Proceedings of the International Workshop
Future Directions in Difference Equations.
June 13-17, 2011, Vigo, Spain.
PAGES 61–75

Existence of solutions of discrete equations via critical point theory

Alberto Cabada[1], Antonio Iannizzotto[2] and Stepan Tersian[3]

[1]Universidade de Santiago de Compostela, Spain
[2]Università degli Studi di Catania, Italy
[3]University of Ruse, Bulgaria

alberto.cabada@usc.es, iannizzotto@dmi.unict.it, sterzian@uni-ruse.bg

Abstract

In this paper we make an overview on some classical and recent results related to the critical point theory. Moreover, we use some of the introduced results to show how to prove the existence of solutions of second order difference equations.

1 Introduction

Critical point theory or Calculus of variations is a field of mathematics which deals with extremum values at points of functionals on finite or infinite dimensional spaces. Historically it goes back to *brachistochrone problem*, stated by J. I. Bernoulli in *"Acta Euditorum"* in June, 1696 and solved by G. Leibnitz, J. Bernoulli, G. de L'Hôpital and I. Newton in 1697. The concept of finding extremum points of functionals was developed by L. Euler in 1744.

Modern critical point theory started with the works of M. Morse and L. Ljusternik & L. Schnirelmann in 1934. The critical values of a functional $f : X \to \mathbb{R}$ were characterized as inf max values ($\inf_{A\in S} \max_{u\in A} I(u)$) over a class of sets S in the space X. Further, R. Palais (1963) [35] and R. Palais & S. Smale (1964) [36] introduced the so called (PS) condition which was a base for modern development of critical point theory. A. Ambrosetti & P. H. Rabinowitz in [2] formulated the celebrated *mountain-pass theorem* and applied it to solvability of nonlinear elliptic problems. In finite dimensional case it was obtained by R. Courant [19]. The concept was intensively developed by L. Nirenberg, H. Brezis, I. Ekeland, J. Mawhin, M. Willem, K. C. Chang and many other authors in different directions. During the last three decades, critical point theory has been extended in different directions and applied to various problems in differential and difference equations, mathematical physics, geometry and optimization. In the present paper we outline, in section 2, minimization and mountain-pass theorems and generalizations which are applicable to existence of solutions of boundary value problems for differential and difference equations. Such application is showed in section 3.

2 Minimization theorems and (PS) conditions

A natural extension of the derivative of a function of one variable is the Fréchet derivative of a mapping in a Banach space. Let X and Y be two Banach spaces with norms $||\cdot||_X$ and $||\cdot||_Y$

respectively. Let $U \subset X$ be an open subset and $f : U \to Y$ be a mapping. When $Y = \mathbb{R}$, f is said to be a functional.

Definition 2.1. Let x be a point of the open subset $U \subset X$. The mapping $f : U \to Y$ is Fréchet-differentiable at $x \in U$ if there exists a bounded linear operator $A \in L(X, Y)$ such that

$$\lim_{h \in X, h \to 0} \frac{\|f(x+h) - f(x) - Ah\|_Y}{\|h\|_X} = 0.$$

The operator A is said to be the *Fréchet derivative* of the mapping f at x and denote it as $Df(x)$ or $f'(x)$.

Many equations of Mathematical physics have the operator form $f'(x) = 0$ in an appropriate Banach space X. The equation $f'(x) = 0$ is said to be the *Euler–Lagrange equation* of the energy functional $f : X \to \mathbb{R}$. Its solutions are assumed in the weak sense, i.e.,

$$< f'(x), h >= 0, \quad \forall\, h \in X.$$

X^* is the dual space of X and $\langle \cdot, \cdot \rangle$ is the duality pairing between X^* and X. The weak solutions are considered as critical points of the functional $f : X \to \mathbb{R}$.

Another derivative of a functional f is the directional derivative: the so called Gâteaux derivative of f.

Definition 2.2. Let $f : U \to Y$ be a mapping and $x \in U$. We say that f is Gâteaux-differentiable at x if there exists $A \in L(X, Y)$ such that

$$\lim_{t \to 0} \frac{f(x+th) - f(x)}{t} = Ah, \quad \forall h \in X.$$

The mapping A is uniquely determined. It is called *Gâteaux derivative of f at x* and is denoted by $D_G f(x)$ or $f'_G(x)$.

We say that f is a C^1 functional, and write $f \in C^1(U, \mathbb{R})$, if the Fréchet-derivative $f'(x)$ exists at every point x of U and the mapping $x \mapsto f'(x)$ is continuous from U to X^*, i.e., if $\lim_{j \to \infty} x_j = x \in U$ then

$$\lim_{j \to \infty} \left\langle f'(x_j) - f'(x), v \right\rangle = 0, \quad \text{uniformly on } \{v \in X : ||v|| \leq 1\}.$$

If f is Fréchet-differentiable it is clear that it is Gâteaux-differentiable. The converse is not true. However, if $f : U \to \mathbb{R}$ has a continuous Gâteaux derivative on U, then f is Fréchet-differentiable and $f \in C^1(U, \mathbb{R})$, which follows from the mean value theorem.

Let X be a Banach space and $f : X \to \mathbb{R}$ a functional bounded from below. A sequence $(x_j)_j$ is said to be a *minimizing sequence* if

$$\lim_{j \to \infty} f(x_j) = \inf_{x \in X} f(x).$$

The functional $f : X \to \mathbb{R}$ is said to be *lower semi-continuous* (respectively *weakly lower semi-continuous*) if whenever $\lim_{j \to \infty} x_j = x$ strongly ($\lim_{j \to \infty} x_j = x$ weakly), it follows

$$\liminf_{j \to \infty} f(x_j) \geq f(x).$$

The functional $f : X \to \mathbb{R}$ is *sequentially weakly continuous* if whenever $\lim_{j\to\infty} x_j = x$ weakly, it follows

$$\lim_{j\to\infty} f(x_j) = f(x).$$

We present a criterion for weak lower semi continuity and minimization theorem (see M. Berger [5, Chapter 6] and J. Mawhin & M. Willem [33, Chapter 1]).

Theorem 2.1. *Let X be a reflexive Banach space and $f : X \to \mathbb{R}$ be a functional represented as the sum $f = f_1 + f_2$, where f_1 is continuous and convex and f_2 is sequentially weakly continuous. Then f is weakly lower semi-continuous.*

Theorem 2.2. *Let f be a weakly lower semi-continuous functional on the reflexive Banach space X with a bounded minimizing sequence. Then f has a minimum on X.*

Now we introduce the concept of *coercive functional*

Definition 2.3. A functional f is said to be *coercive* if $f(x) \to +\infty$ as $||x|| \to \infty$.

By the last theorem we have

Corollary 2.3. *Let f be a weakly lower semi-continuous functional bounded from below on the reflexive Banach space X. If f is coercive, then $c = \inf f$ is attained at a point $x_0 \in X$.*

Note that if X is separable and f is sequentially weakly lower semi-continuous then the same conclusion holds.

Let M be a complete metric space and $\Phi : M \to \mathbb{R}$ be a lower semi-continuous functional, bounded from below. If $(u_j)_j$ is a minimizing sequence, then for every $\varepsilon > 0$ there exists j_0 such that for $j > j_0$

$$\Phi(u_j) \le \inf_M \Phi + \varepsilon.$$

We say that u is an $\varepsilon-$*minimum point* of Φ if

$$\Phi(u) \le \inf_M \Phi + \varepsilon.$$

Ekeland theorem [21] considers the existence of $\varepsilon-$minimum points.

Theorem 2.4 (Ekeland Principle, strong form, 1979). *Let M be a complete metric space and $\Phi : M \to \mathbb{R}$ be a lower semi continuous functional which is bounded from below. Let $k > 1$, $\varepsilon > 0$ and $u \in M$ be an $\varepsilon-$minimum point of Φ. Then there exists $v \in M$ such that*

$$\begin{aligned}
\Phi(v) &\le \Phi(u), \\
d(u,v) &\le \frac{1}{k}, \\
\Phi(v) &< \Phi(w) + \varepsilon\, k\, d(w,v), \quad \forall w \ne v.
\end{aligned}$$

Minimizing sequences for differentiable functionals are convergent under certain compactness conditions. We shall use later the so called Palais–Smale condition (PS-condition for short), appeared in the book of J. T. Schwartz [43] as

Definition 2.4. A C^1-functional $f : X \to \mathbb{R}$ satisfies the Palais–Smale (PS) condition if any sequence $(x_j)_j$ in X such that $f(x_j)$ is bounded and $\lim_{j\to\infty} ||f'(x_j)||_{X^*} = 0$ has a convergent subsequence.

From (PS) condition, it follows that the set of critical points for a bounded functional is compact. A variant of (PS) condition, noted as $(PS)_c$, was introduced by H. Brézis, J. M. Coron and L. Nirenberg [9].

Definition 2.5 (Brézis, Coron, Nirenberg, 1980). Let $c \in \mathbb{R}$ be given. A C^1 functional $f : X \to \mathbb{R}$ satisfies the $(PS)_c$ condition if any sequence $(x_j)_j$ *in* X such that $\lim_{j\to\infty} f(x_j) = c$ and $\lim_{j\to\infty} ||f'(x_j)||_{X^*} = 0$ contains a convergent subsequence.

It is clear that the (PS) condition implies the $(PS)_c$ condition for every $c \in \mathbb{R}$. Moreover, the $(PS)_c$ condition implies the compactness of the set of critical points at a fixed level c.

A consequence of Theorem 2.4 for differentiable functionals is the following

Theorem 2.5. *Let $f : X \to \mathbb{R}$ be a C^1 functional bounded from below. Then, for each $\varepsilon > 0$ and $x \in X$ such that*

$$f(x) \le \inf_X f + \varepsilon,$$

there exists $y \in X$ such that

$$\begin{aligned} f(y) &\le f(x), \\ ||x-y|| &\le \sqrt{\varepsilon}, \\ ||f'(y)||_{X^*} &\le \sqrt{\varepsilon}. \end{aligned}$$

By last theorem it follows

Corollary 2.6. *Let $f : X \to \mathbb{R}$ be a C^1 functional bounded from below and $(x_j)_j$ be a minimizing sequence. Then there exists another minimizing sequence $(y_j)_j$ such that*

$$\begin{aligned} f(y_j) &\le f(x_j), \\ \lim_{j\to\infty} ||x_j - y_j|| &= 0, \\ \lim_{j\to\infty} ||f'(y_j)||_{X^*} &= 0. \end{aligned}$$

Now, combining this result with (PS) condition, we get

Theorem 2.7. *Let $f : X \to \mathbb{R}$ be a C^1-functional bounded from below and $c = \inf f$. Assume that f satisfies $(PS)_c$ condition. Then c is achieved at a point $x_0 \in X$ and $f'(x_0) = 0$.*

The last theorem has a generalization based on another (PS) condition introduced by G. Cerami [17].

Definition 2.6 (Cerami, 1978). Let $c \in \mathbb{R}$. We say that the C^1-functional $f : X \to \mathbb{R}$ satisfies the $(PSC)_c$ condition if every sequence $(x_j)_j$ in X such that

$$\lim_{j\to\infty} f(x_j) = c \text{ and } \lim_{j\to\infty} (1 + ||x_j||)\, ||f^{'}(x_j)||_{X^*} = 0,$$

has a convergent subsequence.

Theorem 2.8. *Let $f : X \to \mathbb{R}$ be a C^1-functional bounded from below and $c = \inf f$. If $(PSC)_c$ condition is satisfied, then c is a minimum of f.*

3 Mountain pass and three critical points theorems

The original approach to prove the *Mountain Pass Theorem (MPT)* of A. Ambrosetti & P. H. Rabinowitz [2] is based on the *Deformation Theorem*. Let $f \in C^1(X, \mathbb{R})$ be a functional defined on the open subset X in the Banach space E. We introduce the following notations

$$\begin{aligned} f^c &= \{x \in E : f(x) \le c\}, \quad f_c = \{x \in E : f(x) \ge c\}, \\ f_a^b &= \{x \in E : a \le f(x) \le b\} = f_a \cap f^b, \\ K_c &= \big\{x \in E : f(x) = c, f'(x) = 0\big\}. \end{aligned}$$

A continuous mapping $\eta : (t, x) \in [0,1] \times X \to \eta(t, x) \in X$ is said to be a *homotopy of homeomorphisms* if

(1) $\eta(t, x) : X \to X$ is a homeomorphism for all $t \in [0,1]$,
(2) $\eta(0, x) = x$, for all $x \in X$.

The following theorem is proved by P. H. Rabinowitz (see [38, Appendix A, Theorem A.4]).

Theorem 3.1 (Deformation Theorem). *Let $f \in C^1(E, \mathbb{R})$ satisfy the (PS) condition. If $c \in \mathbb{R}$, $\varepsilon_0 > 0$ and N is any neighborhood of K_c, then there exists an $\varepsilon \in (0, \varepsilon_0)$ and $\eta \in C([0,1] \times E, E)$ such that*

(1) $\eta(0, x) = x$, for all $x \in E$;

(2) $\eta(t, x) = x$, for all $t \in [0,1]$ if $|f(x) - c| \ge \varepsilon_0$;

(3) $\eta(1, f^{c+\varepsilon} \backslash N) \subset f^{c-\varepsilon}$;

(4) if $K_c = \emptyset$ then $\eta(1, f^{c+\varepsilon}) \subset f^{c-\varepsilon}$;

(5) if f is even, then $\eta(t, x)$ is odd in x for all $t \in [0,1]$.

Theorem 3.1 was generalized by M. Willem [46, 47], P. Bartolo, V. Benci & D. Fortunato [4], H. Brezis & L. Nirenberg [10] in various directions. We recommend to the reader the books on the topic for further studies, where detailed proofs of theorems and their applications are presented (see, among others, [26, 29, 33, 38, 47, 49]).

Theorem 3.2 (Mountain Pass Theorem, Ambrosetti & Rabinowitz, 1973). *Let X be a real Banach space and $f \in C^1(X, \mathbb{R})$. Suppose that f satisfies the (PS) condition, $f(0) = 0$ and*

(i) there exist constants $\rho > 0$ and $\alpha > 0$ such that $f(x) \geq \alpha$ if $\|x\| = \rho$;

(ii) there is $e \in X$, $\|e\| > \rho$, such that $f(e) \leq 0$.

Then f has a critical value $c \geq \alpha$ which can be characterized as

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t)), \tag{1}$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}. \tag{2}$$

Geometrically, when $X = \mathbb{R}^2$ the assumptions (i) and (ii) mean that the origin lies in a valley surrounded by a "mountain"

$$\Gamma_f = \left\{(x, f(x)) \in \mathbb{R}^3 : x \in \mathbb{R}^2\right\}.$$

So, there must exist a mountain pass joining $(0,0)$ and $(e, f(e))$ that contains a critical value. Note that the (PS) condition is essential in Theorem 3.2. The function $h(x,y) = x^2 + (x+1)^3 y^2$ satisfies assumptions (i) and (ii) of Theorem 3.2, but does not satisfy the (PS) condition and its unique critical point is $(0,0)$.

Following weaker form of mountain-pass theorem is due to D. G. Figueredo & S. Solimini (see [23, 24]).

Theorem 3.3. *Let $f \in C^1(X, \mathbb{R})$ satisfying the (PS) condition. Suppose that*

$$\inf\{f(x) : ||x|| = r\} \geq \max\{f(0), f(e)\},$$

where $0 < r < ||e||$. Then f has a critical point $x_0 \neq 0$.

Corollary 3.4. *Let $f \in C^1(X, \mathbb{R})$ satisfying the (PS) condition. Suppose that f has two local minima. Then f has at least one more critical point.*

The next result is due to P. H. Rabinowitz [38].

Theorem 3.5 (Linking theorem, Rabinowitz, 1986). *Let $X = X_1 \bigoplus X_2$ be a real Banach space, where $X_1 \neq \{0\}$ is finite dimensional. Suppose that $f \in C^1(X, \mathbb{R})$, satisfies the (PS) condition and*

(I_1) There exist constants $\sigma, \ \rho > 0$ such that $\sup\{f(x) : ||x|| = \rho, \ x \in X_1\} \leq \sigma$,

(I_2) There exist $e \in X_1$, with $\|e\| = \rho$ and a constant $\omega > \sigma$ such that $\inf\{f(e + x) : \ x \in X_1\} \geq \omega$.

Then f possesses a critical value $c \geq 0$ and

$$c = \inf_{\gamma \in \Gamma} \max\{f(\gamma(u)); \ u \in X_1, \|u\| = \rho\},$$

where

$$\Gamma = \{\gamma \in C(B[0,\rho] \cap X_1, X) \text{ such that } \gamma(u) = u, \text{ for all } \ u \in \partial B[0,\rho] \cap X_1\}.$$

The last statement infers the existence of three critical points of a functional. Other three critical points theorems with (PS) condition were proved by P. Pucci & J. Serrin [37], J. Mawhin & M. Willem [33], V. Moroz, A. Vignoli & P. Zabreiko [34], B. Ricceri [39]. We point out the following Ricceri's theorem which has several generalizations (see [6, 11]) and applications to differential and difference equations (see [12, 20, 40]).

Theorem 3.6 (Ricceri [39]). *Let X be a separable and reflexive real Banach space, $I \subseteq \mathbb{R}$ an interval, and $g : X \times I \to \mathbb{R}$ a continuous function satisfying the following conditions:*

(i) for each $x \in X$ the function $g(x, \cdot)$ is concave;

(ii) for each $\lambda \in I$, the function $g(\cdot, \lambda)$ is sequentially weakly lower semi continuous, coercive and continuously Gâteaux differentiable, satisfies the (PS) condition;

(iii) there exists a continuous concave function $h : I \to \mathbb{R}$ such that

$$\sup_{\lambda \in I} \inf_{x \in X} (g(x, \lambda) + h(\lambda)) < \inf_{x \in X} \sup_{\lambda \in I} (g(x, \lambda) + h(\lambda)).$$

Then, there exists an open interval $\Lambda \subseteq I$ and a positive real number σ such that for each $\lambda \in \Lambda$ the equation $g'_x(x, \lambda) = 0$ admits at least three solutions in X whose norms are less than σ.

A recent consequence of Ricceri's theorem due to A. Cabada & A. Iannizzotto [11] as follows

Theorem 3.7. *Let $(X, ||\cdot||)$ be a uniformly convex Banach space with strictly convex dual space, $J \in C^1(X, \mathbb{R})$ be a functional with compact derivative, $x_0, x_1 \in X, p, r \in \mathbb{R}$ be such that $p > 1$ and $r > 0$. Assume that the following conditions are satisfied:*

(i) $\liminf\limits_{||x|| \to \infty} \frac{J(x)}{||x||^p} \geq 0$;

(ii) $\inf\limits_{x \in X} J(x) < \inf\limits_{||x - x_0|| \leq r} J(x)$;

(iii) $||x_1 - x_0|| < r$ *and* $J(x_1) \leq \inf\limits_{||x - x_0|| = r} J(x)$.

Then, there exists a nonempty open set $A \subset (0; +\infty)$ such that for all $\lambda \in A$ the functional

$$x \mapsto \frac{||x - x_0||^p}{p} + \lambda J(x)$$

has at least three critical points in X.

To finish this section, let us note also variants of Mountain Pass Theorem due to H. Brezis & L. Nirenberg [10] without (PS) condition.

Let Q be a compact metric space and let Q_* be a nonempty closed subset $\neq Q$ and p_* be a fixed continuous map on Q. Define

$$\mathcal{A} := \{p \in C(Q, X) : p = p_* \quad \text{on } Q_*\}$$

and

$$c := \inf_{p \in \mathcal{A}} \max_{t \in Q} f(p(t)),$$

so that

$$c \geq \max_{t \in Q_*} f(p_*(t)).$$

Theorem 3.8 (Brezis & Nirenberg, Standard MPT). *Let $f \in C^1(X, \mathbb{R})$. Assume that for every $p \in \mathcal{A}$, $\max_{t \in Q} f(p(t))$ is attained at some point in $Q \backslash Q_*$. Then there exists a sequence $(x_j)_j$ in X such that*

$$\lim_{j \to \infty} f(x_j) = c \quad \textit{and} \quad \lim_{j \to \infty} ||f'(x_j)|| = 0.$$

In addition, if f satisfies the $(PS)_c$ condition, then c is a critical value. Moreover, if $(p_j)_j$ is any sequence in $\mathcal{A}$ such that

$$c = \lim_{j \to \infty} \max_{t \in Q} f(p_j(t)),$$

then there exists a sequence $(t_j)_j$ in Q such that

$$\lim_{j \to \infty} f(p_j(t_j)) = c \quad \textit{and} \quad \lim_{j \to \infty} ||f'(p_j(t_j))|| = 0.$$

Theorem 3.2 is clearly a special case of Theorem 3.8 with $Q = [0, 1], Q_* = \{0, 1\}$ and $p_*(t) = te$. The next Theorem of N. Ghoussoub [25] (see [10]) contains an earlier result of P. Pucci & J. Serrin [37] as a special case.

Theorem 3.9. *Assume the conditions of Theorem 3.8 and that there is a closed set $\Sigma \subset X$, disjoint from $p_*(Q_*)$, on which $f \geq c$ and such that for all $p \in \mathcal{A}$, $p(Q) \cap \Sigma \neq \emptyset$. Then there exists a sequence $(x_j)_j$ in X such that*

$$\lim_{j \to \infty} f(x_j) = c, \quad \lim_{j \to \infty} ||f'(x_j)|| = 0, \quad \textit{and} \quad \lim_{j \to \infty} dist(x_j, \Sigma) = 0.$$

4 Existence results for difference equations

As we have noticed in the previous sections, the variational methods coupled with critical point theory have been extensively used in the literature to deduce existence results for Ordinary and Partial Differential Equations, see for instance the monographs of H. Brezis [8], M. R. Grossinho & S. Tersian [26] and S. Kesavan [32]. However, if we refer to difference equations, this application is relatively new. Some of the first steps on the literature related to this field are referred on the paper by Z. Guo & J. Yu [27], in which the authors survey some results concerning the existence and multiplicity of solutions for several kinds of difference equations. In particular they consider the second order non autonomous difference equation

$$\Delta^2 x_{n-1} + f(n, x_n) = 0, \quad n \in \mathbb{Z}, \tag{3}$$

in which $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ such that $f(n + M, s) = f(n, s)$ for every $n \in \mathbb{N}$, $s \in \mathbb{R}$ and some $M \in \mathbb{N}$.

By assuming, among other conditions, that $f(t, \cdot)$ is superquadratic at infinity, the authors prove the existence of at least one nontrivial periodic solution of period $p\,M$ (for a given $p \in \mathbb{N}$). They prove that the $p\,M$ – periodic solutions of this problem coincide with the critical points of the functional

$$J(x) = \sum_{n=1}^{p\,M} \left(\frac{1}{2} (\Delta\, x_n)^2 + F(n, x_n) \right), \quad x \in \mathbb{R}^{p\,m},$$

where

$$F(n,s) \equiv \int_0^s f(n,\tau)\,d\tau, \quad n \in \mathbb{Z},\ s \in \mathbb{R}.$$

The proofs are deduced from the fact that operator J satisfies the (PS) condition and by applying the (linking) Theorem 3.5.

Using a more sophisticated argument, the same authors prove the existence o three periodic solutions in [28]. The vectorial case has been considered by Z. Zhou, J. Yu & Z. Guo in [50].

Equation (3), coupled with Dirichlet boundary conditions, has been studied by R. P. Agarwal, K. Perera & D. O'Regan in [1]. The authors deduce the existence of at least two nonnegative solutions by assuming some suitable growth conditions on the asymptotic behavior of function f at 0 and at $+\infty$. The results follow from the (mountain pass) Theorem 3.2. A generalization of this results for higher dimensional systems has been done by L. Jiang & Z. Zhou in [30]. Also, F. Faraci & A. Iannizzotto obtained some multiplicity results for discrete Dirichlet problems in [22].

The self–adjoint second order equation

$$\Delta(p_n\,\Delta x_{n-1}) + q_n x_n = f(n,x_n), \quad n \in \mathbb{Z},$$

has been studied by J. Yu, Z. Guo & X. Zou in [45], with periodic boundary conditions, and by Y. Li in [31] for the Dirichlet ones (with $q \equiv 0$ and $f(n,x) \equiv g(n) - h(n,x)$).

Problems with parametric dependence have been considered by G. Zhang & S. Liu in [48]. In particular, they study the autonomous Dirichlet problem

$$\Delta^2 x_{n-1} + \lambda f(x_n) = 0, \quad n \in \{1,\dots,T\}, \quad x(0) = x(T+1) = 0. \tag{4}$$

By means of the (mountain pass) Theorem 3.2, the authors prove that if there is $a > 0$ such that $f(0) = -a$ and $F(\delta) \equiv \int_0^\delta f(s)\,ds > 0$ for some $\delta > 0$, then problem (4) has at least one positive solution on $\{1,\dots,T\}$ for all $\lambda > 0$, provided that function f is sublinear at $+\infty$:

$$\lim_{u\to+\infty} \frac{f(u)}{u} = 0,$$

and there are $\theta \in (0,1/2)$ and $R > 0$ such that

$$F(s) \le \theta s\, f(s), \quad \text{for all} \quad s \ge R.$$

Moreover they prove that if f is sublinear at 0:

$$\lim_{u\to 0} \frac{f(u)}{u} = 0,$$

then there are $0 < \Lambda_0 \le \Lambda_1$, for which problem (4) has no positive solution for all $\lambda \in (0,\Lambda_0)$ and two positive solutions for all $\lambda > \Lambda_1$.

The analogous non autonomous Dirichlet problem

$$\Delta^2 x_{n-1} + \lambda f(n,x_n) = 0, \quad n \in \{1,\dots,T\}, \quad x(0) = x(T+1) = 0, \tag{5}$$

has been studied by D. Bai & Y. Xu in [3].

On this work, the authors prove that if function f is continuous and odd in the second variable and there exists $\alpha > 0$ such that $f(n, \alpha) = 0$ and $f(n, u) > 0$ for $u \in (0, \alpha)$, then there exists $\lambda^* > 0$ such that problem (5) has at least T distinct pairs of nontrivial solutions for all $\lambda > \lambda^*$. Furthermore, each nontrivial solution u satisfies that $|u_n| \le \alpha$ for all $n \in \{1, \dots, T\}$.

The proof follows from the following multiplicity result, known as Clark's theorem, given by P. H. Rabinowitz in [38].

Theorem 4.1. *Let X be a real Banach space, $f \in C^1(X, \mathbb{R})$ even, bounded from below, and satisfying the (PS) condition. Suppose $f(0) = 0$, there is a set $K \subset \Sigma$ such that K is homeomorphic to S^{j-1} (the $j - 1$ dimension unit sphere) by an odd map, and* $\sup\{f(x) : x \in K\} < 0$. *Then the functional f has at least j distinct pairs of nonzero critical points.*

The p – laplacian equation (with $\varphi_p(s) = |s|^{p-2}s$, $p > 1$) has been considered by P. Chen & H. Fang in [18]. On that paper the authors studied the general second order equation

$$\Delta(\varphi_p(\Delta x_{n-1})) + f(n, x_{n+1}, x_n, x_{n-1}) = 0, \quad n \in \mathbb{Z}.$$

By assuming the continuity of the function f and its periodicity on the first variable, using the (linking) Theorem 3.5, they obtain a sufficient condition for the existence of periodic and subharmonic solutions of the previous equation.

The parameter dependence on the p – laplacian equation has been considered by G. Bonanno & P. Candito in [7]. They study the following problem, which generalizes (5):

$$\Delta(\varphi_p(\Delta x_{n-1}) + \lambda f(n, x_n) = 0, \quad n \in \{1, \dots, T\}, \quad x(0) = x(T+1) = 0. \tag{6}$$

The authors obtain some real intervals in which for λ in such intervals, problem (6) has at least three distinct positive solutions. To deduce these multiplicity results they impose some growth assumptions on the behavior of the function f at 0 and/or at $+\infty$. The critical point result employed in this case is the following, proved in [7, Theorem 2.1]:

Theorem 4.2. *Let X be a finite dimensional real Banach space and $I_\lambda : X \to \mathbb{R}$ be a functional such that $I_\lambda(u) := \Phi(u) - \lambda\Psi(u)$ for all $u \in X$. Here $\Phi,\ \Psi : X \to \mathbb{R}$ are two functions of class C^1 on X, with Φ coercive, and λ is a positive real parameter. Assume, moreover that*

1. *Φ is convex and* $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$.
2. *For each $\lambda > 0$ and for every $x_1,\ x_2$ which are local minima for the functional $\Phi - \lambda\Psi$ and such that $\Psi(x_1) \ge 0$ and $\Psi(x_2) \ge 0$ one has*

$$\inf_{t\in[0,1]} \Psi(tx_1 + (1-t)x_2) \ge 0.$$

Further, assume that there are two positive constants ρ_1, ρ_2 and $\bar{v} \in X$, with $\rho_1 < \Phi(\bar{v}) < \rho_2/2$, such that

$$\frac{\sup_{u\in\Phi^{-1}(-\infty,\rho_1)} \Psi(u)}{\rho_1} < \frac{\Psi(\bar{v})}{2\,\Phi(\bar{v})} \quad ; \quad \frac{\sup_{u\in\Phi^{-1}(-\infty,\rho_2)} \Psi(u)}{\rho_1} < \frac{\Psi(\bar{v})}{4\,\Phi(\bar{v})}.$$

Then, for each $\lambda \in \left(\frac{2\Phi(\bar{v})}{\Psi(\bar{v})}, \min\left\{ \frac{\rho_1}{\sup_{u\in\Phi^{-1}(-\infty,\rho_1)} \Psi(u)}, \frac{\rho_2/2}{\sup_{u\in\Phi^{-1}(-\infty,\rho_2)} \Psi(u)} \right\} \right)$ *the functional* I_λ *admits at least three distinct critical points* $u_1,\ u_2,\ u_3$ *such that* $u_1 \in \Phi^{-1}(-\infty,\rho_1)$*,* $u_2 \in \Phi^{-1}(\rho_1,\rho_2/2)$ *and* $u_3 \in \Phi^{-1}(-\infty,\rho_2)$*.*

Different results in this direction have been given by P. Candito & G. D'Agui [15] and P. Candito & N. Giovannelli [16].

In [12], we prove the following multiplicity result for the solutions of the problem (6).

Theorem 4.3. *Let* $r, s \in \mathbb{R}$ *satisfying* $0 < r < s$*. Assume that the following conditions hold:*

1. $\limsup_{|t|\to+\infty} \dfrac{F(k,t)}{|t|^p} \le 0$ *for every* $k \in [1,T]$*;*
2. $\displaystyle\sum_{k=1}^{T} \sup_{|t|\le \frac{s}{c_1}} F(k,t) < \sum_{k=1}^{T} \sup_{t\in\mathbb{R}} F(k,t)$*;*
3. $\displaystyle\sup_{\frac{r}{c_2}\le|t|\le\frac{s}{c_1}} F(k,t) \le -\sum_{h\neq k} \sup_{|t|\le\frac{s}{c_1}} F(h,t)$ *for every* $k \in [1,T]$*.*

Then, there exists $\lambda^* > 0$ *such that* (6) *admits at least three solutions.*

Here c_1 and c_2 denote the embedding constants of the solutions space into $\mathbb{R}^T$, endowed with the maximum norm (see [12] for the exact computation of such constants).

The proof follows by combining the (mountain pass) Theorem 3.2 together with the following theorem of Ricceri (see [42, Theorem A] or [41, Theorem 1]).

Theorem 4.4. *Let* (X,τ) *be a Hausdorff space and* $\Phi, J : X \to \mathbb{R}$ *be functionals; moreover, let* M *be the (possibly empty) set of all the global minimizers of* J *and define*

$$\alpha = \inf_{x\in X} \Phi(x),$$

$$\beta = \begin{cases} \inf\limits_{x\in M} \Phi(x) & \text{if } M \neq \emptyset \\ \sup\limits_{x\in X} \Phi(x) & \text{if } M = \emptyset \end{cases}.$$

Assume that the following conditions are satisfied:

1. *for every* $\mu > 0$ *and every* $\rho \in \mathbb{R}$ *the set* $\{x \in X \,:\, \Phi(x) + \mu J(x) \le \rho\}$ *is sequentially compact (if not empty);*
2. $\alpha < \beta$*.*

Then, at least one of the following conditions holds:

(a) *there exists a continuous mapping* $h : (\alpha,\beta) \to X$ *with the following property: for every* $t \in (\alpha,\beta)$*, one has*

$$\Phi(h(t)) = t$$

and for every $x \in \Phi^{-1}(t)$*,* $x \neq h(t)$

$$J(x) > J(h(t));$$

(b) *there exists $\mu^* > 0$ such that the functional $\Phi + \mu^* J$ admits at least two global minimizers in X.*

In [12] it is pointed out that while the first condition on Theorem 4.3 is a standard coercivity assumption, the next two ones are rather unusual; however such assumptions cannot be, in general, removed or weakened. Theorem 4.3 can be improved using Theorem 3.7:

Theorem 4.5. *Assume that the following conditions are satisfied:*

(i) $\limsup_{|t|\to\infty} \dfrac{F(k,t)}{|t|^p} \leq 0$ *for all* $k \in [1,T]$;

(ii) $\sum_{k=1}^{T} \sup_{|t|\leq s} F(k,t) < \sum_{k=1}^{T} \sup_{t\in\mathbb{R}} F(k,t)$ *for some* $s > 0$;

(iii) *there exists* $\xi \in \mathbb{R}^T$ *such that* $|\xi| < s$ *and* $\sum_{k=1}^{T} F(k,\xi_k) = \sup_{|\eta|\leq s} \sum_{k=1}^{T} F(k,\eta_k)$.

Then, there exists a nonempty open set $A \subseteq (0,+\infty)$ such that for all $\lambda \in A$ problem (6) admits at least three solutions.

In [14], A. Cabada & S. Tersian prove the existence of heteroclinic solutions of the equation

$$\Delta(\varphi_p(\Delta x_{n-1}) + \lambda f(n,x_n) = 0, \quad n \in \{1,\dots,T\}, \quad x(-\infty) = -1; x(+\infty) = 1.$$

The proof follows by constructing a sequence of solutions of related Dirichlet problems, and from the application of some existence results of P. Candito and N. Giovannelli in [16].

In [13], by means of the (mountain pass) Theorem 3.2, A. Cabada, C. Li & S. Tersian prove the existence of homoclinic solutions for the $p-$Laplacian difference equation

$$\Delta(\varphi_p(\Delta x_{n-1}) - V_n\varphi_q(x_n) + \lambda f(n,x_n) = 0, \quad x_n \to 0, |n| \to \infty. \tag{7}$$

Here functions V and $f(\cdot,x)$ are T-periodic and satisfy the following assumptions:
The potential function $F(n,t)$ of $f(n,t)$ satisfies the Rabinowitz's type condition:

(F_1) There exist $\mu > p \geq q > 1$ and $s > 0$ such that

$$\mu F(k,t) \leq t f(k,t), \ k \in \mathbb{Z}, \ t \neq 0$$
$$\text{and } F(k,t) > 0, \ \forall k \in \mathbb{Z}, \text{ for } t \geq s > 0.$$

(F_2) $f(k,t) = o\left(|t|^{q-1}\right)$ as $|t| \to 0$.

Let $\Phi_q(t) = \frac{|t|^q}{q}$. The main obtained result is the following

Theorem 4.6. *Suppose that the function $V : \mathbb{Z} \to \mathbb{R}$ is positive and T-periodic and the functions $f(k,\cdot) : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}$ satisfy assumptions (F_1)–(F_2). Then, for each $\lambda > 0$, the equation (7) has a nonzero homoclinic solution $x \in \ell^q$, which is a critical point of the functional $J : \ell^q \to \mathbb{R}$:*

$$J(x) = \sum_{n\in\mathbb{Z}} \Phi_p(\Delta x(n-1)) + \sum_{n\in\mathbb{Z}} V(n)\Phi_q(x(n)) - \lambda \sum_{n\in\mathbb{Z}} F(n,x(n)).$$

Moreover, given a nontrivial solution x of problem (7), there exist two integer numbers $k_\pm$ such that for all $k > k_+$ and $k < k_-$, the sequence $x(n)$ is strictly monotone.

References

[1] R. P. Agarwal, K. Perera and D. O'Regan, *Multiple positive solutions of singular and non-singular discrete problems via variational methods*, Nonlinear Anal. **58** (2004), 69–73.

[2] A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.

[3] D. Bai and Y. Xu, *Nontrivial solutions of boundary value problems of second-order difference equations*, J. Math. Anal. Appl. **326** (2007), 297–302.

[4] P. Bartolo V. Benci and D. Fortunato, *Abstract critical point theory and applications to some nonlinear problems with "strong" resonance at infinity*, Nonlinear Anal. **7** (1983), 981–1012.

[5] M. Berger, *Nonlinearity and Functional Analysis*, Academic Press, 1977.

[6] G. Bonanno, *Some remarks on a three critical points theorem*, Nonlinear Anal. **54** (2003), 651–665.

[7] G. Bonanno and P. Candito, *Nonlinear difference equations investigated via critical point methods*, Nonlinear Anal. **70** (2009), 3180–3186.

[8] H. Brezis, *Analyse fonctionnelle (French)*, Collection Mathématiques Appliquées pour la Maitrise, Masson, Paris, 1983.

[9] H. Brezis, J. M. Coron and L. Nirenberg, *Free vibrations for a nonlinear wave equation and a theorem of P. Rabinowitz*, Comm. Pure and Appl. Math. **33** (1980), 667–689.

[10] H. Brezis and L. Nirenberg, *Remarks on finding critical points*, Comm. Pure and Appl. Math. **44** (1991), 939–963.

[11] A. Cabada and A. Iannizzotto, *A note on a question of Ricceri*, Appl. Math. Letters, to appear.

[12] A. Cabada, A. Iannizzotto and S. Tersian, *Multiple solutions for discrete boundary value problems*, J. Math. Anal. Appl. **356** (2009), 418–428.

[13] A. Cabada, C. Li and S. Tersian, *On homoclinic solutions of a semilinear p-Laplacian difference equation with periodic coefficients*, Adv. Difference. Equ. 2010, Art. ID 195376, 17 pp.

[14] A. Cabada and S. Tersian, *Existence of heteroclinic solutions for discrete p-Laplacian problems with a parameter*, Nonlinear Anal. Real World Appl. **12** (2011), 2429–2434.

[15] P. Candito and G. D'Agui, *Three solutions for a discrete nonlinear Neumann problem involving the p-Laplacian*, Adv. Difference Equ. 2010, Art. ID 862016, 11 pp.

[16] P. Candito and N. Giovannelli, *Multiple solutions for a discrete boundary value problem involving the p-Laplacian*, Comput. Math. Appl. **56** (2008), 959–964.

[17] G. Cerami, *Un criterio di esistenza per punti critici su varieta illimate*, Rend. Acad. Sci. Let. Ist. Lombardo **112** (1978), 332–336.

[18] P. Chen and H. Fang, *Existence of periodic and subharmonic solutions for second-order p-Laplacian difference equations*, Adv. Difference Equ. 2007, Art. ID 42530, 9 pp.

[19] R. Courant, *Dirichlet's Principle, Conformal Mapping and Minimal Surfaces*, Interscience, 1950. Reprinted: Springer, 1977.

[20] Z. Du, X. Lin and C. Tisdell, *A multiplicity result for p-Lapacian boundary value problems via critical points theorem*, Appl. Math. Comput. **205** (2008), 231–237

[21] I. Ekeland, *Nonconvex minimization problems*, Bull. Amer. Math. Soc. **1** (1979), 443–474.

[22] F. Faraci, A. Iannizzotto, *Multiplicity theorems for discrete boundary value problems*, Aequationes Math. **74** (2007), 111–118.

[23] D. G. Figueredo, *Lectures on the Ekeland Variational Principle with Applications and Detours*, Preliminary Lecture Notes, SISSA, 1988.

[24] D. G. Figueredo and S. Solimini, *A variational approach to superlinear elliptic problems*, Comm. Partial Differential Equations **9** (1984), 699–717.

[25] N. Ghoussoub, *Location, multyplicity and Morse indices of mini-max critical points*, J. Reine Angew. Math. **417** (1991), 27–76.

[26] M. R. Grossinho and S. Tersian, *An introduction to Minimax Theorems and their Applications to Differential Equations*, Kluwer Acad. Publ., 2001.

[27] Z. Guo and J. Yu, *The existence of periodic and subharmonic solutions of subquadratic second order difference equations*, J. London Math. Soc. (2) **68** (2003), 419–430.

[28] Z. Guo and J. Yu, *Existence of periodic and subharmonic solutions for second-order superlinear difference equations*, Sci. China Ser. A **46** (2003), 506–515.

[29] Y. Jabri, *The Mountain Pass Theorem. Variants, Generalizations and some Applications*, Cambridge Univ. Press, 2003.

[30] L. Jiang and Z. Zhou, Zhan, *Positive solutions of higher dimensional discrete boundary value problem*, Ann. Differential Equations **22** (2006), 295–298.

[31] Y. Li, *The existence of solutions for second-order difference equations*, J. Difference Equ. Appl. **12** (2006), 209–212.

[32] S. Kesavan, *Topics in Functional Analysis and Applications*, John Wiley & Sons, Inc., 1989.

[33] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, Springer-Verlag, 1988.

[34] V. Moroz, A. Vignoli and P. Zabreiko, *On the three critical point theorem*, Topol. Methods Nonlinear Anal. **11** (1998), 103–113.

[35] R. Palais, *Morse theory on Hilbert manifolds*, Topology **2** (1963), 299–340.

[36] R. Palais and S. Smale, *A generalized Morse theory*, Bull. Amer. Math. Soc. **70** (1964), 165–171

[37] P. Pucci and J. Serrin, *Extensions of the mountain-pass lemma*, J. Functional Anal. **59** (1984), 185–210.

[38] P. H. Rabinowitz, *Minimax methods in Critical Point Theory and Applications to Differential Equations*, CBMS Reg. Conf. 65, AMS, 1986.

[39] B. Ricceri, *On a three critical points theorem*, Arch. Math. (Basel) **75** (2000), 220–226.

[40] B. Ricceri, *Existence of three solutions for a class of elliptic eigenvalue problems*, Math. Comput. Model. **32** (2000), 1485–1494.

[41] B.Ricceri, *Well–posedness of constrained minimization problems via saddle–points*, J. Global Optim. **40** (2008), 389–397.

[42] B. Ricceri, *A multiplicity theorem in* $\mathbb{R}^n$, J. Convex Anal. **16** (2009), 3–4.

[43] J. T. Schwartz, *Nonlinear Functional Analysis*, Notes by H. Fattorini, R. Nirenberg and H. Porta, with an additional chapter by Hermann Karcher. Notes on Mathematics and its Applications, Gordon and Breach Science Publishers, 1969.

[44] M. Struwe, *Variational Methods*, Springer Verlag, 1990.

[45] J. Yu, Z. Guo and X. Zou, *Periodic solutions of second order self-adjoint difference equations*, J. London Math. Soc. (2) **71** (2005), 146–160.

[46] M. Willem, *Lecture Notes on Critical Point Theory*, Fundaçao Universidade de Brasilia, **199**, 1983.

[47] M. Willem, *Minimax Theorems*, Birkhäuser, 1997.

[48] G. Zhang and S. Liu, *On a class of semipositone discrete boundary value problems*, J. Math. Anal. Appl. **325** (2007), 175–182.

[49] W. Zou and M. Schechter, *Critical Point Theory and its Applications*, Springer, 2006.

[50] Z. Zhou, J. Yu and Z. Guo, *Periodic solutions of higher-dimensional discrete systems*, Proc. Roy. Soc. Edinburgh Sect. A **134** (2004), 1013–1022.