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Integrability and dynamical degree of periodic non-autonomous Lyness recurrences.

Anna Cima

Universitat Autònoma de Barcelona, Spain

cima@mat.uab.cat

Abstract

This work deals with non-autonomous Lyness type recurrences of the form

$$x_{n+2} = \frac{a_n + x_{n+1}}{x_n},$$

where $\{a_n\}_n$ is a k-periodic sequence of positive numbers with minimal period k. We treat such non-autonomous recurrences via the autonomous dynamical system generated by the birational map $F_{a_k} \circ F_{a_{k-1}} \circ \cdots \circ F_{a_1}$ where F_a is defined by $F_a(x, y) = (y, \frac{a+y}{x})$. When $k \in \{1, 2, 3, 6\}$ the behavior of the sequence $\{x_n\}_n$ is simple (integrable). In fact, the corresponding mappings have a rational first integral. We show that for $k = 4$ the dynamical system is, generically, no longer rationally integrable, by calculating its dynamic degree. We also show that for $k = \dot{5}$ and for most values of the parameters the dynamical system has no meromorphic first integral.

This paper is based on the joint work with A. Gasull and V. Mañosa, *Integrability and non-integrability of periodic non-autonomous Lyness recurrences* and also on the joint work with S. Zafar, *Dynamical degree of periodic non-autonomous Lyness recurrences*.

1 Introduction

Autonomous recurrences are frequently used to model ecological systems. One of the modifications applied in the models in order to adapt them to more realistic situations consists in converting the recurrences into non-autonomous ones changing one of the constant parameters by a periodic cycle. In this situation it is said that the model takes into account seasonality. For instance a parameter taking values in a cycle of period 2 could model an ecological situation that has different features during summer or winter.

Consider the non-autonomous Lyness difference equations of the form

$$x_{n+2} = \frac{a_n + x_{n+1}}{x_n}, \tag{1}$$

where $\{a_n\}_n$ is a k-periodic sequence of positive numbers. Such recurrences have been studied in [6, 9, 12, 14, 15] and recently in [7].

For each k, the *composition maps* are

$$F_{a_k,\dots,a_2,a_1} := F_{a_k} \circ \cdots \circ F_{a_2} \circ F_{a_1} \tag{2}$$

where each F_{a_i} is defined by

$$F_{a_i}(x,y) = \left(y, \frac{a_i + y}{x}\right)$$

and $a_1, a_2, \dots, a_k$ are the k elements of the cycle. The values $a_1, a_2, \dots, a_k$ will be usually called parameters.

When there is no confusion, for the sake of shortness, we also will use the notation $F_{[k]} := F_{a_k,\dots,a_2,a_1}$. Note that these maps are birational maps, that

$$(x_1,x_2) \xrightarrow{F_{a_1}} (x_2,x_3) \xrightarrow{F_{a_2}} (x_3,x_4) \xrightarrow{F_{a_3}} (x_4,x_5) \xrightarrow{F_{a_4}} (x_5,x_6) \xrightarrow{F_{a_5}} \cdots$$

and in general,

$$F_{[k]}(x_1,x_2) = (x_{k+1}, x_{k+2}).$$

For instance, when $k = 2$, by setting

$$a_n = \begin{cases} a & \text{for} \quad n = 2\ell + 1, \\ b & \text{for} \quad n = 2\ell, \end{cases} \tag{3}$$

we get

$$F_{b,a}(x,y) := F_b \circ F_a(x,y) = \left(\frac{a+y}{x}, \frac{a+bx+y}{xy}\right),$$

and when $k = 3$,

$$a_n = \begin{cases} a & \text{for} \quad n = 3\ell + 1, \\ b & \text{for} \quad n = 3\ell + 2, \\ c & \text{for} \quad n = 3\ell, \end{cases} \tag{4}$$

and

$$F_{c,b,a}(x,y) := F_c \circ F_b \circ F_a(x,y) = \left(\frac{a+bx+y}{xy}, \frac{a+bx+y+cxy}{y\,(a+y)}\right).$$

Clearly the study of the dynamics of the recurrences given by (1) can be deduced from the dynamics generated by the composition maps (2).

It is known that for the cases $k \in \{1,2,3,6\}$ and for all values of the parameters, the mappings $F_a, F_{b,a}, F_{c,b,a}$ and $F_{f,e,d,c,b,a}$ have a rational first integral, see [7].

Next, we will present some typical phase portraits that can be found when doing some numerical explorations. All the pictures of the paper are done with the maps

$$G_a(x,y) = (y, -x + \ln(a + \exp(y)),$$

which are conjugate with $F_a(x,y)$ in the first quadrant Q^+, trough the change $\Phi(x,y) = (\ln(x), \ln(y))$. These new variables allow us to "observe" much better the behavior of the orbits in Q^+.

The typical picture for the orbits of $G_{[k]}$ when $k = 1,2,3,6$ is shown in Fig. 1.

Figure 2 shows a picture of some of the orbits of a map $G_{[4]}$ for concrete values of a, b, c, d.

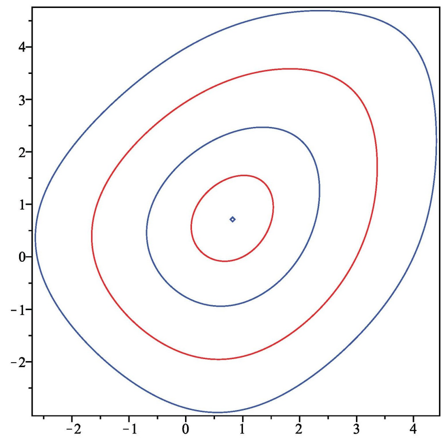

Figure 1: Typical phase portrait of a map $G_{[k]}$ in the *rationally integrable* case, that is, when $k = 1, 2, 3, 6$.

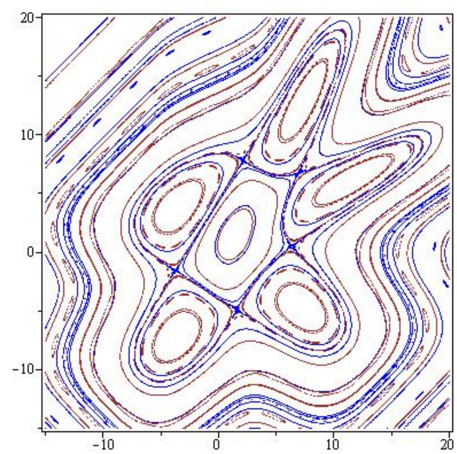

Figure 2: Typical phase portrait of a map $G_{[4]}$.

The picture in Fig. 2 suggests that the map is not integrable. In fact we show that for a generic set of values of the parameters the map $F_{[4]}$ has dynamical degree bigger than 1. From the works [4] and also [11], we conclude that generically $F_{[4]}$ does not have a rational first integral.

Concerning the case $k = 5$ all our numerical approaches seem to show that the maps $F_{[5]}$ are integrable.

As we see in Fig. 3, the phase portrait does not coincide with the ones found in all the rational integrable cases. However, in this case we can prove that for most of the values of the parameters the map $F_{[5]}$ has not a meromorphic first integral.

All the results that we mention in this paper are proved in [7] except the one concerning the dynamical degree for the case $k = 4$ which is proved in the forthcoming paper [8], a joint work

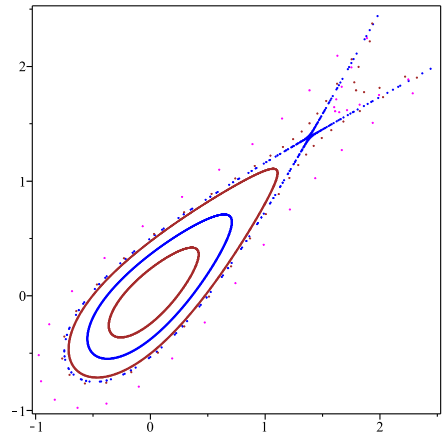

Figure 3: Phase portrait of a map $G_{[5]}$.

with S. Zafar.

2 Rational integrability and associated dynamics

This section deals with the integrable cases. The results are the following:

Theorem 2.1. *The maps $F_{a_k \dots a_2 a_1}$, for $k \in \{1, 2, 3, 6\}$, have the first integrals:*

$$\begin{aligned} V_a(x,y) &= \frac{a+(a+1)x+(a+1)y+x^2+y^2+x^2y+xy^2}{xy} \\ V_{b,a}(x,y) &= \frac{ab+(a+b^2)x+(b+a^2)y+bx^2+ay^2+ax^2y+bxy^2}{xy} \\ V_{c,b,a}(x,y) &= \frac{ac+(a+bc)x+(c+ab)y+bx^2+by^2+cx^2y+axy^2}{xy} \\ V_{f,e,d,c,b,a}(x,y) &= \frac{af+(a+bf)x+(f+ae)y+bx^2+ey^2+cx^2y+dxy^2}{xy} \end{aligned}$$

Theorem 2.2. *Assume that a, b, c are positive constants. Then the following hold:*

(i) The level sets of V_a, V_{ba} and V_{cba} in the first quadrant Q^+ are homeomorphic to a circles surrounding a unique fixed point of F_a, F_{ba} and F_{cba} respectively.

(ii) The action of F_a, F_{ba} and F_{cba} on each level set contained in Q^+ is conjugated to a rotation of the circle.

When $k = 6$ we are confident that the same result holds but we have only been able to prove the result when $F_{[6]}$ has a unique fixed point in the first quadrant.

Theorem 2.3. *Assume that a, b, c, d, e, f are positive constants and that $F_{f,e,d,c,b,a}$ has a unique fixed point in the first quadrant Q^+. Then:*

(i) The level sets of $V_{f,e,d,c,b,a}$ *in the first quadrant* Q^+ *are homeomorphic to a circles surrounding the unique fixed point of* $F_{f,e,d,c,b,a}$.

(ii) The action of $F_{f,e,d,c,b,a}$ *on each level set contained in* Q^+ *is conjugated to a rotation of the circle.*

Next result collects our integrability results for any $k \neq 5$. To state it, we need the following definitions: given a periodic sequence $\{a_n\}_n$ of prime period k we will say that its *rank* is m if

$$\text{Card}\{a_1, a_2, \dots, a_k\} = m \in \mathbb{N}.$$

In our context the recurrence (1) is called *persistent* if for any sequence $\{x_n\}_n$ there exist two real positive constants c and C, which depend on the initial conditions, such that for all n, $0 < c < x_n < C < \infty$.

Theorem 2.4. *(i) For any* $k \geq 15$*, there exist sequences* $\{a_n\}_n$ *of primitive period* k *and rank* k *such that* $F_{[k]} = F_{a_k,\dots,a_2,a_1}$ *is rationally integrable and the corresponding recurrence (1) is persistent.*

(ii) For any $k < 15, k \neq 5$*, there exist sequences* $\{a_n\}_n$ *of primitive period* k *with the ranks as in Table 1, such that* $F_{[k]}$ *is rationally integrable and the corresponding recurrence (1) is persistent.*

(iii) Moreover it is possible to take in all the above cases parameters $a_1, a_2, \dots a_k$ *such that each sequence* $\{x_n\}_n$ *is either periodic, with period a multiple of* k*, or it densely fills at most* k *disjoint intervals of* $\mathbb{R}^+$.

k	1	2	3	4	5	6	7	8	9	$10 \leq k \leq 14$	$k \geq 15$
Rank	1	2	3	4	-	6	3	4	5	$k-5$	k

Table 1. Possible ranks for integrable $F_{[k]}$.

3 Algebraic non-integrability for the case $\mathbf{k} = 4$

As usual we say that a set of k parameters $a_1, a_2, \dots, a_k$ is generic if $\{(a_1, a_2, \dots, a_k) \in \mathbb{R}^k\}$ is an open and dense subset of $\mathbb{R}^k$ with the usual topology.

We notice that the maps $F_{a_k,\dots,a_2,a_1} : \mathbb{R}^2 \to \mathbb{R}^2$ are birational maps. A birational map is a map F with rational components such that there exists an algebraic curve V and another rational map G such that $F \circ G = G \circ F = \text{id}$ in $\mathbb{C}^2 \setminus V$. We are going to use the embedding $(x_1, x_2) \longmapsto [x_0 : x_1 : x_2] \in P\mathbb{C}^2$ to extend the maps to $P\mathbb{C}^2$ in the usual way, by getting a homogeneous map which has an associated degree, called the degree of the map. Let d_n be the degree of the iterates $F^n = F \circ \cdots \circ F$. The *dynamical degree* of F is defined as

$$\delta(F) = \lim_{n \to \infty} (d_n)^{\frac{1}{n}}.$$

And its logarithm is called the *algebraic entropy* of F.

It is known (see Corollary 2.2 of [11] for instance) that given a birational $f : P\mathbb{C}^2 \to P\mathbb{C}^2$ and calling d_n the degree of f^n then the sequence d_n satisfies a homogeneous linear recurrence with constant coefficients:

$$d_{n+k} = -(c_{k-1}d_{n+k-1} + \cdots + c_0 d_n).$$

Theorem 3.1. *For a generic set of the values of the parameters $a, b, c, d \in \mathbb{R}$, the dynamical degree of the map $F_{d,c,b,a} := F_d \circ F_c \circ F_b \circ F_a$ is the largest root of the polynomial $z^3 - 2z^2 - 3z - 1$, which aproximatelly is* 3.079595623.

It is also known that the existence of a foliation of the space by algebraic invariant curves implies that the dynamical degree is one, see [4] for instance, also [11]. So, Theorem 2.1 implies that for generic values of $a, b, c, d \in \mathbb{R}$, the map $F_{d,c,b,a}$ is not rationally integrable.

We deal with the maps:

$$F_{d,c,b,a}(x,y) := F_d \circ F_c \circ F_b \circ F_a(x,y) = \left(\frac{bxy + cx + d + y}{y\,(d+y)}, \frac{x\left(ayd + ay^2 + bxy + cx + d + y\right)}{(d+y)\,(cx+d+y)} \right).$$

By extending it to $P\mathbb{C}^2$ we get the map $f[x_0 : x_1 : x_2]$ with components

$$\begin{aligned}
f_1[x_0 : x_1 : x_2] &= x_0\, x_2\, (dx_0 + x_2)\,(cx_1 + dx_0 + x_2)\,,\\
f_2[x_0 : x_1 : x_2] &= x_0\, \left(bx_1\, x_2 + cx_0\, x_1 + dx_0{}^2 + x_0\, x_2\right)(cx_1 + dx_0 + x_2)\,,\\
f_3[x_0 : x_1 : x_2] &= x_1\, x_2\, \left(ax_2\, dx_0 + ax_2{}^2 + bx_1\, x_2 + cx_0\, x_1 + dx_0{}^2 + x_0\, x_2\right).
\end{aligned}$$

Notice that $d_1 = 4$. In [8] we prove that the sequence of the degrees d_n of $f[x_0 : x_1 : x_2]$ satisfies the recurrence

$$d_{n+3} = 2d_{n+2} + 3d_{n+1} + d_n$$

and since $d_1 = 4, d_2 = 12$ and $d_3 = 37$, the sequence of the degrees is

$$4\,,\ 12\,,\ 37\,,\ 114\,,\ 351\,,\ 1081\,,\ 4059\,,\ 11712\,, \ldots.$$

4 Meromorphic non-integrability for the case $k = \dot{5}$

Two analytic functions $P, Q : \mathcal{U} \subset \mathbb{C}^2 \to \mathbb{C}$ are said to be coprime if the points of the set $\{(x,y) \in \mathcal{U} : P(x,y) = Q(x,y) = 0\}$ are isolated. A function $H = P/Q$, with P and Q coprime, will be called a *meromorphic function*. A *meromorphic first integral* of an analytic map $F : \mathcal{U} \to \mathbb{C}^2$ is a meromorphic function $H = P/Q$ such that

$$P(F(x,y))Q(x,y) = P(x,y)Q(F(x,y)) \quad \text{for all} \quad (x,y) \in \mathcal{U}.$$

Observe that from this definition $H(F(x,y)) = H(x,y)$ for all points of $\mathcal{U}$ for which both terms of this last equality are well-defined. When P and Q are polynomials then it is said that H is a *rational first integral*. Similarly we can talk about *meromorphic or rational invariants*. The first result is a necessary condition for the meromorphic integrability of planar maps near a fixed point. Our approach follows the guidelines of Poincaré when he studied the same problem for ordinary differential equations, see [16] and the references there in for the approach to ordinary differential equations. We apply the results below to study the case $k = \dot{5}$.

Theorem 4.1. *Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ an analytic map defined in $\mathcal{U}$, an open neighborhood of the origin, such that $F(0,0) = (0,0)$ and $DF(0,0)$ is diagonalizable with eigenvalues λ and μ. Assume that F has a meromorphic first integral H in $\mathcal{U}$.*

(i) If $\lambda\mu \neq 0$ then there exists $(p,q) \in \mathbb{Z}^2$, $(p,q) \neq (0,0)$, such that $\lambda^p \mu^q = 1$.

(ii) If $\lambda \neq 0$ and $\mu = 0$ then there exists $n \in \mathbb{N}^+$ such that $\lambda^n = 1$.

When the map $F : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}^2$ is real valued and of class $\mathcal{C}^2(\mathcal{U})$ the proof of Theorem 4.1 can be adapted following the same steps. Taking into account that in this case, when $\lambda \in \mathbb{C}$ is an eigenvalue of $DF(0,0)$, then $\bar{\lambda}$ it is so, and we have to deal with the resonant condition $\lambda^p \bar{\lambda}^q = 1$, we obtain the following result:

Corollary 4.2. *Let $F : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a $\mathcal{C}^2(\mathcal{U})$ map such that $F(0,0) = (0,0) \in \mathcal{U}$ and $DF(0,0)$ is diagonalizable, with eigenvalues λ and μ. Assume that F has a meromorphic first integral H in $\mathcal{U}$.*

(i) If $\lambda, \mu \in \mathbb{R}$, $\lambda\mu \neq 0$, then there exists $(0,0) \neq (p,q) \in \mathbb{Z}^2$ such that $\lambda^p \mu^q = 1$.

(ii) If $0 \neq \lambda \in \mathbb{C} \setminus \mathbb{R}$ (hence $\mu = \bar{\lambda}$), then either $|\lambda| = 1$ or $\lambda = |\lambda| e^{i\theta}$ and there exists $0 \neq n \in \mathbb{N}$ such that $(e^{i\theta})^{2n} = 1$.

(iii) If $\lambda \neq 0$ and $\mu = 0$ then there exists a $n \in \mathbb{N}^+$ such that $\lambda^n = 1$.

Our main result is the following theorem:

Theorem 4.3. *For $k = \dot{5}$ and for most values of the parameters $\{a_n\}_n$ the map $F_{[k]}$ has no meromorphic first integral.*

In the above result, for "most values of the parameters" we exactly mean that the set of the parameters for which the theorem does not necessarily hold has Lebesgue measure equal zero.Its proof is a consequence of the following result:

Proposition 4.4. *For $k = \dot{5}$ let ϕ_i, $i = 1, \dots, 5$ be defined by*

$$\phi_i = \prod_{\substack{n \equiv i \pmod 5 \\ n = 1, \dots, k}} a_n.$$

Then if $\{\phi_2, \phi_3, \phi_4, \phi_5\} \not\subset \{\phi_1^r,\ r \in \mathbb{Q}\}$ the map $F_{[k]}$ has no meromorphic first integral.

References

[1] D.K. Arrowsmith and C.M. Place, An Introduction to Dynamical Systems, Cambridge University Press, Cambridge 1990.

[2] E. Bedford and KH Kim, *Periodicities in linear fractional recurrences: degree growth of birational surface maps*, Michigan Math. J. **54** (2006), 647–670.

[3] E. Bedford and KH Kim, *On the degree growth of birational mappings in higher dimension*, J. Geom. Anal. **14** (2004), 567–596.

[4] M.P. Bellon, *Algebraic entropy of birational maps with invariant curves*, Lett. Math. Phys. **50** (1999), 79–90.

[5] A. Cima, A. Gasull and V. Mañosa, *Studying discrete dynamical systems through differential equations*, J. Differential Equations **244** (2008), 630–648.

[6] A. Cima, A. Gasull and V. Mañosa, *On 2- and 3- periodic Lyness difference equations*, J. Differ. Equations Appl. (2011) DOI:10.1080/10236198.2010.524212

[7] A. Cima, A. Gasull and V. Mañosa, *Integrability and non-integrability of periodic non-autonomous Lyness recurrences*, Preprint arXiv:1012.4925 [math.DS].

[8] A. Cima and S. Zafar, *Dynamical degree of periodic non-autonomous Lyness recurrences*, Preprint.

[9] V. de Angelis, *Notes on the non–autonomous Lyness equation*, J. Math. Anal. Appl. **307** (2005), 292–304.

[10] J. Diller, *Dynamics of birational maps of* $P\mathbb{C}^2$, Indiana Univ. Math. J. **45**, (1996) 721–772.

[11] J. Diller and C. Favre, *Dynamics of bimeromorphic maps of surfaces*, Amer. J. Math. **123** (2001), 1135–1169.

[12] J. Feuer, E.J. Janowski and G. Ladas, *Invariants for some rational recursive sequences with periodic coefficients*, J. Differ. Equations Appl. **2** (1996), 167–174.

[13] J-E. Fornaess and N. Sibony, *Complex dynamics in higher dimension II*, in T. Bloom et al. eds, *Modern Methods in Complex Analysis*, Annals of Mathematics Studies, 137, Princeton University Press, 1995, pp. 135–182.

[14] E.J. Janowski, M.R.S. Kulenović and Z. Nurkanović, *Stability of the* k*th order Lyness' equation with period–*k *coefficient*, Int. J. Bifurcations & Chaos **17** (2007), 143–152.

[15] M.R.S. Kulenović and Z. Nurkanović, *Stability of Lyness' equation with period–three coefficient*, Radovi Matematički **12** (2004), 153–161.

[16] W. Li, J. Llibre and X. Zhang, *Planar analytic vector fields with generalized rational first integrals*, Bull. Sci. Math. **125** (2001), 341–361.