IOP Publishing | London Mathematical Society Nonlinearity
Nonlinearity **36** (2023) 507–536 https://doi.org/10.1088/1361-6544/aca50d

The peak-end rule and its dynamic realization through differential equations with maxima*

Olena Trofymchuk[1], Eduardo Liz[2] and Sergei Trofimchuk[3,]**

[1] Department of Mathematical Physics and Differential Equations, Igor Sikorsky Kyiv Polytechnic Institute, Kyiv, Ukraine
[2] Departamento de Matemática Aplicada II, Universidade de Vigo, 36310 Vigo, Spain
[3] Instituto de Matemática, Universidad de Talca, Casilla 747, Talca, Chile

E-mail: trofimch@inst-mat.utalca.cl

Received 20 January 2022; revised 2 August 2022
Accepted for publication 22 November 2022
Published 9 December 2022

Recommended by Dr Jonathan Touboul

Abstract
In the 1990s, after a series of experiments, the behavioural psychologist and economist Daniel Kahneman and his colleagues formulated the following peak-end evaluation rule: *the remembered utility of pleasant or unpleasant episodes is accurately predicted by averaging the peak (most intense value) of instant utility (or disutility) recorded during an episode and the instant utility recorded near the end of the experience* (Kahneman *et al* 1997 *Q. J. Econ.* **112** 375–405). Based on this rule, we propose a mathematical model for the time evolution of the experienced utility function $u = u(t)$ given by the scalar differential equation $u'(t) = au(t) + b\max\{u(s) : s \in [t-h, t]\} + f(t)$ $(*)$, where f represents exogenous stimuli, h is the maximal duration of the experience, and $a, b \in \mathbb{R}$ are some averaging weights. In this work, we study equation $(*)$ and show that, for a range of parameters a, b, h and a periodic sine-like term f, the dynamics of $(*)$ can be completely described in terms of an associated one-dimensional dynamical system generated by a piece-wise continuous map

[*] To memory of Anatoly Samoilenko (1938–2020).
[**] Author to whom any correspondence should be addressed.

from a finite interval into itself. We illustrate our approach with two representative examples. In particular, we show that the utility $u(t)$ (e.g. 'happiness', interpreted as hedonic utility) can exhibit chaotic behaviour.

Keywords: peak-end rule, differential equations with maxima, return map, turbulent (chaotic) behaviour

Mathematics Subject Classification numbers: 34K13, 34K23, 37E05, 91E45

(Some figures may appear in colour only in the online journal)

1. Introduction

Utility theory plays a key role in decision making, which influenced important fields of study, including behavioural economics, consumer psychology, well-being studies and political science, among others [29]. Although in economics and modern decision theory utility is understood as *utility of outcomes* and refers to their weight in decisions, the original Bentham's concept of utility [4] is hedonic quality and aims at measuring the value of pleasure or pain. This latter interpretation is referred to as *experienced utility* (or hedonic utility) by Kahneman *et al* in [28], an influential paper in which the authors provide some key ideas to measure experienced utility, which in turn can be interpreted as an assessment of a person's objective happiness [26].

Fredrickson and Kahneman introduced in [14] a moment-based approach to make retrospective evaluations of affective episodes (an episode is here understood as a connected time interval); see also [29, chapter 38]. This approach assumes that evaluations of experienced utility require two types of utility concepts: instant utility and remembered utility. Instant utility refers to the sign and intensity of a hedonic experience at a given moment time, which can be characterized by a value on a good/bad dimension. Even if pleasure and pains are attributes of a moment experience, the outcomes that people value extend over time. Thus, evaluation of experienced utility must include evaluation of past episodes, which leads to the concept of remembered utility. The model of evaluation by moments (or *snapshot* model) [14] is based on the principle that people evaluate the utility of an episode by retrieving a representative moment and by evaluating the utility of that moment. The *peak-end* rule establishes that two moments can determine the global evaluation of an entire episode: the moment of most extreme affect experienced during the episode (peak) and the affect experienced at the end. For further reading, we refer to the nice reviews by Fredrickson [13] and Kahneman and Tversky [29, chapters 37 and 38]. In particular, they include some discussions on empirical evidence, robustness, and applications of the peak-end rule.

The aforementioned seminal paper [28] also proposed a complex research agenda for the theoretical and empirical studies of experienced utility. The first part of this agenda concerns relations between stream of outcomes and instant utility, one of the related questions being *'What are the dynamics of experienced utility?'* Note that a continuous utility profile can be constructed from the momentary good-bad ratings, assigning to each moment t the scalar value of its instant utility $u(t)$. Thus, the design of a suitable dynamic model for the time evolution of $u(t)$ is of clear applied interest. By [28, p 381], *'the system that forms and stores evaluations of situations is not designed to optimize experienced utility'*, a key observation indicating that every appropriate evolution model should have relatively complex dynamics. For instance, it cannot be given by a scalar autonomous ordinary differential equation. Having in mind that decisions about the future are made based on present and past affective experience, the above comments together suggest that the general evolution model leading to an adequate dynamic realization of the peak-end rule is given by the differential equation with maxima

$$u'(t) = F\left(t, u(t), \max_{s\in[t-h,t]} u(s)\right), \tag{1}$$

where $F : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous map and the delay h is chosen so as it captures a temporal episode of suitable length. One consequence of the peak-end rule is that the duration of affective episodes is largely neglected (*duration neglect principle*) [13]. This allows us for certain flexibility, and in particular we can work with a constant delay h. Equation (1) is generally nonautonomous, and the dependence on t usually comes in the form of external stimuli.

The simplest choice of (1) would be the linear autonomous map $F(t,x,y) = ax + by$, with constant real coefficients a, b, leading to equation

$$u'(t) = au(t) + b \max_{s\in[t-h,t]} u(s). \tag{2}$$

The consideration of constant external stimuli in (2) yields

$$u'(t) = au(t) + b \max_{s\in[t-h,t]} u(s) + c, \tag{3}$$

with $c \in \mathbb{R}$, which can be reduced to (2) by a simple change of variables. Some implications of equation (3) in the context of utility theory have been discussed in [37].

However, it is more realistic to assume that exogenous stimuli are not constant in time. As a first approximation for their mode of variation, having in mind periodic repetition of an established daily or weekly routine in a person's life, it is quite natural to consider a choice of equation (1) with the right-hand side depending periodically on the time variable t. Accordingly, the simplest mathematical model for time evolution of the experienced utility function $u = u(t)$ can be given by the scalar differential equation

$$u'(t) = au(t) + b \max_{s\in[t-h,t]} u(s) + f(t), \tag{4}$$

where f represents the action of periodically changing exogenous stimuli, $h > 0$ is the maximal duration of the experience, and a, b are some real coefficients. The determination of qualitatively plausible psychological parameters f, a, b seems to be a rather difficult task which we do not address here. For simplicity, we start considering the case when the T-periodic continuous function f has sine-like shape (see definition 1 below), assuming that the rate of change of the utility $u(t)$ is affected by linear decay (with coefficient $\alpha > 0$) and is proportional (with coefficient $\beta > 0$) to the difference between its instant value and its peak on the precedent fixed time interval:

$$u'(t) = -\alpha u(t) + \beta(u(t) - \max_{s\in[t-h,t]} u(s)) + f(t). \tag{5}$$

The first two terms in the right-hand side of (5) provide a feedback loop, which is in accordance with the concept of *hedonic adaptation*. Indeed, the first term just represents a linear decay of happiness (see, e.g. [19]); the second one comes from a comparison with a recent peak. Thus, our model agrees with the principles of Brickman *et al* [6], which establish that adaptation level theory offers two general mechanisms to explain the decay of happiness after a salient experience: habituation and contrast.

Similar evolutionary rules, with the term $u(t-h)$ instead of $\max\{u(s) : s \in [t-h,t]\}$, can be found in other related models: see, for instance, the celebrated Kalecki difference-differential equation describing a macroeconomic model of business cycles [12, 30, 32] or the mathematical model of emotional balance dynamics proposed in [47]. The works [7, 10, 17] show how the psychology of agents trading the foreign currency generates a similar dynamical mechanism expressed by the equation

$$u'(t) = -b|u(t)|u(t) + a(u(t) - u(t-1)), \quad a,b > 0.$$

Comparing (4) and (5), we obtain that $a = \beta - \alpha,\ b = -\beta$. In this way, the situation when $b < 0$ and $a + b < 0$ may appear as more appealing from the applied point of view, and, as we manifest in the present paper, it is certainly more interesting by its mathematical implications. In particular, we will show that for a range of parameters a, b, h and a periodic term f, the dynamics of (4) can exhibit chaotic behaviour. Since 'happiness' [18] is one of the possible interpretations of the experienced utility and, from own individual experience [23], everyone knows that happiness can be unpredictable, we thus obtain an additional theoretical evidence in favour of the analysis based on the 'peak-end' rule. Accordingly, our phenomenological model (5) could be considered as another attempt to use mathematics to understand the behaviour of happiness, a topic that goes back at least to Edgeworth's calculus of pleasure or 'hedonimetry' [9].

A side effect of our studies is the elaboration of a satisfactory mathematical framework to deal with the quasilinear functional differential equation (4). As far as we know, the first article dedicated to equations with maxima appeared in 1964 [42]. In his survey on the theory of functional differential equations, Myshkis [39, section 12] highlighted systems with maxima as differential equations with deviating argument of complex structure. He also noted that *'the specific character of these equations is not sufficiently clear yet'* [39, p 199]. Let $C[-h,0]$ be the set of continuous functions from $[-h,0]$ to $\mathbb{R}$. Equation (4) can be written in the form $u'(t) = F(t, u_t)$, where $u_t(s) = u(t+s)$, for all $s \in [-h,0]$, and the functional $F : \mathbb{R} \times C[-h,0] \to \mathbb{R}$ defined by $F(t,\phi) = a\phi(0) + b\max\{\phi(s) : s \in [-h,0]\} + f(t)$ is globally Lipshitzian in ϕ, which guarantees the existence, uniqueness, global continuation and continuous dependence on initial data of the solutions to (4). However, this functional is not differentiable in ϕ. By using a representation $\max\{\phi(s), s \in [-h,0]\} = \phi(-\tau(\phi))$ with some $\tau(\phi) \in [0,h]$, we see that (4) can be considered as a functional differential equation with state-dependent delay. Independently of the choice of the value $\tau(\phi) \in [0,h]$ for a given $\phi \in C[-h,0]$, the function $\tau : C[-h,0] \to [0,h]$ is clearly discontinuous at each constant element.

We will call (4) the Magomedov equation, honouring the mathematician who introduced this model in the late 70s and since then has analysed several particular cases of it with periodic forcing term $f(t)$ [1, 3, 38, 44, 45]. On pages 4–7 of his monograph [38], Magomedov explains how the periodic equation (4) can be used for modelling automatic control of voltage in a generator of constant current.

Besides the above mentioned applications, the periodic equation (4) plays an important role in the stability theory for the functional differential equation

$$u'(t) = au(t) + bf(t, u_t),$$

where $a, b < 0$, and the continuous functional $f : \mathbb{R} \times C[-h,0] \to \mathbb{R}$ satisfies either the following (sublinear) Yorke condition [50]

$$-\max_{s\in[-h,0]}(-\phi(s)) \leqslant f(t,\phi) \leqslant \max_{s\in[-h,0]}\phi(s),\ t \geqslant 0,\ \phi \in C[-h,0],$$

or its nonlinear version introduced in [35]. In this context, model (4) was used as a key test equation whose analysis determines the optimal stability regions for equations satisfying one of the aforementioned Yorke conditions. For instance, in the simplest situation when $a = 0$, equation (4) has a uniformly asymptotically stable periodic solution for every periodic function

$f(t)$ if and only if $0 < -bh < 3/2$ (that constitutes a variant of the so-called Myshkis–Wright–Yorke 3/2-stability criterion [11, 34, 36, 43]).

Among other mathematical objects closely related to equation (4), we would like to mention the Hausrath equation $u'(t) = b(\max\{|u(s)| : s \in [t-h,t]\} - u(t))$, analysed in [22, pp 73–74], and the Halanay inequality $u'(t) \leqslant au(t) + b\max\{u(s) : s \in [t-h,t]\}$, which became an important tool in the stability theory of functional differential equations [2, 3, 21, 25].

The present work extends previous studies [3, 43] where, in particular, the existence of multiple periodic solutions to equation (4) was established by using Krasnoselsky's rotation number and introducing a substitute of the variational equation for the non-smooth model (4). Our approach here is cardinally different, its workhorse is an associated selfmap $\mathcal{R}$ of an interval called 'the return map' in the paper. This function allows to reproduce the sequence of consecutive 'qualified peaks' $p_j = u(q_j,p)$ of each solution $u(t,p)$ to (4) with initial condition $u(s,p) = p,\ s \in [q-h,q]$, where q will be chosen in a suitable way. By a qualified peak we mean a local maximum of $u(\cdot,p)$ satisfying $u(q_j,p) = \max_{s\in[q_j-h,q_j+\varepsilon_j]} u(s,p)$ for some $\varepsilon_j > 0$. Specifically, we define $\mathcal{R}$ by $\mathcal{R}(p) = u(q_1,p)$. As we will show, the information stored in $\mathcal{R}$ is sufficient to describe the dynamics in (4). Now, analysing the dependence of the 'qualified' peak $u(q_1,p)$ on p, one can observe that at some specific values of p this peak disappears due to a cusp catastrophe. Accordingly, the return map $\mathcal{R}$ has a discontinuity at each such point so that important efforts in section 2 are focused on the studies of the continuity and differentiability properties of $\mathcal{R}$.

Next, in section 3 we show that, in spite of the uniqueness of T-periodic solution to (4) for all sufficiently small and large values of hT^{-1}, in general equation (4) can exhibit complex dynamics. Indeed, in section 3.2, we show that the restriction of the map $\mathcal{R}$ to an appropriate compact subset of its continuity domain can have a generalized horseshoe. This fact implies the existence of an infinite number of different periodic solutions to (4) as well as sensitive dependence on the initial values (chosen in some subset of continuous functions). Our example in section 3.2 extends a relatively small set of delay differential equations coming from applications where the existence of 'chaotic' behaviour has been proved analytically, cf [49] and its references. As usual, this requires elementary but laborious evaluations of some auxiliary smooth functions on compact sets. To make the paper more readable, this work and the proofs of some results are realized in three Appendices. Finally, in section 4 we discuss some qualitative features of our model, such as its adequacy, versatility and robustness, and emphasize the potential impacts of our results on hedonic utility studies.

2. Associated one-dimensional dynamics

Let us recall that an initial value problem for the functional differential equation (4) at the moment τ with the initial function (initial history) $\phi \in C := C[-h,0]$ has the form $u(s+\tau) = \phi(s),\ s \in [-h,0]$. As we have mentioned, the solution $u(t,\tau,\phi)$ of this problem exists for all $t \geqslant \tau$. The set C of initial functions forms the state space for the T-periodic equation (4), and the first recurrence map $\Pi : C \to C$ defined by $\Pi(\phi)(s) := u(T+s,0,\phi),\ s \in [-h,0]$, is a standard tool to study the dynamics of (4). An obvious complication in the use of this instrument is that Π acts on an infinite-dimensional space. Nevertheless, as we will show in this section, the dynamics generated by equation (4) is essentially one-dimensional. In particular, the detailed analysis of the solutions of (4) realized in section 2.1 shows that we can find an alternative scalar first recurrence map adapted to the particular situation of equation (4). Such an interval map, constructed in section 2.2 and denoted as $\mathcal{R}$, can be regarded as the quintessence of the evolution system (4). In section 3, we show on two particular examples that the dynamical properties of $\mathcal{R}$ depend strongly on the system parameters, so that their satisfactory description constitutes a separate task for each quartet $(a,b,h,f(t))$. In any case, the basic analytical properties of $\mathcal{R}$ should first be considered: we do this work in section 2.3 (continuity of $\mathcal{R}$) and section 2.4 (differentiability of $\mathcal{R}$). In theorem 17 of section 2.4, we also give some dynamically relevant information about the general geometric structure of $\mathcal{R}$.

2.1. *Some properties of the solutions to (4)*

We will consider sine-like T-periodic functions in the sense of the following definition:

Definition 1. We say that a T-periodic continuous function $f:\mathbb{R}\to\mathbb{R}$ has sine-like shape if there exist t_0, t_1 such that $0 < t_1 - t_0 < T$, f is strictly monotone on $[t_0, t_1]$ and on $[t_1, t_0 + T]$, and t_1 is a turning point of f.

Our main aim in this section consists of defining a one-dimensional map that allows us to study some important aspects of the dynamics of (4). Roughly speaking, the iterates of the map for a given initial condition correspond to the sequence of *qualified peaks* of an associated solution of (4). The concept of qualified peak is established in the following definition:

Definition 2. Let $u:[t_0,\infty)\to\mathbb{R}$ be a solution of (4). If there exists a point $\nu > t_0 + h$ such that $u(\nu) = \max\{u(s) : s\in[\nu - h, \nu+\varepsilon]\}$ for some $\varepsilon > 0$, then we say that $u(\nu)$ is a *qualified peak* of u.

The following result is a consequence of lemmas 22 and 23 in appendix A, and plays a key role in the definition of the return map associated to (4):

Theorem 3. *If f has sine-like shape and either of the following conditions holds:*

$$ah \leqslant 1 \text{ and } b + a < 0, \quad \text{or} \quad ah > 1 \text{ and } bh < -\exp(ah - 1), \tag{6}$$

then all solutions of (4) are bounded. Moreover, for each solution $u:[t_0,\infty)\to\mathbb{R}$ of (4) there exist $\nu > t_0 + h$ and $\varepsilon > 0$ such that $u(\nu) = \max\{u(s) : s\in[\nu-h, \nu+\varepsilon]\}$, that is, $u(\nu)$ is a qualified peak of u.

We notice that (6) provides the necessary and sufficient conditions for which the trivial solution of the delay equation (2) is uniformly asymptotically stable [48].

It was shown in [3, 25, 43] that if $a + b \neq 0$, then equation (4) has at least one T-periodic solution. Moreover, there is a unique globally attracting periodic solution if $a + b < 0$ and one of the following four conditions is satisfied: (a) $b \geqslant 0$; (b) $b < 0$, $a > 0$ and $(a-b)h < 1$; (c) $b < 0$, $a = 0$ and $-bh < 3/2$; (iv) $b < 0$, $a < 0$ and $(a/b)e^{ah} > \ln\left((b^2 + ab)/(b^2 + a^2)\right)$. However, as it was proved in [43] (see also section 3.1 below), equation (4) can have several periodic solutions for other parameter values. Moreover, in section 3.2 of the paper we will show that (4) can even possess an infinite set of periodic solutions as well as some solutions with 'chaotic' behaviour.

In the sequel, we assume that $0 < h < T$, f is T-periodic with sine-like shape, and either of the two conditions in (6) holds. In particular, this implies that $a + b < 0$. Indeed, if $ah > 1$ and $bh < -e^{ah-1}$, then, since $e^x > 1 + x$ for all $x > 0$:

$$(a+b)h = ah + bh < (ah-1) + 1 - e^{(ah-1)} < 0.$$

2.2. Construction of the return map

We introduce the function

$$\tilde{f}(t) = \frac{-f(t)}{a+b} = \frac{f(t)}{|a+b|}. \tag{7}$$

After a change of variables $u \to v + \min_{t\in\mathbb{R}}\tilde{f}(t)$ and $t \to s + \text{const}$, without loss of generality, we can also assume the following condition:

(**H**) f is a continuous $T-$periodic function, strictly decreasing on the interval $I_1 = [0,\beta]$ and strictly increasing on $I_2 = [\beta, T]$, with $\min_{t\in\mathbb{R}} f(t) = 0$.

Clearly, if $p \in \tilde{f}(I_1)$ then $p = \tilde{f}(q)$ for a unique $q \in I_1$. Let $u(\cdot,p) : [q,+\infty) \to \mathbb{R}$ be the solution of the initial value problem $u(s,p) = p,\ s \in [q-h,q]$, for equation (4). By theorem 3, there exist $\nu = \nu(q) > q$ and $\varepsilon > 0$ such that

$$u(\nu,p) = \max_{r\in[\nu-h,\nu+\varepsilon]} u(r,p). \tag{8}$$

Let ν^* be the smallest $\nu > q$ satisfying (8) and set $\mathcal{R}(p) = u(\nu^*,p)$. We refer the reader to figure 1 below for an illustration of the definition of $\mathcal{R}$ and some characteristic points involved in our results. The next statement says that $\mathcal{R}(\tilde{f}([0,\beta])) \subset \tilde{f}([0,\beta))$, in other words, that $\mathcal{R}(p) > 0$ for each $p \in \tilde{f}([0,\beta])$ and the application

$$\mathcal{R} : \tilde{f}([0,\beta]) \to \tilde{f}([0,\beta]) \tag{9}$$

is well defined.

Lemma 4. *Let $u : [-h,+\infty) \to \mathbb{R}$ be a solution of (4), and let $\tau > 0$ be a point of local maximum for u; moreover, assume that, for some $\varepsilon > 0$, $u(\tau) \geqslant u(t)$ for all $t \in [\tau - h, \tau + \epsilon)$. Then $u(\tau) = f(\tau)/|a+b|$ and $\tau^* := \tau \pmod{T} \in [0,\beta)$.*

Proof. The first conclusion of the lemma is evident. We prove the second one by contradiction. Suppose that $\tau^* \notin [0,\beta)$. Then there exists an interval $E = (0,\varepsilon)$, $0 < \varepsilon < h$, such that $f(s+\tau) - f(\tau) > 0$ and $M = \max_{r\in[\tau-h,\tau]} u(r) \geqslant u(s+\tau)$ for all $s \in E$. This implies that the function $d(s) = u(s+\tau) - u(\tau)$ satisfies the equation

$$d'(s) = ad(s) + f(s+\tau) - f(\tau)$$

for all $s \in E$. Using the variation of constants formula and the equality $d(0) = 0$, we get

$$0 \geqslant d(s)\exp(-as) = \int_0^s \exp(-ar)(f(r+\tau) - f(\tau))\mathrm{d}r > 0,$$

for all $s \in E$. This contradiction proves that actually $\tau^* \in [0,\beta)$. □

Note that a partial converse of lemma 4 is also true:

Lemma 5. *let $u : [-h,+\infty) \to \mathbb{R}$ be a solution of (4). If $u(\tau) = \tilde{f}(\tau) = \max_{s\in[\tau-h,\tau]} u(s)$, where $\tau^* = \tau \pmod{T} \in [0,\beta)$, then u strictly decreases on $I_r := (\tau, \tau + r)$, where $r = \min\{h, \beta - \tau^*\} > 0$. In particular, $\tau > 0$ is a point of local maximum for u.*

Proof. Indeed, consider the initial value problem $v(\tau) = \tilde{f}(\tau)$ for the equation $v'(t) = av(t) + bv(\tau) + f(t)$. The difference $m(t) = v(t) - v(\tau)$ satisfies the equation

$$m'(t) = am(t) + f(t) - f(\tau), \quad m(\tau) = 0.$$

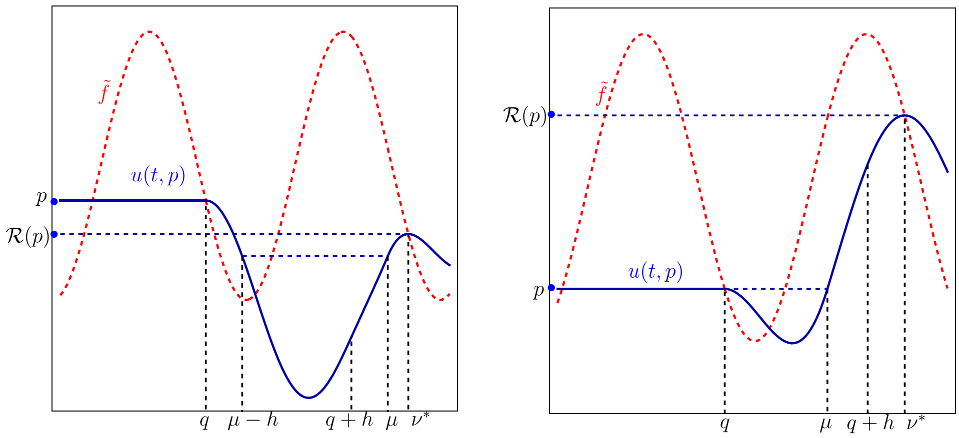


Figure 1. Schematic representation of two U-shaped solutions and their characteristic points. On the left, $\lambda(q)=q+h$; on the right, $\lambda(q)=\mu(q)$ and therefore $I_2=\emptyset$ (see definition 7). The graph of the map $\tilde{f}$ is given by the dashed curve in red colour.

Thus, by the variation of constants formula, for all $t\in I_r$,

$$(v(t)-v(\tau))\exp(-at)=\int_\tau^t \exp(-as)(f(s)-f(\tau))\mathrm{d}s<0,$$

proving that $\tau>0$ is a point of local maximum for v and therefore $u(t)=v(t)$ for all $t\in I_r$. The same computation shows that $u(t_1)=v(t_1)>v(t_2)=u(t_2)$ if $t_1,t_2\in I_r$ and $t_1<t_2$, and therefore u is strictly decreasing on I_r. □

The first recurrence map $\mathcal{R}$ plays the same role as the Poincaré map in the case of periodic differential equations. The following evident statement summarizes the relations between the delay differential equation (4) and the one-dimensional dynamical system defined by (9).

Lemma 6. *For a given solution u of equation (4), the set of all points at which a qualified peak is reached (in the sense of definition 2) forms a strictly increasing unbounded sequence $\{\tau_j, j\in\mathbb{N}\}$. Furthermore, $u(\tau_{n+j})=\mathcal{R}^j(u(\tau_n))$ for all $j\geqslant 0$ and $n\geqslant 1$.*

Clearly, there is a correspondence between the periodic solutions of (4) and the set of periodic points of $\mathcal{R}$. In particular, by [3], $\mathcal{R}$ has at least one fixed point.

The following definition plays a key role in the study of the regularity properties of the map $\mathcal{R}$.

Definition 7. We will say that the solution $u(t,p)$ is U-shaped if on the interval $\Omega_q=(q,\nu^*(q))$ it has only one critical point, in which it reaches its minimal value, and if in some left-side neighbourhood of ν^*, $u(t,p)=\max\{u(s,p), s\in[t-h,t]\}=:U(t,p)$.

If $u(t,p)$ is U-shaped, then the interval Ω_q can be represented as the disjoint union of the subintervals $I_1=(q,\lambda(q)]$, $I_2=(\lambda(q),\mu(q)]$ and $I_3=(\mu(q),\nu^*(q))$, where either $\lambda(q)=q+h$, or $\lambda(q)=\mu(q)$ and $I_2=\emptyset$, such that $U(t,p)=p$ on I_1, $U(t,p)=u(t-h,p)$ on I_2, $U(t,p)=u(t,p)$ on I_3.

2.3. Continuity of the return map

Set $K = \tilde{f}([0,\beta])$. The goal of this subsection is to describe the continuity properties of the map $\mathcal{R}: K \to K$. The proofs and some auxiliary results can be found in appendix B. We first state an assumption that will guarantee good continuity properties of $\mathcal{R}$ and admits practical verification:

($\mathbf{M_p}$) For a given $p \in K$, it holds that $u(s,p) < u(\nu^*,p)$ for all $s \in [\nu^* - h, \nu^*)$.

Note that the last assumption needs to be checked only for those p satisfying $\mathcal{R}(p) < p$ because of the following simple result:

Lemma 8. *Suppose that $\mathcal{R}(p) \geqslant p$ for some $p \in K$. Then ($\mathbf{M_p}$) is satisfied.*

Proof. If $\mathcal{R}(p) > p$, the result is a consequence of the definition of $\mathcal{R}$. Assume that $\mathcal{R}(p) = p$, $p = \tilde{f}(q)$, and ($\mathbf{M_p}$) does not hold. Consequently, for some $\hat{s} \in [\nu^* - h, \nu^*)$, it holds $u(\hat{s},p) = u(\nu^*,p)$ and $u(s,p) \leqslant u(\hat{s},p)$, $s \in [\hat{s} - h, \nu^*]$. The latter two properties show that $\hat{s}$ should coincide with ν^* unless $\hat{s} \in [q-h,q]$. Since $\hat{s} < \nu^*$, this means that $\nu^* - q \leqslant \nu^* - \hat{s} \leqslant h$. On the other hand, $\mathcal{R}(p) = u(\nu^*,p) = u(q,p) = p$ and thus lemma 4 ensures that $\nu^* - q = jT > h$ with some integer j, a contradiction. □

Assuming that ($\mathbf{M_p}$) holds for some $p = \tilde{f}(q) \in K$, in lemma 25 of appendix B, we will establish the existence of a maximal non-empty interval $(\mu(q),\nu^*)$ such that $u(t,p) > u(s,p), s \in [t-h,t)$ for all $t \in (\mu(q),\nu^*)$. Note that, in the special case of a U-shaped solution, this interval coincides with the interval I_3 defined in the paragraph below definition 7.

Theorem 9. *Let $p_0 = \tilde{f}(q_0)$ be a point of discontinuity for $\mathcal{R}$ and ($\mathbf{M_{p_0}}$) hold. If $\beta < h$ then $u(\beta + jT, p_0) = 0$ for some $\beta + jT \in [\mu(q_0), \nu^*(q_0))$ with $j \in \mathbb{N}$. Furthermore,*

$$\mathcal{R}(p_0) = \mathcal{R}(0), \quad \liminf_{p \to p_0} \mathcal{R}(p) = 0.$$

Theorem 9 allows us to find sufficient conditions for the continuity of $\mathcal{R}$ in some subsets of its domain K. We address this task in the next corollaries. For the proof of theorem 9 and corollaries 11–13, see appendix B.

Corollary 10. *If $h \in (\beta, T)$, $\mathcal{R}(p_0) \geqslant p_0$, and $\mathcal{R}(p_0) \neq \mathcal{R}(0)$, then $\mathcal{R}$ is continuous at p_0.*

Proof. By lemma 8, ($\mathbf{M_{p_0}}$) is satisfied. Since $\mathcal{R}(p_0) \neq \mathcal{R}(0)$, theorem 9 implies the continuity of $\mathcal{R}$ at p_0. □

Corollary 11. *Suppose that $h \in (\beta, T)$, and let $p = \tilde{f}(q) \in K$. If the following inequality holds:*

$$\int_0^h e^{as}(f(q+h-s) - f(q))\mathrm{d}s \geqslant 0, \tag{10}$$

then $\mathcal{R}$ is continuous at p, $\mathcal{R}(p) > p$, and $\nu^*(q) \in (T, T+\beta)$. Moreover, there exists $r \in (q, q+h]$ such that $u(r,p) = p$, $u'(r,p) > 0$, and $u(t,p) < p$ for all $t \in (q,r)$.

Corollary 12. *Suppose that $h \in (\beta, T)$, $b < 0$. Then $\mathcal{R}$ is continuous at the point p, if*

$$\mathcal{R}(p) \neq \mathcal{R}(0) \quad \text{and} \quad \int_0^h e^{as}(f(\nu^*(q) - s) - f(\nu^*(q)))\mathrm{d}s < 0. \tag{11}$$

Corollary 13. *Assume that either of the stability conditions in (6) holds and suppose that $h \in (\beta, T)$. Then there are $\delta > 0$ and p^* such that $\mathcal{R}(p^*) = p^*$ and $\mathcal{R}(p) > p$ for all $p \in [0, p^*)$. Furthermore, $\mathcal{R}$ is continuous on $[0, p^* + \delta)$. If, in addition, ($\mathbf{M_p}$) is satisfied for all $p \in K$ and*

$[0,c) \subset K$ is the maximal half-open interval where $\mathcal{R}$ is continuous then either $[0,c] = K$ and $\inf \mathcal{R}(K) > 0$ or $\mathcal{R}(c-) = 0,\ \mathcal{R}(c) = \mathcal{R}(0) > 0$.

Corollary 13 provides an alternative proof of the existence of at least one $T-$periodic solution for equation (4) with sine-like T-periodic continuous function $f(t)$. In [3] this result was obtained by using the topological degree method.

2.4. Differentiability of the return map

In this subsection, we assume that $\beta < h < T, f$ is T-periodic with sine-like shape, and either of the two conditions in (6) holds. It is not difficult to prove the differentiability (possibly, one-side differentiability) of the return map $\mathcal{R}$ in the case when the graph of $u = u(t,p)$ on the interval $(q, \nu^*(q))$ is U-shaped in the sense of definition 7.

Due to theorem 9, $\mathcal{R}$ is continuous at p if $u(t,p)$ is U-shaped and its graph does not intersect the set $\{(\beta + kT, 0), k \geqslant 1\} \subset \mathbb{R}^2$. Assuming $u(t,p)$ is U-shaped, we introduce the following variational equation along $u(t,p)$:

$$w'(t) = \begin{cases} aw(t) + bw(q), & \text{if } q \leqslant t \leqslant \lambda(q), \\ aw(t) + bw(t-h), & \text{if } \lambda(q) \leqslant t < \mu(q), \\ (a+b)w(t), & \text{if } \mu(q) \leqslant t < \nu^*(q), \end{cases} \tag{12}$$

where $\lambda(q) \in \{q+h, \mu(q)\}$.

Let $v(t),\ t \geqslant 0$, denote the solution of the initial value problem

$$u'(t) = au(t) + bu(t-h), \tag{13}$$

$$u(s) = 0,\ s \in [-h, 0),\ u(0) = 1. \tag{14}$$

Such $v(t)$ is called the fundamental solution of the linear delay-differential equation (13), see [22, section 1.5]. If $w(t)$ satisfies the variational equation (12), we obtain (see [22, chapter 1, theorem 6.1]) that $w(\nu^*(q)) = \Delta(q) w(q)$, where

$$\Delta(q) = \left(v(\mu - q) + b \int_{\mu-q-h}^{\mu-q} v(s) \mathrm{d}s \right) e^{(a+b)(\nu^* - \mu)}.$$

To simplify and shorten our proofs, hereafter we make the additional assumption

(**T**) f is a C^1-smooth T-periodic function having exactly two critical points on each half-open interval of length T. Moreover, $a > 0$, $a + b < 0$ and $h \in (\beta, T)$.

For instance, (**T**) is fulfilled in the example considered in section 3.2.

Using (**T**), we can easily establish that $u(t,p)$ has at most one critical point on the time interval $(q, T] \cap (q, q+h)$. If $p = 0$, this fact follows from corollary 27 in appendix B. Next, lemma 5 shows that $u(t,p)$ with $p > 0$ decreases on some maximal non-empty interval $I \supset (q, \min\{q+h, \beta\})$. In fact, if $\hat{q} > \beta$ is the leftmost point satisfying $\tilde{f}(\hat{q}) = p$, then

$$u'(t,p) = au(t,p) + bp + f(t) < (a+b)p + f(q) = 0,$$

for all $t \in (q, \min\{q+h, \hat{q}\}]$.

If $u(t,p)$ has a leftmost critical point $t_m \in (\hat{q}, q+h)$ then $0 \leqslant u''(t_m) = f'(t_m)$ implying that $\beta < t_m \leqslant T$ and $0 < u''(t_m)$ if $t_m < T$. In particular, $u(t,p)$ can have at most one critical point on (q, T). Now, suppose that $u'(T,p) = 0$ and $t_m < T < q+h$. Then $u(t,p) < p$ for all $t \in (q, T)$ so that $v(t) = u'(t,p)$ satisfies $v'(t) = av(t) + f'(t),\ v(T) = 0$, in some small neighbourhood of T. Since $f'(t)$ is changing its sign at T from positive to negative, $v(t)$ is negative in some small punctured vicinity of T. Thus $u(t,p)$ should have an additional local maximum point between t_m and T, a contradiction. In this way, $u(t,p)$ can have at most one critical point on $(q,T] \cap (q,q+h)$.

The above reasoning is useful in proving the following result (graphically presented on the right frame of figure 1):

Lemma 14. *Let (**T**) and (10) hold. Then the graph of $u = u(t,p)$ is U-shaped.*

Proof. With the notations of corollary 11 and the above comments, it suffices to establish that t_m is the unique critical point of $u(t,p)$ on the interval (q,r). Indeed, if $u(t,p)$ has another critical point $t_* > t_m$, then $t_* \in (T,r) \subset (T,\beta+T)$ where $u''(t_*,p) = f'(t_*) < 0$ (recall that $f'(t) < 0$ for $t \in (T,\beta+T)$). Therefore t_* is the unique critical point of $u(t,p)$ on (T,r) where a local maximum is reached. Thus $u'(r,p) < 0$, which is impossible by corollary 11. □

By the same arguments, if $q+h > T$ then $u(t,p)$ can have at most one additional critical point on $(T,q+h]$ where a local maximum is reached. Clearly, this can happen only when $u(t,p) < p$ for $t \in (q,q+h]$. Furthermore, suppose that there exists the leftmost point $r \in (q,q+h]$ such that $u(r,p) = p$. We claim that then the inequality (10) is necessarily satisfied. Indeed, otherwise the solution $u_p(t)$ of the initial value problem $u'(t) = au(t)+bp+f(t)$, $u_p(q) = p$ satisfies $u_p(q+h) < p$ (observe that the inequality $u_p(q+h) \geqslant p$ amounts to (10), see computations in (B.2)). Thus $u_p(t)$ reaches its absolute maximum on $[r,q+h]$ at some point $t^* \in (T,q+h]$. Since $h < T$ this implies that $f(t^*) > f(q)$ and, consequently, $0 = u_p'(t^*) = au(t^*)+bp+f(t^*) > ap+bp+f(t^*) = f(t^*)-f(q) > 0$, a contradiction.

Hence, under the assumptions of lemma 14, $\mu(q) \leqslant q+h$ if and only if the inequality (10) holds. For simplicity, it is convenient to consider the following assumption:

(**C**) The set of all $q \in [0,\beta)$ satisfying inequality (10) is a nonempty interval $S = [\beta_1,\beta)$.

For example, condition (**C**) holds for equation (19) considered in section 3.2, with $\beta_1 \approx 0.392\,89$.

By the implicit function theorem, if (10) and (**C**) hold then the equation $u(t,p(q)) = p(q)$, where we denote $p = \tilde{f}(q) = p(q)$, has a unique solution $t = \lambda(q) \in (q,q+h]$, smoothly depending on $q \in [\beta_1,\beta)$. Also $\lambda(q) = \mu(q)$ if $q \in [\beta_1,\beta)$ and $\lambda(q) = q+h$, if $q \leqslant \beta_1$.

Next, if $q \in (\beta_1,\beta)$, then

$$p = e^{a(\lambda(q)-q)}p + \int_q^{\lambda(q)} e^{a(\lambda(q)-s)}(bp+f(s))\mathrm{d}s, \tag{15}$$

so that

$$1 = (p(a+b)+f(\lambda(q)))\,\partial_p\lambda(q) + \left(1+\frac{b}{a}\right)e^{a(\lambda(q)-q)} - \frac{b}{a},$$

where $\partial_p\lambda(q)$ denotes the partial derivation with respect to p of the composite function $\lambda \circ q(p)$. Next, if $q \in (\beta_1,\beta)$, then $\max_{s\in[t-h,t]} u(s,p) = u(t,p)$ for $t \in [\lambda(q),\nu^*(q)]$ and therefore

$$\mathcal{R}(p) = u(\nu^*(q),p) = pe^{(a+b)(\nu^*(q)-\lambda(q))} + \int_{\lambda(q)}^{\nu^*(q)} e^{(a+b)(\nu^*(q)-s)}f(s)\mathrm{d}s.$$

Here ν^* is C^1-smooth function of q as the solution of the equation $F(\nu,q) = 0$,where $F(\nu,q) = au(\nu,p(q))+bp(q)+f(\nu)$, $\partial_\mu F(\nu,p) = au'(\nu,p)+f'(\nu) = f'(\nu) < 0$. A straightforward computation shows that

$$\mathcal{R}'(p) = u'(\nu^*(q),p)\partial_p\nu^*(q) + e^{(a+b)(\nu^*(q)-\lambda(q))} -$$
$$p(a+b)e^{(a+b)(\nu^*(q)-\lambda(q))}\partial_p\lambda(q) - e^{(a+b)(\nu^*(q)-\lambda(q))}f(\lambda(q))\partial_p\lambda(q) =$$
$$e^{(a+b)(\nu^*(q)-\lambda(q))} - e^{(a+b)(\nu^*(q)-\lambda(q))}\,(p(a+b)+f(\lambda(q)))\,\partial_p\lambda(q) =$$

$$e^{(a+b)(\nu^*(q)-\lambda(q))}\left(\left(1+\frac{b}{a}\right)e^{a(\lambda(q)-q)}-\frac{b}{a}\right)=\Delta(q(p)).$$

As a consequence, we have the following result:

Theorem 15. *Assume (**T**) and (**C**) hold, and let $q\in(\beta_1,\beta)$. Then $\mu(q)=\lambda(q)$ and*

$$\mathcal{R}'(p)=e^{(a+b)(\nu^*(q)-\mu(q))}\left(\left(1+\frac{b}{a}\right)e^{a(\mu(q)-q)}-\frac{b}{a}\right)=\Delta(q(p)). \tag{16}$$

Therefore, $b\geqslant a/(e^{-ah}-1)$ implies that $\mathcal{R}'(p)>0$ for all $p\in(0,\tilde f(\beta_1))\subset(0,p^*)$. On the other hand, if $b<a/(e^{-ah}-1)$ and the only root of equation

$$f(\tau)+b\int_0^{\frac{1}{a}\ln\frac{b}{a+b}}e^{-au}f(u+\tau)\mathrm{d}u=0 \tag{17}$$

is $\tau=q_0\in(\beta_1,\beta)$, then $\mathcal{R}'(p)>0$ for $p\in(0,\tilde f(q_0))$ and $\mathcal{R}'(p)<0$ for $p\in(\tilde f(q_0),\tilde f(\beta_1))$.

Proof. Since $\lambda(q)-q<h$ for all $q\in(\beta_1,\beta)$, it follows that $\Delta(q)>0$ for all $q\in(\beta_1,\beta)$ if $h\leqslant(1/a)\ln(b/(b+a))$. Note that, because of $a+b<0<a$, the latter inequality is equivalent to $b\geqslant a/(e^{-ah}-1)$.

On the other hand, if $h>(1/a)\ln(b/(b+a))$, then by the intermediate value theorem, there is $q_0\in(\beta_1,\beta)$ such that

$$\lambda(q_0)-q_0=\frac{1}{a}\ln\left(\frac{b}{b+a}\right)$$

and therefore $\mathcal{R}'(p(q_0))=0$. Furthermore, recalling that $p_0=\tilde f(q_0)$ and using the above expression for $\lambda(q_0)-q_0$, we obtain from (15) that q_0 satisfies (17), which proves the uniqueness of q_0 (under our assumptions). Hence,

$$\lambda(q)-q<\frac{1}{a}\ln\left(\frac{b}{b+a}\right),\ \text{for } q\in(q_0,\beta);$$
$$\lambda(q)-q>\frac{1}{a}\ln\left(\frac{b}{b+a}\right),\ \text{for } q\in(\beta_1,q_0),$$

which finalizes the proof. □

Example 16. For equation (19) in section 3.2, we get $\beta=0.5\pi$, $b<a/(e^{-ah}-1)$, and we numerically find $\beta_1\approx0.39289$, $\tilde f(\beta_1)\approx0.90754$, $q_0\approx1.18459422$, $\tilde f(q_0)\approx0.10831425$. Thus, the return map $\mathcal{R}$ is C^1-smooth on the interval $[0,0.9]$, where it has a unique critical point $p_0=\tilde f(q_0)$. Moreover, $\mathcal{R}$ reaches its absolute maximum at p_0 (see figure 3).

It is quite remarkable that the expression for $\mathcal{R}'(p)$ in (16) does not depend on the derivatives $\partial_q\nu^*(q)$ and $\partial_q\mu(q)$. As one can see in the proof of our next result, it is due to the following three circumstances: a) that $u(s,p)=p$ for all $s\in[q-h,q]$ (this plays a key role in eliminating the dependence on $\partial_q\mu(q)$); b) that $u'(\nu^*(q),p)=0$ (this eliminates the dependence on $\partial_q\nu^*(q)$); and c) that the graph of $u(t,p)$ is U-shaped.

The next result can be viewed as a natural extension of theorem 15 for $q\leqslant\beta_1$. The proof is given in appendix B.

Theorem 17. *Suppose that assumptions (**T**) and (**C**) are satisfied, equation (17) has a unique root $q_0\in(\beta_1,\beta)$, and there is $\alpha\in(0,\beta)$ such that the solutions $u(t,p(q))$ of equation (4) are U-shaped for all $q\in(\alpha,\beta]$. If $\Delta(q)<0$ for $q\in(\alpha,\beta_1]$, then there is an increasing sequence (either finite or infinite) of real numbers p_i, with $0<\tilde f(q_0)<p_1<\cdots<p_j<\cdots<\tilde f(\alpha)$, such* *that $\mathcal{R}$ is differentiable on the intervals $D_1 = [0,p_1),\ldots,D_j = [p_{j-1},p_j),\ldots$, strictly increasing on the interval $(0,\tilde{f}(q_0))$, and strictly decreasing on the interval $(\tilde{f}(q_0),p_1)$ and on every D_j, $j > 1$. Moreover, $\mathcal{R}(p)$ is right continuous at p_j and $\mathcal{R}(p_j-) = 0$, $\mathcal{R}(0) = \mathcal{R}(p_j)$. Finally, $\mathcal{R}'$ is continuous on every D_j and $\mathcal{R}'(p) = \Delta(q(p))$, $\mathcal{R}'(p_j-) < \mathcal{R}'(p_j+)$.*

The following corollary provides additional information on the number and periods of the periodic solutions of (4). The case $m = 2$ is illustrated in figure 3.

Corollary 18. *Let D_j be the intervals defined in the statement of theorem 17. Suppose that $\mathcal{R}(0) \in D_m$ for some $m \geqslant 1$. Then equation (4) has m sine-like periodic solutions $p_j(t)$ with minimal periods jT and such that $\zeta_j := \max_{\mathbb{R}} p_j(t) < \max_{\mathbb{R}} p_k(t)$ for each pair of indices $j < k$.*

Proof. If $\mathcal{R}(0) \in D_m$ then $\mathcal{R}$ has exactly m fixed points $\zeta_i \in D_i$, $i = 1,\ldots,m$. □

Remark 19. Let v denote the fundamental solution of (13), and consider the function

$$V(t) = v(t) + b\int_{t-h}^{t} v(s)\mathrm{d}s, \quad t \geqslant 0.$$

Note that $V(0) = 1$. Suppose that $V(t) > 0$ for $t \in [0,\alpha_*)$ and $V(t) < 0$ for $t \in (\alpha_*,\beta_*)$. Assume that all conditions of theorem 15 hold, and let each $u(t,p)$ be U-shaped. Then the inequalities $\alpha_* < \mu(q) - q < \beta_*$ clearly guarantee that $\Delta(q) < 0$.

As an application, consider equation (19) in section 3.2, for which $\alpha_* \approx 1.2$, $\beta_* \approx 12.11$. Since $\mu(q) - q > 1.5\pi > \alpha_*$ for $q < q_0$, and $\mu(q) - q < 12.11$ if $\mu(q) \leqslant 1.5\pi$, $q \geqslant -0.5\pi$, we can conclude that $\mathcal{R}'(p) < 0$ for all $p \in (\tilde{f}(q_0),p_1)$ (in appendix C, we will prove that the corresponding solutions $u(t,p)$ are U-shaped).

3. Two examples

In this section, we give two applications of our results.

3.1. Equation with multiple attracting solutions

The equation

$$u'(t) = -\max_{s\in[t-3\pi/2,t]} u(s) + f(t), \tag{18}$$

with $f(t) = -\sin t + \max_{t-3\pi/2\leqslant\tau\leqslant t}\cos\tau$ was studied in [43]. The function $u_1(t) = \cos t$ is an evident solution of (18) and the existence of another 8π-periodic solution u_2 was established in the cited work. However, the full description of the dynamics of (18) was not provided in [43]. This can be easily done by analysing the return map $\mathcal{R}$ for (18), whose graph is presented in figure 2. We see that, in fact, the minimal period of u_2 is 4π. Moreover, u_1, u_2 and $u_3(t) = u_2(t + 2\pi)$ exhaust the set of all periodic solutions to (18), and u_2, u_3 attract all solutions to (18) (clearly, excepting u_1). Set $q = \pi/2$ so that, for this particular value of q and $\tilde{f}(t) = f(t - \pi/2)$, we have $\mathcal{R}(1) = 1 = \tilde{f}(\pi/2)$, $\nu^* = 5\pi/4$, $\mu = 9\pi/4$. Since $v(s) = 1.5\pi + 1 - s$, $s \in [1.5\pi, 3\pi]$, we easily find that $\mathcal{R}'(1) = (1 - 7\pi/4 + \pi^2/32)\exp(-\pi/4) \approx -1.91$, which coincides with the unique non-zero characteristic multiplier determined by the variational equation along $u_1(t) = \cos t$ (see [43, theorem 1.2] for more details).

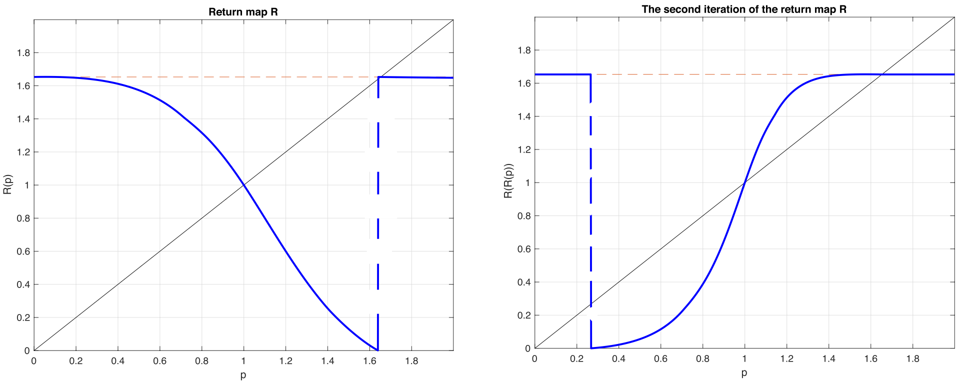


Figure 2. Return map $\mathcal{R}$ for equation (18) (on the left) and its second iteration $\mathcal{R}^2$ (on the right). The dashed horizontal line corresponds to $\mathcal{R}(0)$. Note that $\mathcal{R}^2$ and $\mathcal{R}$ share one point of discontinuity (not appreciable on the right frame).

3.2. Chaotic behaviour in the Magomedov equation

Our main example is given by the equation

$$u'(t) = 0.32u(t) - \max_{t-3\pi/2 \leqslant s \leqslant t} u(s) + 1 - \sin t. \tag{19}$$

The forcing term $f(t)$ in (18) is close to the function $g(t) = 1 - \sin t$. In fact, the replacement of $f(t)$ with $1 - \sin t$ in (18) produces dynamically insignificant changes in the return map so that the modified system has simple dynamics. However, by adding the linear term $0.32u(t)$ to (18), the behaviour of the solutions changes dramatically. Indeed, as we show below, equation (19) exhibits chaotic behaviour.

We note that the specific choice of the parameters $a = 0.32$ and $b = -1$ is mostly motivated by some advantages in the graphical representation of the solutions and in establishing the continuity properties of the return map $\mathcal{R} : K \to K$, $K = [0, 2(a+b)^{-1}] = [0, 2.94\ldots]$, shown in figure 3. In particular, our numerical simulations show that if we fix $b = -1$ and let $a \in [0.32, 0.54]$, for which the admissibility condition (6) holds, the map $\mathcal{R}$ has turbulent trajectories. However, they can coexist with an attracting cycle possessing a large basin of attraction. The latter possibility is excluded by choosing $a = 0.32$. In such a case, the rightmost continuous branch of the graph of $\mathcal{R}$ does not intersect the diagonal, compare with the left part of figure 2.

Since the second condition in (6) holds and $\beta = \pi < h = 1.5\pi < T = 2\pi$, we can apply corollaries 11 and 12 to conclude that $\mathcal{R}$ is continuous on the interval $[0, 0.90\ldots] \subset K$, where $\mathcal{R}(p) > p$ (by corollary 11) and $\mathcal{R}$ is continuous at each point of the set $\mathcal{R}^{-1}([1.43\ldots, 2.94\ldots] \setminus \mathcal{R}(0))$ (by corollary 12). Note that, after integrating, inequalities (10) and (11) take the form $A\sin q + B\cos q \geqslant 0$ with some real coefficients A, B. Thus the above numerical values (like $0.90\ldots$) can be found in closed form: e.g. $0.90\ldots = (1 - \sin q_0)/0.68$, where $\cot q_0 = ((e^{ah} - 1)/a + 1 - a)(a + e^{ah})^{-1}$, $a = 0.32$, $h = 1.5\pi$. The graph of the return map for equation (19) is numerically plotted in figure 3. Corollary 13 shows that the graph in this figure is a continuous curve at least until its first intersection with the diagonal. Theorem 20 below describes in more detail the main continuity properties of the return map for (19). For its proof, see appendix C.

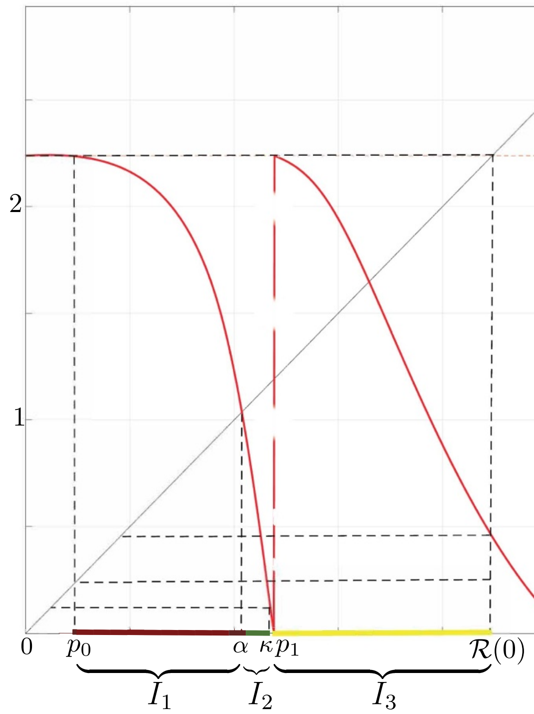


Figure 3. Graph of the return map $\mathcal{R}$ for equation (19) (discontinuous red solid curve). The upper dashed horizontal line corresponds to $\mathcal{R}(0)$. The closed subintervals I_1 (brown), I_2 (green), I_3 (yellow) of K have pairwise disjoint interiors and satisfy the relations $I_2 \cup I_3 \subset \mathcal{R}(I_1)$, $I_1 \subset \mathcal{R}(I_2)$, $I_2 \cup I_3 \subset \mathcal{R}(I_3)$.

Theorem 20. *The return map $\mathcal{R}: K \to K$ for (19) has exactly two points of discontinuity $p_1 \approx 1.2$ and $p_2 \approx 2.61$ on the interval $[0, p_2] \supset \mathcal{R}(K)$, where*

$$\mathcal{R}(p_1) = \mathcal{R}(p_2) = \mathcal{R}(0) \approx 2.23, \quad \mathcal{R}(p_1-) = \mathcal{R}(p_2-) = 0.$$

Furthermore, $\mathcal{R}$ is differentiable on $[0, 1.316] \setminus \{p_1\}$ and has a unique critical point $p_0 \approx 0.108$ on this interval, where it reaches its absolute maximum. Finally, $\mathcal{R}'(p_1-) < \mathcal{R}'(p_1+)$, $\mathcal{R}(p) > p$ for all $p \in [0, 0.9]$ and $\mathcal{R}(\mathcal{R}(0)) < 0.9$.

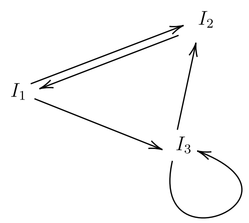
This theorem implies the existence of a leftmost fixed point $\alpha \in (0.9, p_1)$ for $\mathcal{R}$. Let $p_0 \in (0, p_1)$ be defined by $\mathcal{R}(p_0) = \mathcal{R}(0)$ and let $\kappa \in (\alpha, p_1)$ be sufficiently close to p_1 to satisfy $\mathcal{R}(\kappa) < p_0$. Consider the following closed subintervals of K with pairwise disjoint interiors

$$I_1 = [p_0, \alpha], \quad I_2 = [\alpha, \kappa], \quad I_3 = [p_1, \mathcal{R}(0)].$$

These intervals are shown in figure 3. Clearly, the return map is continuous on each of these intervals and

$$I_2 \cup I_3 \subset \mathcal{R}(I_1), \quad I_1 \subset \mathcal{R}(I_2), \quad I_2 \cup I_3 \subset \mathcal{R}(I_3).$$

Writing the inclusion $I_1 \subset \mathcal{R}(I_2)$ in the form $I_2 \to I_1$, and similarly the others, we obtain the following directed Markov graph associated with the collection I_1, I_2, I_3:

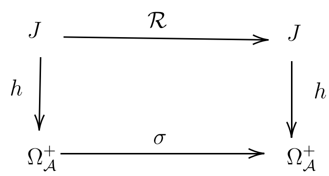


The adjacency matrix $\mathcal{A} = \{a_{ij}\}$ of the graph is defined as follows: $a_{ij} = 1$ if and only if there is an edge from vertex I_i to vertex I_j; otherwise, $a_{ij} = 0$. Thus:

$$\mathcal{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Consider the space $\Omega_{\mathcal{A}}^+$ of all one-sided paths on the above Markov graph (for example, $\omega = (I_3, I_3, I_2, I_1, I_2, I_1, I_3, \ldots)$) provided with the metrizable topology of component-wise convergence. It is easy to realize that $\Omega_{\mathcal{A}}^+$ is a closed perfect subspace of the product space $\{I_1, I_2, I_3\}^{\mathbb{N}}$ so that it is a Cantor set. Let $\sigma : \Omega_{\mathcal{A}}^+ \to \Omega_{\mathcal{A}}^+$ denote the one-sided shift defined by $\sigma(\{I_{n_k}\}) = \{I_{n_{k+1}}\}$ (e.g. $\sigma(I_3, I_3, I_2, I_1, I_2, I_1, I_3, \ldots) = (I_3, I_2, I_1, I_2, I_1, I_3, \ldots)$). Since all elements of the matrix $\mathcal{A}^3$ are positive (hence, the matrix $\mathcal{A}$ is transitive [31, definition 1.9.6]), the dynamical system $\sigma : \Omega_{\mathcal{A}}^+ \to \Omega_{\mathcal{A}}^+$ is topologically mixing and its periodic points are dense in $\Omega_{\mathcal{A}}^+$, see [31, proposition 1.9.9]. Then, an application of theorem 15.1.5, corollaries 1.9.5, 15.1.6, 15.1.8, and proposition 3.2.5 in [31] yields the following result (notice that the greatest eigenvalue of $\mathcal{A}$ is $\lambda_{\mathcal{A}}^{\max} = (\sqrt{5} + 1)/2$).

Theorem 21. *There is a closed subset $J \subset I_1 \cup I_2 \cup I_3$ and a continuous surjection $h : J \to \Omega_{\mathcal{A}}^+$ such that $\mathcal{R}(J) \subset J$ and the following diagram is commutative.*



Moreover, to each periodic orbit $\omega \in \Omega_{\mathcal{A}}^+$ corresponds at least one periodic point of the same period in $h^{-1}(\omega)$ so that $\mathcal{R} : J \to J$ has an infinite set of periodic solutions. In fact, the number of different n-periodic orbits of $\mathcal{R}$ is bigger than or equal to the trace $Tr\mathcal{A}^n$, and the topological entropy of $\mathcal{R} : J \to J$ is at least $\log((\sqrt{5} + 1)/2) > 0$.

In this way, equation (19) has an infinite set of periodic solutions. In particular, figure 3 shows that it has a 2π-periodic solution u_1 with $\alpha = \max_{t \in \mathbb{R}} u_1(t) \approx 1.037$, and a 4π-periodic solution u_2 with $\gamma = \max_{t \in \mathbb{R}} u_2(t) \approx 1.65$. The graph of the second iteration $\mathcal{R}^2$ restricted to the interval $[\alpha, \gamma]$ suggests that $\mathcal{R}$ has an infinite set of unstable periodic solutions. In figure 4, we represent two particular solutions of equation (19): the curve $(t, u(t))$ of the solution $u = u(t, 1)$, $t < 350$, and the projection of the solution $u = u(t, 0)$ on the plane $(u(t), u(t-h))$.

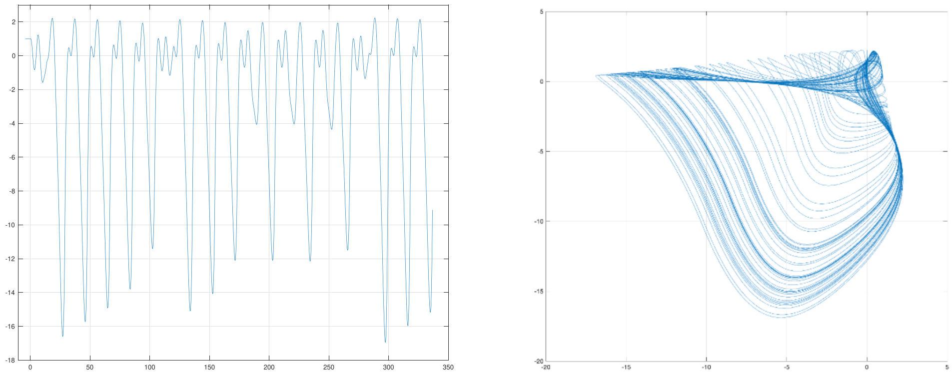


Figure 4. On the left, we represent the solution $u = u(t,1)$ of (19); on the right, the projection of the solution $u = u(t,0)$ of (19) on the plane $(u(t,0), u(t-h,0))$.

4. Discussion

Scalar delay differential equations proved to be particularly useful for understanding the dynamics of key indicators for complex 'non-mechanistic' systems whose evolution at each instant is determined not only by their current state but also by some specific states in their immediate pre-history. Hutchinson's equation [24] and Nicholson's blowflies equations [20] in ecology, the Mackey–Glass equations in physiology [49], and the Kalecki equation in economics [30] are outstanding examples of such applications. Even if the action of recent history is incorporated in the mentioned equations by means of very simple mechanisms, they allowed to explain qualitative and, in some cases, quantitative dynamical characteristics of the respective systems.

In this work, we show that the framework of the experienced utility theory, with its concepts of instant and remembered utilities, and the peak-end rule suggesting the mechanism of delayed action, is ideally adapted to the use of delay differential equations in the particular form (1) of differential equations with maxima. Recall that the peak-end rule is a psychological theory that states that two moments can serve as a proxy to evaluate and make decisions based on certain types of past affective episodes: the moment of peak affect intensity and the ending. Applying this theory to model the dynamics of experienced utility, we see that a suitable mathematical evolution equation must take into account the peak and the end of each episode. We assume here that one person's life consists of a continuum of episodes.

Now, the studies [20, 49] also show that the actions of present and past states combine better in an additive form (and not in a multiplicative form as in [24]). The latter consideration, together with the hedonic adaptation principle, leads to the more specific form (5) of quasilinear equation with maxima. Equation (5) is non-autonomous and it is natural to assume that in general the action $f(t)$ of external stimuli is changing randomly. As an approximation for $f(t)$, we take in this work its periodic non-random component, which can be imposed by certain repeated life routine.

Our studies of the periodic model (4), which considers only a linear combination of the peak and the end states, show its robustness and versatility. Indeed, the return map $\mathcal{R}$, that keeps all essential information about the dynamics of the solutions, depends continuously on parameters $a, b, f(t)$. On the other hand, equation (4) can generate not only globally stable periodic dynamics as in the example of section 3.1 (note there the period duplication effect) but also turbulent behaviour as in the example of section 3.2.

Importantly, the proposed approach to equation (4) can be extended to general *dissipative periodic* equations with maxima (1) once the algebraic equation $F(t,u,u)=0$ admits a unique continuous periodic solution $u=\tilde{f}(t)$. Moreover, the sine-like shape of $\tilde{f}(t)$ assumed in this paper can be replaced by the less restrictive condition of its piece-wise monotonicity. Certainly, such generalizations have significant computational, technical and expository costs; e.g. the compact interval K in $\mathcal{R}:K\to K$ should be replaced by a disjoint union of compact segments. At the same time, the more simple model (4) already has a sufficiently representative character. The mathematical method of this paper can be also useful in the case of aperiodic piece-wise monotone continuous functions $\tilde{f}(t)$. In such a case, the return map $\mathcal{R}:K\to K$ should be replaced by an appropriate non-autonomous discrete dynamical system [33].

To close this section, we would like to emphasize the importance of the measurement of the hedonic experienced utility ('happiness'), which constitutes not only a crucial issue in psychological research, but also a great advance for economics [16]; moreover, *'happiness research represents a case of a productive cross–fertilization of two otherwise isolated fields: economics and psychology'* [15]. But happiness is itself valuable without having to contribute to anything else [41], and therefore it is an important objective to try to explain its behaviour, not only with empirical studies [8], but also constructing and analysing mathematical models [19, 46, 47]. Our model represents one step more in this direction and connects some theoretical conclusions and experimentally observed dynamics of happiness. For instance, it gives a purely mathematical argument in favour of the possibility of unpredictable behaviour of happiness within a well-established life's routine. Note also that the U-shape of the utility profile represented in figure 1 is usually observed in empirical research [5, 19].

Acknowledgments

We dedicate this work to the memory of Anatoly Samoilenko, one of influential Soviet and Ukrainian experts in the field of ordinary differential equations, cf [40, sections 2.43, 2.49], professor and doctoral adviser of the first and third authors.

We express our appreciation to Rafael Ortega for suggesting the present simple proof of lemma 23. We also thank L'ubomír Snoha and Hans-Otto Walther for valuable discussions. We are indebted to Alexander Rezounenko for providing the monograph [38]. O Trofymchuk and S Trofimchuk were supported by FONDECYT (Chile), project 1190712.

Finally, we sincerely thank the reviewers and the editors for their comments and suggestions, which helped to improve the paper.

Appendix A. Properties of the solutions of the equation with periodic external stimuli

First we prove two important results concerning the monotonicity and boundedness properties of the solutions to (4).

Lemma 22. *Assume that $h<T$, $a+b\neq 0$ and the T-periodic continuous function f has sine-like shape. Let $u:[\alpha,+\infty)\to\mathbb{R}$ be a solution to (4). Then at least one of the following options is satisfied:*

(a) there exists $\tau>\alpha$ such that u strictly increases on $[\tau,+\infty)$ and $u(t)\to+\infty$ as $t\to+\infty$;

(b) there exists $\tau_1 > \alpha + h$ *such that* $U(t) := \max_{s\in[t-h,t]} u(s)$ *decreases on* $[\tau_1, +\infty)$ *and* $u(t) \to -\infty$ *as* $t \to +\infty$*;*

(c) there exist $\tau_2 > \alpha + h$ *and* $\varepsilon > 0$ *such that* $\max_{s\in[\tau_2-h,\tau_2+\varepsilon]} u(s) = u(\tau_2)$.

Proof. Consider the function $U : (\alpha + h, +\infty) \to \mathbb{R}$ defined in (b). We have the following three alternatives:

(a) U is decreasing on some interval $(\sigma, +\infty)$. If, in addition $U(+\infty) = -\infty$, then the second option of the lemma is satisfied. So, suppose that $U(+\infty) = U_*$ is finite. Then u satisfies the differential equation

$$u'(t) = au(t) + bU_* + bg(t) + f(t),$$

where $g(t) := U(t) - U_* \geqslant 0$ for $t \geqslant \sigma$, $g(+\infty) = 0$.

If, in addition, $a = 0$, then $b \neq 0$ and

$$u(t) = u(\sigma) + \int_\sigma^t (f(s) - \bar{f})\mathrm{d}s + b\int_\sigma^t (U_* + b^{-1}\bar{f} + g(s))\mathrm{d}s,$$
$$\bar{f} := T^{-1}\int_0^T f(s)\mathrm{d}s.$$

Evidently, $U_* + b^{-1}\bar{f} = 0$ (otherwise $u(+\infty)$ exists and is infinite) so that

$$u(t) = p_1(t) + g_1(t),\ t \geqslant \sigma, \tag{A.1}$$

where

$$g_1(t) = C + b\int_\sigma^t g(s)\mathrm{d}s, \quad p_1(t) = u(\sigma) - C + \int_\sigma^t (f(s) - \bar{f})\mathrm{d}s,$$

C is some constant and g_1 is a monotone function, which is bounded because u and p_1 are bounded. We set $C = -b\int_\sigma^{+\infty} g(s)ds$, implying $g_1(+\infty) = 0$. Moreover,

$$p_1'(t) = f(t) - f(\theta), \quad \min_{s\in\mathbb{R}} f(s) < f(\theta) < \max_{s\in\mathbb{R}} f(s),$$

for some fixed $\theta \in [0, T]$, so that p_1 has exactly two critical points on each half-closed interval of length T. Thus p_1 is a sine-like T-periodic function. However, since $h < T$, this implies that $U(t)$ cannot be monotone, a contradiction.

Consider now the case when $a \neq 0$. Similarly, we find that representation (A.1) is true in this situation, with $g_1(+\infty) = 0$ and p_1 being the unique $T-$periodic solution of the equation

$$x'(t) = ax(t) + f_1(t), \quad f_1(t) := bU_* + f(t). \tag{A.2}$$

This will produce again a contradiction once it is established that the $T-$periodic function p_1 is sine-like. So, set $f_1(T_1) = \max_{\mathbb{R}} f_1(s)$, $f_1(T_2) = \min_{\mathbb{R}} f_1(s)$ for some $T_1 < T_2 < T_1 + T$. First consider $a < 0$, then

$$-\frac{f_1(T_2)}{a} = \int_{-\infty}^t e^{a(t-s)} f_1(T_2)\mathrm{d}s < p_1(t) = \int_{-\infty}^t e^{a(t-s)} f_1(s)\mathrm{d}s < -\frac{f_1(T_1)}{a}.$$

Therefore the graph of the solution $x(t)$, $T_1 \leqslant t \leqslant T_1 + T$, belongs to the rectangle

$$[T_1, T_1 + T] \times (-a^{-1}f_1(T_2), -a^{-1}f_1(T_1))$$

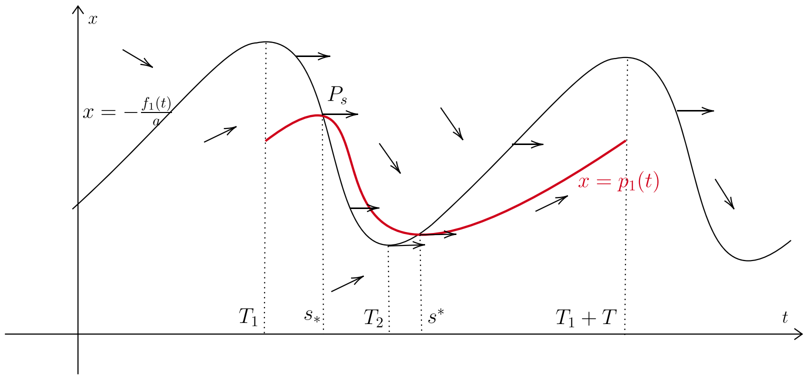


Figure A1. Schematic representation of the zero isocline for (A.2) (black curve) and its unique periodic solution p_1 (red curve).

of the extended phase plane, see figure A1. The zero isocline for (A.2) is given by the graph of $x = -a^{-1}f_1(t)$.

In the open region below this isocline, the solutions of (A.2) are increasing, while they are decreasing above the nullcline. Take any point $P_s = (s, -a^{-1}f_1(s))$ for $s \in (T_1, T_2)$; it is easy to see that each trajectory of (A.2) through P_s is strictly decreasing in the backward direction and therefore has a unique intersection with the zero isocline on $[T_1, s]$. This proves that $x = p_1(t)$ has a unique intersection with $x = -a^{-1}f_1(t)$ on the interval $[T_1, T_2]$ (say, at some point $s_* \in (T_1, T_2)$), it is strictly increasing on $[T_1, s_*]$ and strictly decreasing on some maximal interval $[s_*, s^*]$, where $s^* \in (T_2, T_1 + T)$ and $p_1(s^*) = -a^{-1}f_1(s^*)$. By the same argument as before, we obtain that $x(t) = p_1(t)$ cannot cross the zero isocline for $t \in (s^*, T_1 + T]$ a second time and therefore p_1 is strictly increasing on $[s^*, T_1 + T]$. This means that p_1 has sine-like form.

To complete the analysis of the first alternative, we should consider $a > 0$. This case can be easily reduced to the previous one since the periodic function $q(t) := p_1(-t)$ satisfies the equation $q'(t) = -aq(t) - f_1(-t)$ so that q has sine-like shape.

(b) Next, we consider the alternative when U is increasing on some interval $(\sigma, +\infty)$. Evidently, if U is eventually strictly increasing, then $U(t) = u(t)$ for all sufficiently large values of t. This implies that u satisfies the equation $u'(t) = (a+b)u(t) + f(t)$ possessing a unique $T-$periodic solution $p(t)$ (as we have established in (a), $p(t)$ is sine-like). Therefore $u(t) = ce^{(a+b)t} + p(t)$ for some $c \in \mathbb{R}$ so that $u(t)$ increases only if $a + b > 0$, $c > 0$, with $u(t) \to +\infty$. This is the first option in the statement of lemma 22.

So assume that U is increasing on $(\sigma, +\infty)$ and there exists a sequence of maximal intervals $[a_j, b_j]$, $a_j < b_j < a_{j+1}$, $\lim a_j = +\infty$, such that U is constant on each of them. Then clearly the third option of the lemma is satisfied for each $\tau_2 = a_j$.

(c) Finally, if U is not eventually monotone then there exist $\alpha + 2h < s_1 < s_2 < s_3$ such that $U(s_1) < U(s_2) > U(s_3)$. If $\hat{s}$ is the leftmost point where the absolute maximum of $u(t)$ on the interval $[s_1, s_3]$ is attained, then $\hat{s} \in (s_1, s_3)$ and the third option of the lemma is satisfied with $\tau_2 = \hat{s}$.

This completes the proof of lemma 22. □

Lemma 23. *Assume that the trivial solution of the delay equation*

$$u'(t) = au(t) + b \max_{s \in [t-h, t]} u(s) \tag{A.3}$$

is uniformly asymptotically stable, that is, either of the following conditions holds:

$$ah \leqslant 1 \text{ and } b+a<0 \quad \text{or} \quad ah>1 \text{ and } bh<-\exp(ah-1). \tag{A.4}$$

Then every solution of (4) is bounded on each interval $[r,+\infty)$ *belonging to its domain.*

Proof. First, we notice that, by [43, theorem 2.1], every non-trivial solution of (A.3) is eventually strictly monotone. This implies that the zero solution to (A.3) is uniformly exponentially stable if and only if the characteristic function $z-a-be^{-zh}$ associated to the linear delay-differential equation (13) does not have nonnegative real zeros (hence, the exponential stability of equation (13) implies the uniform exponential stability of (A.3)). It can be proved that this property holds if and only if either of conditions in (A.4) holds, a stability result established in [48].

In the following, we assume that equation (A.3) is uniformly exponentially stable. Then, for every solution $v:\mathbb{R}_+\to\mathbb{R}$ of (A.3) there is a real number μ such that $\mu>2h>0$ and $\|v_\mu\|\leqslant 0.5\|v_{2h}\|$, where $\|\phi\|=\max\{|\phi(s)|: s\in[-h,0]\}$, $v_d(s)=v(d+s), s\in[-h,0]$.

Assume that there is an unbounded solution u of (4). Then there exists a sequence $t_n\to+\infty$ such that $|u(t_n+\mu)|=\max_{r\in[t_n,t_n+\mu]}|u(r)|\to\infty$ as $n\to\infty$. The sequence

$$v^{(n)}(t)=\frac{u(t+t_n)}{\max_{s\in[t_n,t_n+\mu]}|u(s)|},\ t\in[0,\mu],$$

is relatively compact in $C[0,\mu]$, therefore it has a subsequence $\{v^{(n_j)}\}$ uniformly converging to an element $v\in C[0,\mu]$. Finally, v satisfies equation (A.3) on the interval $[h,\mu]$ and $1=|v(\mu)|=\|v_\mu\|=\max_{r\in[0,\mu]}|v(r)|\geqslant\|v_{2h}\|$, a contradiction with the definition of μ. □

Appendix B. Auxiliary results for the regularity of $\mathcal{R}$

We begin by considering the initial value problem $u(\tau)=\tilde{f}(\tau)$ for the ordinary differential equation

$$u'(t)=(a+b)u(t)+f(t). \tag{B.1}$$

Clearly, there exists some $\delta>0$ such that $f(t)-f(\tau)$ does not change sign on each of the open intervals $(\tau-\delta,\tau)$, $(\tau,\delta+\tau)$. A straightforward computation shows that the solution u of the mentioned initial value problem satisfies, for all $0<|t-\tau|<\delta$,

$$\frac{u(t)-u(\tau)}{t-\tau}(f(t)-f(\tau))=\frac{1}{t-\tau}\int_\tau^t e^{(a+b)(t-s)}(f(t)-f(\tau))(f(s)-f(\tau))\mathrm{d}s>0.$$

This relation implies the following result:

Lemma 24. *If* $\tau\in(0,\beta)$ *[respectively,* $\tau\in(\beta,T)$*] then the solution* u *of the initial value problem* $u(\tau)=\tilde{f}(\tau)$ *for (B.1) has a strict local maximum [respectively, strict local minimum] at* τ*. Moreover,* τ *is the unique critical point of* u *in some open neighbourhood of* τ*. If* $\tau=\beta$*, then* $u'(t)>0$ *for all* t *in some punctured neighbourhood of* β*. If* $\tau=T$*, then* $u'(t)<0$ *for all* t *in some punctured neighbourhood of* T*.*

Proof. Suppose, for example, that $\tau=\beta$. Then $u(\tau)=0$ and $u(t)<0$ for $t\in(\tau-\delta,\tau)$. Since $a+b<0$ this implies that $u'(t)>0$ for $t\in(\tau-\delta,\tau)$. Similarly, $u(t)>0$ for $t\in(\tau,\delta+\tau)$. If we suppose that $u'(s_0)=0$ for some $s_0\in(\tau,\delta+\tau)$ then $u(t)$ is strictly decreasing on (β,s_0)

(since $f(t)$ is strictly increasing on the same interval), a contradiction. The other cases can be established in a similar fashion. □

Next, we will analyse the trajectory of the solution $u(s,p)$ on the interval (q,ν^*) (see section 2.2 and figure 1 for the notation related to the definition of $\mathcal{R}$).

Lemma 25. *Assume that* $(\mathbf{M_p})$ *holds for some* $p=\tilde{f}(q)\in K$*. Then there exists a maximal non-empty interval* $(\mu(q),\nu^*)$ *such that* $u(t,p)>u(s,p), s\in[t-h,t)$ *for all* $t\in(\mu(q),\nu^*)$*. In particular,* $u'(t,p)>0$ *(possibly, except for one point* $s=\beta+jT$ *where* $u(s,p)=u'(s,p)=0$*) and* $u(t,p)$ *satisfies (B.1) on* $(\mu(q),\nu^*)$*.*

Proof. Indeed, otherwise there exists an increasing sequence $t_n<\nu^*,\ t_n\to\nu^*$ such that $u(t_n)\leqslant u(s_n):=\max\{u(s),s\in[t_n-h,t_n]\}$. Since $u(\nu^*-h)<u(\nu^*)$, we conclude that there is $\gamma>0$ such that $u(s_n)=\max\{u(s),s\in[\nu^*-h-\gamma,t_n]\}$ for all large n. Furthermore, $s_n\to\nu^*$: indeed, if $s_{n_j}\to\sigma\in[\nu^*-h,\nu^*)$ for some subsequence s_{n_j}, then $u(\sigma)=u(\nu^*)$ that is not allowed by $(\mathbf{M_p})$. Thus $u(s_n)=\max\{u(s),s\in[s_n-h,t_n]\}$.Since $s_n<t_n$, this contradicts the definition of ν^*.

As a consequence, $u(t,p)$ satisfies (B.1) on $(\mu(q),\nu^*)$. By lemma 24, $u'(s,p)=0$ if and only if $s=\beta+jT$ for some integer j and $u(s,p)=\tilde{f}(\beta)=0$. □

Corollary 26. *For each* $p\in K$*, the inequality* $\mathcal{R}(p)<\max\{\tilde{f}(t),\ t\in\mathbb{R}\}$ *holds.*

Proof. Indeed, if $\mathcal{R}(p)=\max\{\tilde{f}(t),\ t\in\mathbb{R}\}\geqslant p$ for some $p\in K$, lemmas 8 and 25 imply that $u(t,p)$ satisfies (B.1) and increases on some left-hand side neighbourhood of $\nu^*=0\pmod T$. However, this contradicts the last assertion of lemma 24. □

Corollary 27. *With the notations of lemma 25,* $\mu(\beta)=\beta$*.*

Proof. Lemma 24 implies that the solution of the initial problem $u(\beta,0)=0$ increases on some right-hand side neighbourhood of β. Therefore the graph of $u(t,0)$ increases until its first intersection at some point $(z,\mathcal{R}(0))$, $z\in(\beta,\beta+T)$, with the decreasing part of the graph of the function $\tilde{f}:[T,\beta+T)\to(0,+\infty)$. For better visualization of this situation, see the graph of the solution $u=u(t,p_0)$ with $p_0=0$ on the left frame of figure C1. □

Now we are in a condition to prove the main results in sections 2.3 and 2.4.

Proof of theorem 9. Due to the continuous dependence of solutions on the initial values, $u'(t,p)$ converges uniformly to $u'(t,p_0)$ on $[\mu(q_0),\nu^*(q_0)]$ as $p\to p_0$. By lemma 4 and corollary 26, $\nu^*(q_0)\in(j_0T,j_0T+\beta)$ for some j_0. Set $\mu_*(q_0)=\max\{j_0T,\mu(q_0)\}$. Then lemma 25 and corollary 27 assure that for every fixed small $\delta>0$ it holds that $u'(t,p)>0$ for all $t\in[\mu_*(q_0)+\delta,\nu^*(q_0)-\delta)$ if p is sufficiently close to p_0. This implies that $u(t,p)>u(s,p)$, $s\in[t-h,t)$, for $t\in(\mu_*(q_0)+\delta,\nu^*(q_0)-\delta)$ and therefore $u(t,p)$ has a local maximum point $\hat{\nu}(p)$ such that $\hat{\nu}(p)\to\nu^*(p_0)$ as $p\to p_0$. In addition, $u(t,p)$ is strictly monotone in some left and in some right neighbourhoods of $\hat{\nu}(p)$ and $u(\hat{\nu}(p),p)=\max\{u(s,p),\ s\in[\hat{\nu}(p)-h,\hat{\nu}(p)]\}$.

Now, since $\beta<h$, by lemma 5, $\hat{\nu}(p)$ is the absolute maximum point of $u(t,p)$ on the interval $(j_0T,\beta+j_0T)$ containing $\nu^*(q_0)$. Therefore, if we suppose that $\mathcal{R}$ has a discontinuity at p_0, it should exist a sequence $p_k=\tilde{f}(q_k)\to p_0=\tilde{f}(q_0)$ and a positive integer $j_1<j_0$ such that $\nu^*(q_k)\in[j_1T,\beta+j_1T)$ and $\nu^*(q_k)\to\mu^*\neq\nu^*(q_0)\pmod T$. Since $u(\nu^*(q_k),p_k)=\max\{u(s,p_k),\ s\in[\nu^*(q_k)-h,\nu^*(q_k)]\}$, in view of the continuous dependence of $u(t,p)$ on the initial data, we conclude that

$$u(\mu^*,p_0)=\max\{u(s,p_0),\ s\in[\mu^*-h,\mu^*]\}.$$

Now, if $\mu^* < \beta + j_1 T$ (i.e. $u(\mu^*, p_0) \neq 0$), then by lemma 5, $u(\mu^*, p_0) = \max\{u(s, p_0),\ s \in [\mu^* - h, \beta + j_1 T]\}$ and $[j_1 T, \beta + j_1 T) \ni \mu^* = \nu^*(q_0) \in [j_0 T, \beta + j_0 T)$, a contradiction.

Therefore $\mu^* = \beta + j_1 T$,

$$\max\{u(s, p_0),\ s \in [\mu^* - h, \mu^*]\} = u(\mu^*, p_0) = u(\beta + j_1 T, p_0) = 0,$$

so that $\mathcal{R}(p_0) = \mathcal{R}(0)$ and $\mathcal{R}(p_k) = u(\nu^*(q_k), p_k) \to u(\mu^*, p_0) = u(\beta + j_1 T, p_0) = 0.$ □

Proof of corollary 11. Note that, for $p = 0$, all conclusions of corollary 11 follow easily from corollary 27 and theorem 9. Thus it suffices to take $p > 0$. Consider the solution $u(t, p)$ on the interval $I_q = (q, q+h]$. If $u(t, p) \leqslant p$ for all $t \in I_q$, and (10) holds, we obtain

$$\begin{aligned} u(q+h, p) &= e^{ah} p + \int_q^{q+h} e^{a(h+q-s)} [bp + f(s)] \mathrm{d}s = \\ &= e^{ah} p + p\frac{b}{a}(e^{ah} - 1) + \int_0^h e^{as} f(q+h-s) \mathrm{d}s \geqslant p. \end{aligned} \tag{B.2}$$

This shows that $u(r, p) = p$ at some leftmost point $r \in (q, q+h]$ where $u'(r, p) \geqslant 0$. If $u'(r, p) = 0$, then $p = u(r, p) = \tilde{f}(r)$, so that $r \in (\beta, T)$. Now, since $u'(t, p) = au(t, p) + bp + f(t)$ for $t \in [q, r]$, the difference $m(t) = u(t, p) - p$ satisfies the equation

$$m'(t) = am(t) + f(t) - f(r), \quad m(r) = 0,\ t \in [q, r].$$

Thus, by the variation of constants formula, for all $t \in (q, r)$,

$$(u(t, p) - p)\exp(-at) = \int_r^t \exp(-as)(f(s) - f(\tau))\mathrm{d}s > 0,$$

proving that $u(t, p) > p$ for $t \in (q, r)$, a contradiction. Thus $u'(r, p) > 0$ so that $u(t, p)$ is increasing on the non-empty interval $(r, \nu^*(q))$, $r = \mu(q)$, and $\mathcal{R}(p) > p$. Clearly, $\nu^*(q) \in (T, \beta + T)$ (recall that $h < T$ and $u(t, p) > p \geqslant 0$ on $(r, \nu^*(q))$) so that graph of the solution $u(t, p)$, $t \in [q, \nu^*(q)]$ does not intersect the set $\{(\beta + jT, 0) : j \geqslant 1\}$. Hence, by theorem 9, $\mathcal{R}$ is continuous at p if (10) holds. □

Proof of corollary 12. We claim that $u(s, p) < u(\nu^*, p)$ for all $s \in [\nu^* - h, \nu^*)$. Indeed, otherwise $\max\{u(s, p), s \in [t-h, t]\} \geqslant \mathcal{R}(p)$ for all $t \in [\nu^* - h, \nu^*]$ so that

$$\begin{aligned} \mathcal{R}(p) &= e^{ah} u(\nu^* - h, p) + \int_{\nu^* - h}^{\nu^*} e^{a(\nu^* - s)} [b \max_{w \in [s-h, s]} u(w, p) + f(s)] \mathrm{d}s \\ &\leqslant e^{ah} \mathcal{R}(p) + \int_{\nu^* - h}^{\nu^*} e^{a(\nu^* - s)} [b\mathcal{R}(p) + f(s)] \mathrm{d}s \\ &= e^{ah} \mathcal{R}(p) + \int_0^h e^{as} [b\mathcal{R}(p) + f(\nu^* - s)] \mathrm{d}s < \mathcal{R}(p), \end{aligned}$$

a contradiction. Thus condition $(\mathbf{M_p})$ is satisfied and theorem 9 implies the continuity of $\mathcal{R}$ at p once $\mathcal{R}(p) \neq \mathcal{R}(0)$. □

Proof of corollary 13. In view of corollary 27, the graph of $u(t, 0)$ increases until its first intersection at some point $(z, \mathcal{R}(0))$, $z \in (T, \beta + T)$ with the decreasing part of the graph Γ of $\tilde{f} : (T, \beta + T) \to (0, +\infty)$ (note that $z > T$ due to corollary 26)). Since $u(t, 0)$ has a strict maximum at z, it follows that

$$u(z, 0) > u(s, 0), \text{ for all } s \in [z - h, z + \epsilon] \setminus \{z\},$$

for all small $\varepsilon > 0$, and hence we conclude that for $p > 0$ close to 0 the solutions $u(t,p)$ have also strict maxima at some points close to z. Therefore the continuous dependence of $u(t,p)$ on the variables (t,p) implies that the point $(\nu^*(q), \mathcal{R}(p))$ changes continuously belonging to Γ while $\mathcal{R}(p) \geqslant p$. By the continuity of $u(t,p)$, our argument still works when p belongs to some right-hand neighbourhood of the least fixed point p^*.

Finally, if $(\mathbf{M_p})$ holds for all $p \in K$, then condition $\mathcal{R}(p) \geqslant p$ can be omitted in the above argumentation and $\mathcal{R}$ has a continuous graph until the first intersection of its closure with the real axis at some point c, where $\mathcal{R}(c-) = 0$, $\mathcal{R}(c) = \mathcal{R}(0) > 0$.

□

Proof of theorem 17. By theorem 9, $\mathcal{R}$ is continuous at p if $u(t,p)$ is U-shaped and if the graph of $u(t,p)$ does not intersect the set $\{(\beta + kT, 0) : k \in \mathbb{N}, k \geqslant 1\}$ on the interval $(q, \nu^*(q))$. Suppose that $\mathcal{R}$ is continuous at a point $\bar{p} = p(\bar{q})$, with $\bar{q} < \beta_1$. We claim that $\mathcal{R}'(\bar{p}) = \Delta(\bar{q})$. Indeed, since $\bar{q} < \beta_1$, we have that $\lambda(\bar{q}) = \bar{q} + h < \mu(\bar{q})$ and therefore, for all p close to $\bar{p}$, it holds

$$\mathcal{R}(p) = u(\nu^*(q),p) = u(\mu(q),p)e^{(a+b)(\nu^*(q)-\mu(q))} + \int_{\mu(q)}^{\nu^*(q)} e^{(a+b)(\nu^*(q)-s)}f(s)\mathrm{d}s,$$

where, for brevity, we preserve the notations $\nu^*(q), \mu(q)$ for the composite functions $\nu^* \circ q(p)$ and $\mu \circ q(p)$. As we have mentioned, the equality $u'(\nu^*(q),p) = 0$ eliminates the dependence of $\mathcal{R}'(p)$ on $\partial_q \nu^*(q)$ and leads to the following expression:

$$\mathcal{R}'(p) = e^{(a+b)(\nu^*(q)-\mu(q))}\left(\partial_p(u(\mu(q),p))\right) - \left((a+b)u(\mu(q),p) + f(\mu(q))\partial_p\mu(q)\right),$$

Now, we have to calculate the partial derivative $\partial_p(u(\mu(q),p))$. A key observation here is that, since $u(t,p)$ is U-shaped, it satisfies the following delay differential equation on $[q,\mu(q)]$:

$$u'(t) = au(t) + bu(t-h) + f(t),\ t \in [q,\mu(q)], \quad u(s) = p,\ s \in [q-h,q].$$

Thus, using the above mentioned fundamental solution v and taking into account the initial condition $u(s,p) = p$, $s \in [q-h,q]$, from [22, section 1.6] we obtain that

$$u(\mu(q),p) = v(\mu(q)-q)p + bp\int_{q-h}^{q} v(\mu(q)-s-h)\mathrm{d}s + \int_{q}^{\mu(q)} v(\mu(q)-s)f(s)\mathrm{d}s.$$

As a consequence, since $u(\mu(q),p) = u(\mu(q)-h,p)$, $v'(t) = av(t) + bv(t-h)$, $t > 0$, we find that

$$\begin{aligned}
\partial_p(u(\mu(q),p)) &= u'(\mu(q),p)\partial_p\mu(q) + v(\mu(q)-q) + b\int_{q-h}^{q} v(\mu(q)-s-h)\mathrm{d}s \\
&\quad + (-v'(\mu(q)-q)p - v(\mu(q)-q)f(q) \\
&\quad + bpv(\mu(q)-q-h) - bpv(\mu(q)-q))\partial_p q \\
&= u'(\mu(q),p)\partial_p\mu(q) + v(\mu(q)-q) + b\int_{q-h}^{q} v(\mu(q)-s-h)\mathrm{d}s \\
&= [(a+b)u(\mu(q),p) + f(\mu(q))]\,\partial_p\mu(q) + v(\mu(q)-q) \\
&\quad + b\int_{q-h}^{q} v(\mu(q)-s-h)\mathrm{d}s.
\end{aligned}$$

In this way,

$$\mathcal{R}'(p) = e^{(a+b)(\nu^*(q)-\mu(q))}\left[v(\mu(q)-q) + b\int_{q-h}^{q} v(\mu(q)-s-h)\mathrm{d}s\right]$$
$$= \Delta(q). \tag{B.3}$$

Next, integrating equation (12) we find that $\Delta(q)$ is a combination of some elementary functions depending on a, b, h, $\lambda(q)$, $\mu(q)$ and $\nu^*(q)$. The continuous dependence of $\lambda(q)$, $\mu(q)$ and $\nu^*(q)$ on q belonging to some small neighbourhood $\mathcal{O}$ of $\bar{q}$ implies the continuity of $\mathcal{R}'(p) = \Delta(q)$ in $\mathcal{O}$.

Observe also that the sign of $\mathcal{R}'(p)$ is completely defined by the factor given in brackets in (B.3). In view of the U-shaped form of $u(t,p)$, the function $\mu(q)$ is C^1-smooth so that the aforementioned factor depends continuously on p. Differently, $\nu^*(q)$ is discontinuous at the pre images of the discontinuity points $p_j = \tilde{f}(q_j)$ of $\mathcal{R}$. Assuming that there exist $\mathcal{R}'(p_j+)$ and $\mathcal{R}'(p_j-)$, we find immediately that

$$\mathcal{R}'(p_j+) = \mathcal{R}'(p_j-)e^{(a+b)(\nu^*(\beta)-\beta)} < \mathcal{R}'(p_j-).$$

By corollary 13, either $\mathcal{R}$ is continuous on K or there exists a leftmost discontinuity point p_1. In the first case, $\mathcal{R}$ has a unique critical point $\tilde{f}(\tau)$ on K and $\nu^*(q) < T+\beta$ for all q. In the second case, $\mathcal{R}$ is continuous and strictly decreasing on $[\tilde{f}(\tau), p_1)$, with $\mathcal{R}(p_1-) = 0$, $\mathcal{R}(0) = \mathcal{R}(p_1)$.

Next, we claim that $\nu^*(q) > \beta + T$ for $p > p_1$. Indeed, if $\nu^*(\hat{q}) < \beta + T$ for some $\hat{p} = \tilde{f}(\hat{q}) > p_1$, then the negativity of $\mathcal{R}'(p) = \Delta(q)$ for $q \in [\hat{q}, q_0]$, yields $\mathcal{R}(p) \geqslant \mathcal{R}(\hat{p}) > 0$ for all $p \in [\tilde{f}(q_0), \hat{p}]$. Since $p_1 \in [\tilde{f}(q_0), \hat{p}]$, this contradicts the property $\mathcal{R}(p_1-) = 0$. Therefore, considering the U-shaped form of $u(t,p)$ and the inequality $\mathcal{R}'(p) < 0$, we conclude that the graph of $u(t,p)$ does not contain the point $(\beta + T, 0)$ for $p > p_1$. This allows us to repeat the argumentation of corollary 13 for the case when $\nu^*(q) \in (2T, \beta + 2T)$. In particular, we obtain that $\mathcal{R}$ is continuous and strictly decreasing on some maximal open right neighbourhood $\mathcal{O}_1$ of p_1 and that, if $p_2 := \sup \mathcal{O}_1$ is an interior point of K, then $\mathcal{R}(p_2-) = 0$, $\mathcal{R}(0) = \mathcal{R}(p_2)$.

By applying repeatedly the above procedure, we construct the sequence $\{p_j\}$ with the properties mentioned in the statement of the theorem. □

Appendix C. Proof of theorem 20

Here, we present a proof of theorem 20 based on the analysis of the explicit formulae for the solutions of the initial value problem

$$u'(t) = au(t) - u(t-h) + 1 - \sin t,\ u(s) \equiv p$$
$$= (1-\sin q)/d,\ s \in [q-h, q], \tag{C.1}$$

where $a = 0.32$, $h = 3\pi/2$, $d = 0.68$ and $q \in [-0.5\pi, 0.4]$ (note that, by example 16, $\beta_1 \approx 0.39289 < 0.4$). For our purposes, it suffices to integrate (C.1) on two steps: $[q, q+h]$ and $[q+h, q+2h]$. Equation (C.1) is linear inhomogeneous so that the following simple observation will be used repeatedly in the sequel: the unique periodic solution $p(t)$ of the ordinary differential equation $u'(t) = au(t) + k\sin(t+\varphi)$ has the form

$$u(t) = -\frac{k}{\sqrt{a^2+1}}\sin(t+\varphi+\theta_0),\ \text{where } \theta_0 := \arcsin\frac{1}{\sqrt{a^2+1}}.$$

Using this observation and integrating (C.1) on $[q, q+h]$, we easily find that

$$u(t) = C_0 \sin(t+\theta_0) + C_1 + C_2 e^{a(t-q)}, \text{ where } C_0 = \frac{1}{\sqrt{a^2+1}}, \ C_1 = \frac{p-1}{a}, \quad \text{(C.2)}$$

$$C_2 = p - C_1 - C_0 \sin(q+\theta_0).$$

Hence, solving (C.1) on $[q+h, q+2h]$ amounts to the integration of the linear inhomogeneous ordinary differential equation

$$u'(t) = au(t) - (C_0 \sin(t-h+\theta_0) + C_1 + C_2 e^{a(t-h-q)}) + 1 - \sin t, \quad \text{(C.3)}$$

subject to the initial condition

$$u(q+h) = -C_0 \cos(q+\theta_0) + C_1 + C_2 e^{ah} =: C_3. \quad \text{(C.4)}$$

The solution of (C.3) and (C.4) is given by

$$u(t) = C_0^* \cos(t+2\theta_0) + C_0 \sin(t+\theta_0) + C_1^* - C_2(t-h-q)e^{a(t-h-q)} + C_2^* e^{a(t-h-q)}, \quad \text{(C.5)}$$

where

$$C_0^* = \frac{1}{a^2+1}, \ C_1^* = \frac{C_1 - 1}{a}, \ C_2^* = C_3 - C_0^* \sin(q+2\theta_0) + C_0 \cos(q+\theta_0) - C_1^*.$$

This implies that the first derivative $u'(t)$ is an analytic function of the variables $q \in [-0.5\pi, 0.4]$ and $t \in [q+h, q+2h]$:

$$u'(t) = -C_0^* \sin(t+2\theta_0) + C_0 \cos(t+\theta_0) + [-C_2 a(t-h-q) + (C_2^* a - C_2)] e^{a(t-h-q)}.$$

Note also that

$$u(q+2h) - u(q+h) = -C_0^* \cos(q+2\theta_0) - C_0 \sin(q+\theta_0) + C_1^* - C_2 h e^{ah} + C_2^* e^{ah} - C_3.$$

Lemma 28. *For all $q \in [0.105, 0.4]$, $t \in [q+h, q+2h]$, it holds that $u'(t) > 0$. Furthermore, $u(q+2h) - u(q+h) > 0$ for all $q \in [-0.12, 0.4]$.*

Proof. It is convenient to introduce the new variable $s = t - q - h \in [0, h] = [0, 1.5\pi]$. Then we have to evaluate the elementary function

$$\Psi(s,q) = C_0 \sin(s+q+\theta_0) + C_0^* \cos(s+q+2\theta_0) + [-C_2 as + (C_2^* a - C_2)] e^{as}$$

on the rectangle $\Pi = [0, 1.5\pi] \times [0.105, 0.4]$. Since $\min\{\Psi(s,q), (s,q) \in \Pi\} = 0.0086\ldots$, the first assertion of the lemma is proved. Similarly, the second conclusion follows from the computation of $\min\{u(q+2h) - u(q+h), \ q \in [-0.12, 0.4]\} \approx 0.02057$. □

Since $[0, 1.316] \subseteq [0, \tilde{f}(0.105)]$ and the inequality $u'(t) > 0$, $t \in [q+h, q+2h]$, guarantees that the only critical point of $u(t)$ on the interval $[q, q+h]$ is a minimum point and that $\mu(q) < \nu^*(q)$, we obtain the following result:

Corollary 29. *for all $p \in [0, 1.316]$, the solution $u(t,p)$ is U-shaped on the interval $(q, \nu^*(q))$ (see figure C1). Moreover, $\mu(q) - q < 3\pi$ so that, by remark 19, it holds $\mathcal{R}'(p+) < 0$ for all $p \in (\tilde{f}(q_0), 1.316]$, where $q_0 \approx 1.1845$ is computed in example 16.*

Lemma 30. *The graph of $u(t,p)$ does not contain the point $(2.5\pi, 0)$ and condition $(\mathbf{M_p})$ is satisfied whenever $p \in \mathcal{I} = [1.26, 2/0.68]$.*

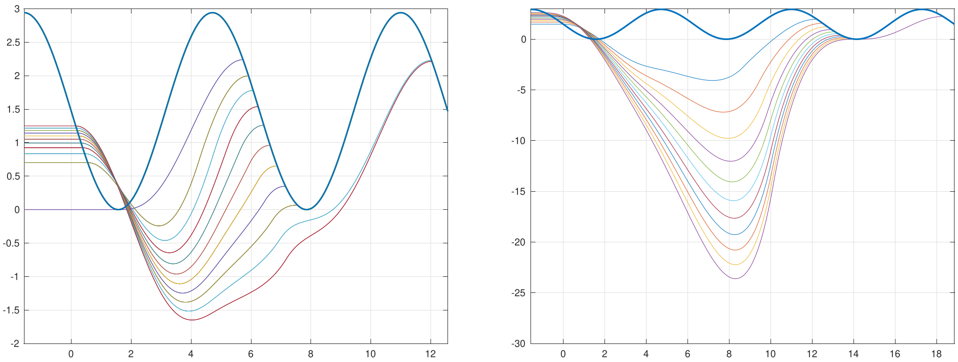


Figure C1. Graphs of solutions $u = u(t,p_j)$ on $t \in [q_j, \nu^*(q_j)]$ together with the sine-like function $\tilde{f}$. On the left: $p_j = \tilde{f}(q_j) = 0.125\sqrt[4]{1000j}$, $j = 0,\ldots,10$. On the right: $p_j = \tilde{f}(q_j) = 1 + 0.125\sqrt{15i}$, $i = 1,\ldots 11$.

Proof. First, note that $u(t,p) = u(t),\ t \in [q, q+1.5\pi]$, for all $p \in \mathcal{I}$. In particular, $u(t)$ has a unique critical point (global minimum point) on the interval $(q, q+1.5\pi]$ so that $u(t) < 0$ on $[\pi, q+1.5\pi]$ if $u(\pi) < 0$ and $u(q+1.5\pi) = C_3 < 0$. It is easy to check that these inequalities hold for all $q \in [-0.5\pi, 0.15]$.

Next, for all $t \in [q+1.5\pi, q+3\pi]$, we find that

$$u'(t,p) = 0.32u(t,p) - U(t,p) + 1 - \sin t \leqslant 0.32u(t,p) - u(t-h,p) + 1 - \sin t.$$

Thus a standard comparison argument shows that $u(t,p) \leqslant u(t)$, $t \in [q+1.5\pi, q+3\pi]$, where $u(t)$ is given by (C.5). Now, setting $s = t - q - h \in [0,h] = [0, 1.5\pi]$, we present $u(t)$ as

$$\Phi(s,q) = C_0^* \sin(s+q+2\theta_0) - C_0\cos(s+q+\theta_0) + C_1^* + [-C_2 s + C_2^*]\,e^{as}.$$

Then the inequality $u(t) < 0,\ t \in [q+1.5\pi, 2.5\pi]$, holds for all $q \in [-0.5\pi, 0.15]$ if $\Phi(s,q) < 0$ on the set $\Pi_2 = \{(s,q) : s+q \leqslant \pi, q \in [-0.5\pi, 0.15], s \geqslant 0\}$. Now, we find that

$$\max\{\Phi(s,q),\ (s,q) \in \Pi_2\} = -0.0615\ldots < 0.$$

Thus $u(t,p) < 0$ for all $t \in [\pi, 2.5\pi]$ whenever $q \in [-0.5\pi, 0.15]$. This proves the first assertion of the lemma. Finally, since $\mathcal{R}(p) > 0$, condition $(\mathbf{M_p})$ is satisfied for each $p \in [\tilde{f}(0.15), \tilde{f}(-\pi/2)] = [1.25\ldots, 2/0.68]$. □

Now we are in a position to prove theorem 20.

Proof of theorem 20. Since the computation of $\mathcal{R}(\bar{p})$ for each given $\bar{p} = \tilde{f}(\bar{q}) \in K$ amounts to the explicit integration of some first order inhomogeneous linear differential equations with constant coefficients on a finite interval $[\bar{q}, \nu^*(\bar{q})]$, and finding zeros of simple elementary functions on the same interval, we will assume that the value of $\mathcal{R}(\bar{p})$ can be found with the required accuracy. For example, the value of $\mathcal{R}(0) \approx 2.2$ can be found by solving the equation $u(t) = \tilde{f}(t)$ on the interval $[1.5\pi, 2.5\pi]$, where $u(t)$ is given by (C.2). In a similar way, we can compute the value of $\mathcal{R}(\mathcal{R}(0)) \approx 0.45$.

Next, corollary 29 and remark 19 allow to apply theorem 17 on the q-interval $(\alpha, \beta] = (0.105, 0.5\pi]$. In order to prove that the associated p-interval $[\tilde{f}(\beta), \tilde{f}(\alpha)) = [0, 1.3\ldots)$ contains one point p_1 of discontinuity, it suffices to take q such that $\tilde{f}(q) = 1.25$ and to check that

$\nu^*(q) \in (3.5\pi, 4.5\pi)$ (such q corresponds to q_{10} in the left frame of figure C1). This establishes all stated properties of $\mathcal{R}$ on the interval $[0, 1.316]$. Concerning the computation of the approximate value of p_1, note that $p_1 \in (a_1, a_2) \subset [0, 1.316]$ if $\mathcal{R}(a_1) < \mathcal{R}(a_2)$ (particularly, we obtain immediately that $p_1 \in (0.9, 1.25)$ while a more accurate similar estimate implies that $p_1 \approx 1.1$).

Finally, lemma 30 shows that condition $(\mathbf{M_p})$ is satisfied for all $p \in \mathcal{I} = [1.26, 2/0.68]$. Then theorem 9 and the proof of corollary 13 imply that the restriction $\mathcal{R} : [p_1, p_2) \to K$ has continuous graph until the first eventual intersection of its closure with the real axis at some point p_2, where $\mathcal{R}(p_2-) = 0$, $\mathcal{R}(p_2) = \mathcal{R}(0) > 0$. In order to establish the existence of such p_2 and find its approximate value, it is enough to take $p_i^* = \tilde{f}(q_i^*) = 1 + 0.125\sqrt{15k} \in \{2.53\ldots, 2.60\ldots\}$, with $k = 10, 11$, and to note, after standard numerical-analytical estimations, that $\nu^*(q_{11}^*) \in (5.5\pi, 6.5\pi)$ while $\nu^*(q_{10}^*) \in (3.5\pi, 4.5\pi)$. See the right frame of figure C1. □

ORCID iD

Eduardo Liz https://orcid.org/0000-0003-1975-5182

References

[1] Bainov D D and Hristova S 2011 *Differential Equations With Maxima* (*CRC Pure and Applied Mathematics*) (Boca Raton, FL: Chapman & Hall)
[2] Baker C T H 2010 Development and application of Halanay-type theory: evolutionary differential and difference equations with time lag *J. Comput. Appl. Math.* **234** 2663–82
[3] Bantsur N and Trofimchuk E 1998 Existence and stability of the periodic and almost periodic solutions of quasilinear systems with maxima *Ukr. Math. J.* **50** 847–56
[4] Bentham J 1948 *An Introduction to the Principle of Morals and Legislations (1789)* (Oxford: Blackwell)
[5] Blanchflower D G and Oswald A J 2008 Is well-being U-shaped over the life cycle? *Soc. Sci. Med.* **66** 1733–49
[6] Brickman P, Coates D and Janoff-Bulman R 1978 Lottery winners and accident victims: is happiness relative? *J. Pers. Soc. Psychol.* **36** 917–27
[7] Brunovský P, Erdélyi A and Walther H-O 2004 On a model of a currency exchange rate - local stability and periodic solutions *J. Dyn. Differ. Equ.* **16** 393–432
[8] Easterlin R A 2003 Explaining happiness *Proc. Natl Acad. Sci.* **100** 11176–83
[9] Edgeworth F Y 1881 *Mathematical Psychics: An Essay on the Application of Mathematics to the Moral Sciences* (reprinted 1967) (New York: M. Kelly)
[10] Erdélyi A 2003 A delay differential equation model of oscillations of exchange rates *PhD Thesis* Bratislava
[11] Faria T, Liz E, Oliveira J J and Trofimchuk S 2005 On a generalized Yorke condition for scalar delayed population models *Discrete Contin. Dyn. Syst.* A **12** 481–500
[12] Franke R 2018 Reviving Kalecki's business cycle model in a growth context *J. Econ. Dyn. Control* **91** 157–71
[13] Fredrickson B L 2000 Extracting meaning from past affective experiences: the importance of peaks, ends and specific emotions *Cogn. Emot.* **14** 577–606
[14] Fredrickson B L and Kahneman D 1993 Duration neglect in retrospective evaluations of affective episodes *J. Pers. Soc. Psychol.* **65** 45–55

[15] Frey B S and Stutzer A 2000 Maximising happiness? *Ger. Econ. Rev.* **1** 145–67
[16] Frey B S and Stutzer A 2002 What can economists learn from happiness research? *J. Econ. Lit.* **40** 402–35
[17] Garab A, Kovács V and Krisztin T 2016 Global stability of a price model with multiple delays *Discrete Contin. Dyn. Syst.* A **36** 6855–71
[18] Gilbert D 2006 *Stumbling on Happiness* (New York: Knopf)
[19] Graham L and Oswald A J 2010 Hedonic capital, adaptation and resilience *J. Econ. Behav. Organ.* **76** 372–84
[20] Gurney W, Blythe S and Nisbet R 1980 Nicholson's blowflies revisited *Nature* **287** 17–21
[21] Halanay A 1966 *Differential Equations: Stability, Oscillations, Time Lags* (New York: Academic)
[22] Hale J K and Verduyn Lunel S M 1993 *Introduction to Functional Differential Equations, Applied Mathematical Sciences* vol 99 (New York: Springer)
[23] Huijer H 2020 Tracking happiness (available at: www.trackinghappiness.com/happiness-diary-review-2020/) (Accessed January 2022)
[24] Hutchinson G E 1948 Circular causal systems in ecology *Ann. New York Acad. Sci.* **50** 221–46
[25] Ivanov A, Liz E and Trofimchuk S 2002 Halanay inequality, Yorke 3/2 stability criterion and differential equations with maxima *Tohoku Math. J.* **54** 277–95
[26] Kahneman D 1999 Objective Happiness *Well-Being: The Foundations of Hedonic Psychology* ed D Kahneman, E Diener and N Schwarz (New York: The Russell Sage Foundation) ch 1, pp 3–25
[27] Kahneman D, Fredrickson B L, Schreiber C A and Redelmeier D A 1993 When more pain is preferred to less: adding a better end *Psychol. Sci.* **4** 401–5
[28] Kahneman D, Wakker P P and Sarin R 1997 Back to Bentham? Explorations of experienced utility *Q. J. Econ.* **112** 375–405
[29] Kahneman D and Tversky A (eds) 2000 *Choices, Values and Frames* (Cambridge: Cambridge University Press)
[30] Kalecki M 1935 A macroeconomic theory of the business cycle *Econometrica* **3** 327–44
[31] Katok A and Hasselblatt B 1995 *Introduction to the Modern Theory of Dynamical Systems, Encyclopedia of Mathematics and its Applications* vol 54 (Cambridge: Cambridge University Press)
[32] Keller A A 2011 *Time-Delay Systems: With Applications to Economic Dynamics and Control* (Saarbrücken: LAP LAMBERT Academic Publishing)
[33] Kolyada S, Misiurewicz M and Snoha L 1999 Topological entropy of nonautonomous piecewise monotone dynamical systems on the interval *Fundam. Math.* **160** 161–81
[34] Krisztin T 1991 On stability properties for one-dimensional functional differential equations *Funkc. Ekvacioj Ser. Int.* **34** 241–56
[35] Liz E, Tkachenko V and Trofimchuk S 2003 A global stability criterion for scalar functional differential equations *SIAM J. Math. Anal.* **35** 596–622
[36] Liz E, Tkachenko V and Trofimchuk S 2003 Yorke and Wright 3/2-stability theorems from a unified point of view *Discrete Contin. Dyn. Syst.* **2003** 580–9
[37] Liz E and Trofimchuk S 2022 On a dynamical model of happiness (in preparation)
[38] Magomedov A R 1991 *Ordinary Differential Equations With Maxima* (Baku: Elm) (in Russian)
[39] Myshkis A D 1977 On certain problems in the theory of differential equations with deviating argument *Russ. Math. Surv.* **32** 181–213
[40] Myshkis A D 2007 *Soviet Mathematicians: My Memories* (Moscow: LKI Publ.) (in Russian)
[41] Ng Y K 1999 Utility, informed preference, or happiness: following Harsanyi's argument to its logical conclusion *Soc. Choice Welf.* **16** 197–216
[42] Petukhov V 1964 Questions about qualitative investigations of differential equations with "maxima" *Izv. Vyssh. Uchebn. Zaved. Mat.* **3** 116–19 (in Russian)
[43] Pinto M and Trofimchuk S 2000 Stability and existence of multiple periodic solutions for a quasilinear differential equation with maxima *Proc. R. Soc.* A **130** 1103–18
[44] Samoilenko A, Trofimchuk E and Bantsur N 1998 Periodic and almost periodic solutions of the systems of differential equations with maxima *Rep. Natl Acad. Sci. Ukr.* **1** 47–52 (in Ukrainian)
[45] Sarafova G H and Bainov D D 1981 Application of A. M. Samoilenko's numerical-analytic method to the investigation of periodic linear differential equations with maxima *Rev. Roumaine Sci. Tech. Ser. Mec. Appl.* **26** 595–603 (in Russian)
[46] Sprott J C 2005 Dynamical models of happiness *Nonlinear Dyn. Psychol. Life Sci.* **9** 23–36

[47] Touboul J, Romagnoni A and Schwartz R 2017 On the dynamic interplay between positive and negative affects *Neural Comput.* **29** 897–936
[48] Voulov H and Bainov D 1991 On the asymptotic stability of differential equations with 'maxima *Rend. Circ. Mat. Palermo* **40** 385–420
[49] Walther H-O 2020 The impact on mathematics of the paper "Oscillation and chaos in physiological control systems", by Mackey and Glass in Science, 1977 (arXiv:2001.09010)
[50] Yorke J A 1970 Asymptotic stability for one dimensional differential-delay equations *J. Differ. Equ.* **7** 189–202