical (respectively, subcritical) Neimark-Sacker bifurcation at $\alpha = \alpha_1(r) = 1 + 1/r$.*

Note that, since $r < -1$, this result refines Theorem 2.1(a) when $k = 1$.

In view of the previous theorems it is tempting to conjecture that belonging to the class S and local attraction still imply global attraction in the cases $k = 1, 2$. Indeed, our numerical simulations suggest that such is the case for the map (8). Needless to say, according to Theorem 2.3 a subcritical bifurcation (and hence a local, but not global attractor) may occur even when $k = 1$ if no conditions on the Schwarzian derivative are imposed. It is easy to check that happens, for instance, for the decreasing map with fixed point $u = 1$

$$h(x) = \frac{419 + 722x + 6859x^2}{(1 + 19x)^3}.$$

While in the above discussion $k = 1$ or $k = 2$ makes no difference at all, there is an important issue where a distinction arises. As is well known, if a map h belongs to the class S, then $x_{n+1} = h(x_n)$ has at most one metric attractor (see, e.g., [2]). By a *metric attractor* we mean a compact set containing the limit sets of the orbits of a positive Lebesgue measure set of points, and having no a (strict) subset with the same properties. In fact, the equation has exactly one metric attractor, attracting the orbits of almost all points, except in the case when h is decreasing and all orbits escape to the endpoints of I (e.g., $h : (0, \infty) \to (0, \infty)$ given by $h(x) = 1/x^2$), that is, $x_{n+1} = h(x_n)$ is not permanent.

Now, numerical experiments suggest that when a decreasing map h belongs to the class S and $k = 1$, (1) has exactly one metric attractor, attracting the orbits of almost all vectors (x_0, x_{-1}) of initial conditions. Moreover, this attractor is either a periodic orbit (possibly a fixed point) or an invariant curve. This seems to be true even if $x_{n+1} = h(x_n)$ is not permanent. (We emphasize that if $h : I \to I$ has negative Schwarzian derivative and $h'(x) < 0$ for any $x \in I$, then it is not difficult to show that $I \neq \mathbb{R}$, hence, as said before, (1) is permanent.) On the other hand, for $k = 2$, an example of equation (1) having two metric attractors, with $h : (0, \infty) \to (0, \infty)$ being a bounded decreasing map such that $Sh(x), h'(x) < 0$ for any $x \in (0, \infty)$, is given in [5].

Detailed proofs of all above-mentioned results will appear elsewhere [9].

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