

ON A DYNAMICAL MODEL OF HAPPINESS*

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Abstract. It is now recognized that the personal well-being of an individual can be evaluated numerically. The related utility (“happiness”) profile would give at each instant t the degree $u(t)$ of happiness. The moment-based approach to the evaluation of happiness introduced by the Nobel laureate Daniel Kahneman establishes that the experienced utility of an episode can be derived from real-time measures of the pleasure and pain that the subject experienced during that episode. Since these evaluations consist of two types of utility concepts: instant utility and remembered utility, a dynamic model of happiness based on this approach must be defined by a delay differential equation. Furthermore, the application of the peak-end rule leads to a class of delay-differential equations called differential equations with maxima. We propose a dynamical model for happiness based on differential equations with maxima and provide results which shed some new light on important experimental observations. In particular, our model supports the U-shaped profile of the age-happiness curve, which is a widely observed pattern: well-being is high in youth, falls to a minimum in midlife (midlife crisis), and rises again in old age.

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1. INTRODUCTION

This paper aims at dealing with an evolutionary model for happiness. We recognize that this target is tremendously ambitious, because even the proper notion of happiness, and how to measure it, is not clear yet [17]. Anyway, we argue that a class of functional differential equations is suitable as a new mathematical model for the evolution of happiness. We hope the reader will find the topic interesting and the paper will motivate her/him to devote some more time to think about this important (perhaps the most important) topic in our lives. As Dolan [14] claims, “*the pursuit of happiness is a noble and very serious objective for us all,*” or as emphasized in [20], “*The United States Declaration of Independence of 1776 takes it as a self-evident truth that the pursuit of happiness is an unalienable right, comparable to life and liberty.*”

To begin with, we need a definition of happiness that permits to evaluate it. Efforts in this direction go back at least to Bentham’s book published in 1789 [6]. He introduced the concept of (hedonic) utility to measure pleasure and pain, which are, in his words, the *sovereign masters* that govern mankind. He defines utility as

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follows: “*By utility, is meant that property in any object, whereby it tends to produce benefit, advantage, pleasure, good or happiness (all this in the present case comes to the same thing).*” Chapter IV in Bentham’s book is devoted to measure the value (utility) of pleasure or pain, according to some properties as its intensity, duration, certainty, proximity, fecundity (the chance it has of being followed by other pleasures), and purity (the chance it has of not being followed by a pain). Later, Edgeworth [15] imagined the hedonimeter: “*Let there be granted to the science of pleasure what is granted to the science of energy; to imagine an ideally perfect instrument, a psychophysical machine, continually registering the height of pleasure experienced by an individual.*” And he also poetically suggested the idea of units of happiness: “*We cannot count the golden sands of life; we cannot number the ‘innumerable smile’ of seas of love; but we seem to be capable of observing that there is here a greater, there a less, multitude of pleasure-units, mass of happiness; and that is enough.*”

Since this pioneering work, there has been a very intense research on happiness, with contributions of psychologists, economists, sociologists and others, which led to an extensive literature on the topic (in many cases related with economics [20]), and the foundation of a good number of well established journals devoted to happiness studies. There are many definitions of happiness (see, *e.g.*, Feldman’s book [17]), and still new proposals have recently arrived; for example, Dolan [14] adds purpose to Bentham’s list of utility and defines happiness as “*experiences of pleasure and purpose over time.*” We mainly follow here the ideas of Daniel Kahneman [23, 24], who greatly influenced the development of “positive psychology” [17], and was awarded the 2002 Nobel Memorial Prize in Economic Sciences [26].

Kahneman introduced in [23] the concept of *objective happiness*, and his paper begins with the question: “*How happy was Helen in March?*” To ask this question, Kahneman chooses a basic unit of analysis, which is called *instant utility*; he refers to utility in the sense of Bentham as *experienced utility*, to distinguish it from the usual meaning in economics and decision theory [28]. What should a concept of instant utility include? Kahneman’s theory assumes that each moment is uniquely characterized by a value on a Good/Bad (GB) dimension. The GB value can be positive or negative (or zero, neither good nor bad), and its absolute value measures the intensity of the experience. Supported by different approaches from the introspective to the biochemical, Kahneman ([23], p. 8) argues that prospects are reasonably good to get a continuous evaluative process, see [1, 29] for further discussion on measures of emotion. Analyzing Kahneman’s theory, Chapter 3 of Feldman [17] interprets Helen’s happiness in March as a curve in the (t, u) plane, where t is time and u is the instant utility at t . As interpreted by Fredrickson [18]: “*So if, to borrow Kahneman’s example, you wanted to determine how happy was Helen in March, you would calculate the average height of the rescaled utility profile constructed from the momentary good-bad ratings Helen made during that month.*”

Once we accept that a time/happiness curve (or utility profile) can be plotted – see also Section 6.3 of [17], one is tempted to construct an evolutionary model for happiness, that would be able to make predictions on the behavior of future happiness. We notice that this idea is not new; see, for example, the dynamic models described by Graham and Oswald [21] and Sprott [32], the mathematical model of emotional balance dynamics by Touboul *et al.* [33], or the dynamic maximization model of Sherman *et al.* [31]. The latter also uses Kahneman’s concept of experienced utility.

The concepts of experienced utility and objective happiness lead to an evaluation of happiness (*evaluation by moments*). The principle of evaluation by moments asserts that people evaluate the utility of an episode by retrieving or constructing a representative moment and by evaluating the utility of that moment. We next list the main ingredients of this evaluation procedure that we shall use; they are taken from [18, 19, 24, 25].

- Episodes: “*An episode is a connected time interval described by its temporal coordinates. The utility profile of an episode assigns a level of instant utility to each time point.*”
- Instant utility and remembered utility: “*Pleasure and pain are attributes of a moment of experience, but the outcomes that people value extend over time. It is therefore necessary to establish a concept of experienced utility that applies to episodes. We have distinguished two descriptive notions of experienced utility: instant utility is the pleasure or distress of the moment; remembered utility is the retrospective evaluation of an episode.*”

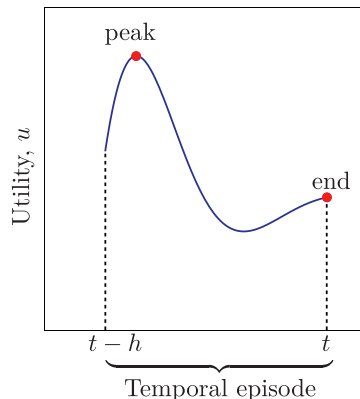


FIGURE 1. Representation of the utility profile of a temporal episode of length h , with its distinctive moments: peak and end.

- Duration neglect: “*The duration of experiences has little or no independent effect on their remembered utility. The remembered utility of an episode is determined by constructing a composite representative moment and assessing the utility of that moment.*”
- The peak-end rule: “*our predictions about future happiness are often based on our past affective experiences. . . people’s global evaluations of past affective episodes can be well predicted by the affect experienced during just two moments: the moment of the most extreme affect experienced during the episode (peak) and the affect experienced at the end.*”

The peak-end rule is supported by empirical evidence and finds applications in many contexts [11, 13, 18, 19, 25, 27]. From that evidence, Fredrickson [18] arrives at the following conclusion, which serves as a summary: “*when people evaluate and make decisions based on certain types of past affective episodes, a few select moments can serve as proxies: The moment of peak affect intensity and the ending. The duration of the episode hardly matters at all.*” In Figure 1 we represent these two crucial points of an episode.

To continue with Kahneman’s example, we introduce our model talking about Helen’s happiness. Assuming that the evolution of Helen’s happiness depends on its current state (instant utility) and her recent past affective episodes (remembered utility), and accepting the peak-end rule, a suitable model for the evolution of happiness belongs to a class of functional differential equations called *differential equations with maxima* [4]:

$$u'(t) = F\left(t, u(t), \max_{s \in [t-h, t]} u(s)\right), \quad (1.1)$$

where $F : [0, \infty) \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous map and the delay h is chosen so as it captures a temporal episode of reasonable length (not too short, not too long). The equation is generally nonautonomous, and the dependence on t comes in the form of external stimuli, which influence happiness. We recall that, although different temporal episodes have different length, the duration neglect principle allows us for certain flexibility. This does not mean that the model completely ignores the duration of the experience utility; actually, we will see how the dynamics of (1.1) and the profile of the solutions strongly depend on the delay h . It can be argued that a happy moment during one second should not have the same impact on current utility than a happy event experienced for one day. This is probably a drawback of the model due to the assumption of the moment-based approach. Another drawback of this model is that it ignores utility from anticipation (sometimes referred to as *forward effect* [16] or *predicted utility* [28]), which also has an impact on current utility. However, a mathematical theory to deal with differential equations involving delayed and advanced arguments is not well developed.

The rest of the paper is organized as follows: in Section 2, we consider the simplest form of (1.1), choosing $F(t, x, y) = ax + by$, with real constants a, b . In some sense, the model in this case does not take into account external stimuli. Our main result (Thm. 2.4) provides sufficient conditions in terms of parameters a, b and h to ensure that solutions exhibit a single minimum point, in agreement with the U-shaped age-happiness curve, which is a widely observed pattern: well-being is high in youth, falls to a minimum in midlife (midlife crisis), and rises again in old age [7–9, 36, 37]. In Section 3, we consider periodic external stimuli, and we show how the dynamics in this case can range from a globally attracting periodic solution to chaos. We give an interpretation of some known theoretical results [5, 22, 30] in the context of our happiness model, and provide a summary of the main ideas in the recent work [34], where rigorous mathematical proofs can be found. Finally, we provide a brief discussion of our main findings in Section 4.

2. THE LINEAR EQUATION WITH A CONSTANT EXTERNAL STIMULUS

The simplest choice of (1.1) consists of choosing the linear autonomous map $F(t, x, y) = ax + by$, with real constants a, b , which leads to the equation

$$u'(t) = au(t) + b \max_{s \in [t-h, t]} u(s). \quad (2.1)$$

We notice that (2.1) is not a linear equation because the functional *maximum* is not linear.

Following [34], we derive (2.1) from the following formulation:

$$u'(t) = -\alpha u(t) + \beta \left(u(t) - \max_{s \in [t-h, t]} u(s) \right), \quad (2.2)$$

with $\alpha > 0, \beta > 0$.

The term $-\alpha u(t)$ in the right-hand side of (2.2) represents a linear decay of happiness with time and it is easy to understand. Everybody should be easily convinced of the fact that happiness is affected by a decay; this is related with the notion of *hedonic adaptation*, which is defined as “*the process by which individuals’ levels of happiness return towards homeostasis after some life experience causes it to become much higher or lower than it had previously been.*” [2]. Much has been written on this topic; see, for example, [21] and its references. In particular, a linear decay is also assumed in the discrete model for hedonic capital introduced in [21].

The term $\beta (u(t) - \max_{s \in [t-h, t]} u(s))$ represents a peak-end evaluation of the temporal episode between $t - h$ and t . Notice that this term cannot be positive because $\beta > 0$, so it also represents a negative reaction, which is less intense if the instant utility at time t is close to the peak affect intensity recorded during the episode (it must be zero if the peak of happiness occurs at the end of the temporal episode).

Our model agrees with the principles of Brickman *et al.* [10], which establish that adaptation level theory offers two general mechanisms to explain the decay of happiness after a salient experience: habituation and contrast. In the context of lottery winners, they argue: “contrast with the peak experience of winning should lessen the impact of ordinary pleasures, while habituation should eventually reduce the value of new pleasures made possible by winning.” This explains why we consider two negative forcing terms, represented by the positive sign of parameters α and β in (2.2).

Comparing (2.1) and (2.2), we get the relations $b = -\beta$ and $a + b = -\alpha$. Thus, although in the paper we generally deal with (2.1) with arbitrary parameters a, b , the specific conditions for the happiness model are the following:

(H1) $b < 0$ and $a + b < 0$.

Notice that the unique equilibrium of (2.1) is $u = 0$. For a real constant $c \neq 0$, it is easy to check that, if $a + b \neq 0$, then equation

$$u'(t) = au(t) + b \max_{s \in [t-h, t]} u(s) + c \quad (2.3)$$

is reduced to (2.1) by the simple change of variables $u \rightarrow u - u^*$, where $u^* = -c/(a + b)$ is the unique equilibrium of (2.3). Since c can be viewed as an external stimulus, it makes sense to assume that u^* is positive if c is positive. This is another argument to assume that $a + b < 0$.

Differential equations with maxima like (2.1) have been studied in several contexts (see, e.g., [3, 12, 22, 30] and references therein). We recall that (2.3) is a retarded functional differential equation, so the initial value problem requires an initial function $\phi : [t_0 - h, t_0] \rightarrow \mathbb{R}$, which is assumed to be continuous. Some basic properties of the solutions are stated in the following result [12]:

Proposition 2.1. *For every continuous function $\phi : [t_0 - h, t_0] \rightarrow \mathbb{R}$, there is a unique solution $u = u(t; \phi)$ of (2.3) defined for all $t \geq t_0 - h$ and satisfying $u(t; \phi) = \phi(t)$, for all $t \in [t_0 - h, t_0]$.*

More subtle results on the behavior of the solutions of (2.3) are given in [30]. We state here some relevant features; for a proof see Theorem 2.1 of [30].

Proposition 2.2. *Assume that $a + b \neq 0$. Then the following properties hold for the solutions of (2.3):*

1. *The unique periodic solution of (2.3) is the constant solution $u \equiv u^* = -c/(a + b)$.*
2. *If $u : [t_0 - h, \infty) \rightarrow \mathbb{R}$ is a solution of (2.3), then there is a value $t_1 > t_0$ such that one of the following conditions hold:*
 - (a) *$u(t) = u^*$ for all $t \geq t_1$;*
 - (b) *u is strictly increasing for $t \geq t_1 - h$ and $u(t) = u^* + (u(t_1) - u^*)e^{(a+b)(t-t_1)} < u^*$, for all $t \geq t_1$.*
 - (c) *u is strictly decreasing for $t \geq t_1 - h$ and u is a solution of the delay differential equation $u'(t) = au(t) + bu(t - h) + c$, for all $t \geq t_1$.*

It also follows from Theorem 2.1 of [30] that the eventually increasing solutions of (2.3) tend to infinity with positive sign if $a + b > 0$, and its eventually decreasing solutions tend to infinity with negative sign if $a + b < 0$ and the characteristic equation $\lambda = a + be^{-\lambda h}$ associated to the linear delay differential equation

$$u'(t) = au(t) + bu(t - h) \quad (2.4)$$

has a real root $\lambda > 0$.

In the long run, unbounded behavior is not realistic, so that we will exclude the two above mentioned possibilities. The corresponding exclusion condition coincides with the criterion for the asymptotic stability of u^* , as it was established in Theorem 3.1 of [35].

Theorem 2.3 ([35]). *The constant solution u^* of (2.3) is uniformly asymptotically stable if and only if the following hypothesis holds:*

- (H2)** *$a + b < 0$ and one of the following conditions is satisfied:*
- i) $ah \leq 1$;
 - ii) $ah > 1$ and $bh < -e^{ah-1}$.

The stability region is represented as the shaded region in Figure 2 (left). Observe that the stability region is unbounded, we only represent it in a bounded square to show its profile.

Next, we will say that a closed interval $[A, B]$ is critical for a function $u(t)$ if $u'(t) = 0$ for all $t \in [A, B]$, and $[A, B]$ is the maximal interval with this property. The critical interval is degenerate if $A = B$.

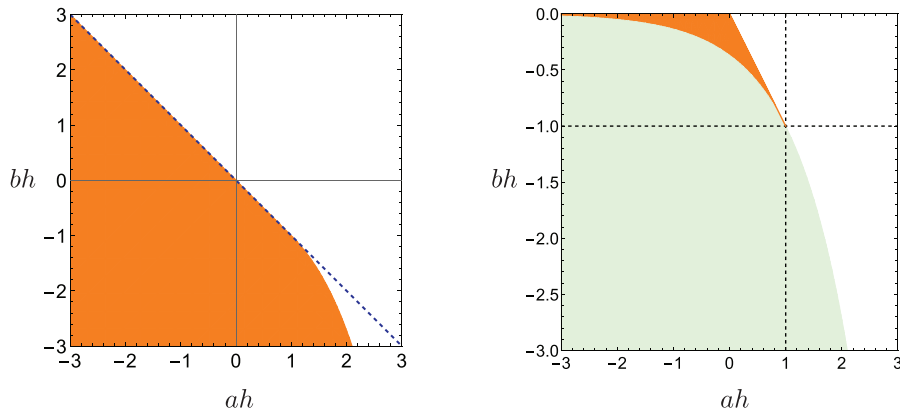


FIGURE 2. *Left:* Stability region for (2.3) in the parameter plane (ah, bh) ; the dashed line represents the graph of $a + b = 0$. *Right:* For $b < 0$ and $a + b < 0$, the green shaded region corresponds to the parameter values for which every solution u of (2.3) with $u(0) > u^*$ is U-shaped, while in the orange region such solutions can be either U-shaped or L-shaped.

Theorem 2.4. *Assume that u is a solution of (2.3) with $u(0) > u^*$. Then:*

- (I) *If $bh < -e^{ah-1}$, then u has U-shape on \mathbb{R}_+ and converges exponentially and monotonically to the equilibrium u_* as $t \rightarrow \infty$. If $a \geq 0$, then u has a unique critical value $u(t_*) < 0$, which is a global minimum of u on \mathbb{R}_+ . If $a < 0$, then u has a unique critical interval I such that $u(t)$ is a negative global minimum of u for all $t \in I$.*
- (II) *If $ah \leq 1$ and $-ah > bh \geq -e^{ah-1}$, then u is either U-shaped or L-shaped. The latter means that u is exponentially decreasing to u^* on \mathbb{R}_+ , with $u'(t) < 0$ for all $t \geq 0$.*

Before proving Theorem 2.4, some remarks are in order.

1. First, we note that $bh < -e^{ah-1}$ implies that $b < 0$ and $a + b < 0$ (since $e^{ah-1} \geq ah$). The parameter values for which $bh < -e^{ah-1}$ holds are represented in the green shaded region on the right of Figure 2
2. Take $a = -2, b = -1, h = 2$, and consider the continuous initial value ϕ for (2.1) defined by $\phi(s) = -s, s \in [-2, -1]$, $\phi(s) = 1, s \in [-1, -1/2]$, $\phi(0) = (e^2 - 5)/4 \approx 0.597$, and $\phi(s)$ is linear on $[-1/2, 0]$. Then we get $u(t) = -1/2$ for all $t \in [1, 1.5]$ so that the critical interval in the statement of Theorem 2.4 can be non-degenerate if $a < 0$.

Proof. Without loss of generality, we can assume that $c = 0$ and therefore $u^* = 0$.

(I) Since $a + b < 0, b < 0$, and $u(0) > 0$, we get $u'(0^+) = au(0) + b \max_{s \in [-h, 0]} u(s) < 0$ and therefore $v(t) := u'(t) < 0$ on some maximal interval $(0, t_1)$. Clearly, while $u(t)$ is decreasing, $u(t)$ and its first derivative $v(t) = u'(t)$ satisfy the same linear delay differential equation (2.4) with initial values $u(s) = \phi(s) := \max_{v \in [s, 0]} u(v)$, $s \in [-h, 0]$, and $v(s) = \phi'(s) \leq 0$ almost everywhere in $[-h, 0]$, $v(0) = u'(0^+) < 0$, respectively.

Suppose first that $a \geq 0$ and t_1 is finite, so that $v(t_1) = 0$. If $a = 0$ it is clear that $u(t_1 - h) = 0$; if $a > 0$ and $u(t_1 - h) > 0$, then $u(t_1) = -(b/a)u(t_1 - h) > u(t_1 - h)$, a contradiction. Thus, $u(t_1 - h) \leq 0$ and $v'(t_1^+) = bv(t_1 - h) > 0$, so that $u(t)$ has a non-degenerate minimum at t_1 . Since $u(t) < 0$ on $(t_1 - h, t_1)$ we also obtain that $v(t) < 0$ on the same interval. Integrating (2.4) for $v(t) = u'(t)$ at a right neighborhood of t_1 , we find that

$$v(t) = \int_{t_1}^t e^{a(t-s)} bv(s-h) ds > 0, \quad t \in (t_1, t_1 + h].$$

Hence, necessarily there is $t_* \in (t_1, t_1 + h)$ such that $u(t_*) = u(t_* - h)$, $u'(t_*) > 0$. Starting from this moment, $u(t)$ considered as a solution of (2.3) will satisfy the ordinary equation $u'(t) = (a + b)u(t)$, so that

$$u(t) = u(t_*)e^{(a+b)(t-t_*)} < 0, \text{ for all } t \geq t_*.$$

Now, if $a < 0$ and t_1 is finite, then $u(t_1 - h) = -(a/b)u(t_1)$ so that $u(t_1) < 0$, $u(t_1 - h) > 0$. Integrating (2.4) for $v(t) = u'(t)$ at a right neighborhood of t_1 , we find now that

$$v(t) = \int_{t_1}^t e^{a(t-s)}bv(s-h)ds \geq 0, \quad t \in (t_1, t_1 + h), \quad v(t_1 + h) > 0.$$

Clearly, this implies the existence of $t_* \in (t_1, t_1 + h)$ such that $u(t_*) = u(t_* - h)$, $u'(t_*) > 0$ so that

$$u(t) = u(t_*)e^{(a+b)(t-t_*)} < 0, \text{ for all } t \geq t_*.$$

Finally, suppose that $u(t)$ decreases on \mathbb{R}_+ . This means that $u(t)$ solves the initial value problem for (2.4) with initial function $\phi(s) = \max_{v \in [s, 0]} u(v)$, $s \in [-h, 0]$. Then $u(+\infty)$ is finite since the characteristic equation $\lambda = a + be^{-\lambda h}$ does not have positive roots. Thus $u(+\infty) = 0$ and $u(t) > 0$ for all $t \in \mathbb{R}_+$, so that u is L-shaped. Since the characteristic equation $\lambda = a + be^{-\lambda h}$ must have at least one negative root, the inequalities $ah \leq 1$ and $-ah > bh \geq -e^{ah-1}$ hold. \square

Theorem 2.4 points out that the delay h plays a key role in the shape of solutions. For example, if $a = 0$ and $bh < -1/e$, then all solutions $u(t)$ of (2.3) with $u(0) > u^*$ are U-shaped. However, it is easy to prove that if $-1/e < bh < 0$ and $\max_{s \in [-h, 0]} u(s) = u(0) > u_*$, then necessarily $u(t)$ is monotonically decreasing to the equilibrium u_* , so that u is L-shaped.

Of course, h also plays a role in the stability properties of the solutions. For example, for equation $u'(t) = u(t) - 2 \max_{s \in [t-h, t]} u(s) + 1$, the equilibrium u^* is uniformly asymptotically stable if and only if $h < h_0 \approx 2.678$, where h_0 is the unique real root greater than 0.5 of equation $e^{h-1} = 2h$. Considering delays greater than h_0 would result in destabilization of the equilibrium and existence of unbounded solutions.

In Figure 3, we represent a U-shaped solution of (2.3) with $a = 0.32$, $b = -1$, $h = 3\pi/2$, $c = 0.68$, and constant initial condition $\phi(t) = 2 > u^* = 1$, for all $t \in [0, h]$. We choose these parameters because the values of a , b and h correspond to those of the main example in [34], that we will revisit later in this paper (see Eq. (3.2)). It is easy to check that condition $bh < -e^{ah-1}$ of Theorem 2.4 holds. Since $a > 0$, this is the typical form of all solutions $u(t)$ of this equation with $u(0) > u^*$.

It has been often argued (see, e.g., [8, 9]) that the typical life-cycle happiness curve is approximately U-shaped. Moreover, Blanchflower [7] found evidence that happiness is U-shaped in age in 145 countries. A recent study shows that “human well-being’s curved shape is not uniquely human and that, although it maybe partly explained by aspects of human life and society, its origins may lie partly in the biology we share with great apes” [36].

Our model offers an additional mathematical argument to support this hypothesis; moreover, Theorem 2.4 provides precise conditions on the model parameters to ensure that happiness profiles are typically U-shaped. Thus, equation (2.3) can predict how happy will be Helen in her life, and the result seems to be, according to Blanchflower and Oswald [8], a typical individual’s happiness profile in many countries.

The U-shape form also appears in the response of happiness to a single event like winning the lottery, which is related with Brickman paradox [10, 31]. See Figure 2 of [32], where a U-shaped solution is obtained from a lottery mathematical model governed by a second-order ordinary differential equation. Similar happiness U-shape profiles to what we obtained in Figure 3 were found by Graham and Oswald when examining responses to different types of shocks (see Figs. 2 and 3 in [21], which “mimic the adaptive pattern often observed in empirical research.”). Moreover, in their model, the solutions also converge exponentially to the steady-state equilibrium.

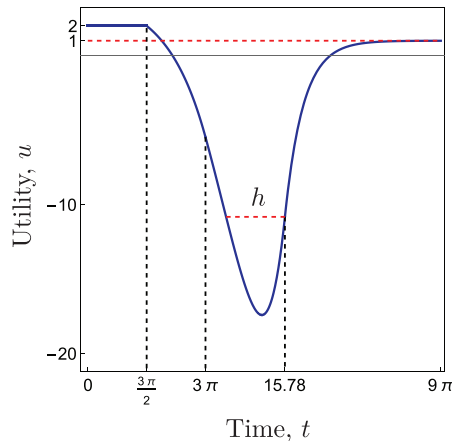


FIGURE 3. Numerical plot of the U-shaped solution of equation (2.3), with $a = 0.32$, $b = -1$, $h = 3\pi/2$, $c = 0.68$, and constant initial condition $\phi \equiv 2$ on $[0, h]$.

We might expect that, since in general more than one important happy event occur in a life, the happiness profile should contain several U-shaped segments following a peak (for example, when someone becomes a lottery winner). Actually, in the next section, solutions that are U-shaped on intervals bounded by two peaks will play an essential role in our analysis. The main reason is that, in that case, it is easy to describe how the equation with maxima works. Looking at Figure 3, we can distinguish three segments in the solution:

- For $t \in [h, 2h] = [3\pi/2, 3\pi]$, $\max\{u(s) : t - h \leq s \leq t\} = u(h) = 2$, and therefore equation (2.3) reads $u'(t) = au(t) + 2b + c$.
- For $t \in [2h, t_1] = [3\pi, 15.78]$, $\max\{u(s) : t - h \leq s \leq t\} = u(t - h)$, and therefore equation (2.3) reads $u'(t) = au(t) + bu(t - h) + c$. Here, t_1 is numerically found as the first point for which $t_1 > 2h$ and $u(t_1) = u(t_1 - h)$ hold.
- For $t > t_1 \approx 15.78$, $\max\{u(s) : t - h \leq s \leq t\} = u(t)$, and therefore equation (2.3) becomes the ODE $u'(t) = (a + b)u(t) + c$.

3. THE LINEAR EQUATION WITH PERIODIC EXTERNAL STIMULI

A more sophisticated model should consider more general (non-constant) external stimuli; for example, Graham and Oswald [21] consider changes in level investment and exogenous shocks, while Sprott [32] adds a forcing term, representing external events and circumstances, to the second-order ODE (damped harmonic oscillator) proposed as a dynamical model of happiness. He gives an example of periodic forcing that leads to chaotic solutions ([32], Fig. 6).

In this section, we consider periodic external stimuli in (2.1), that is, we deal with the following equation:

$$u'(t) = au(t) + b \max_{s \in [t-h, t]} u(s) + f(t), \quad (3.1)$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is T -periodic. The main results for the existence, uniqueness and stability of periodic solutions of (3.1) have been obtained in [5, 22, 30]. We recall them in the following result:

Theorem 3.1. *If $a + b \neq 0$, then (3.1) has at least one T -periodic solution. Moreover, this T -periodic solution is unique and globally asymptotically stable if $a + b < 0$ and one of the following conditions holds:*

- (S1) $b \geq 0$;
- (S2) $b < 0$, $a > 0$, and $(a - b)h < 1$.

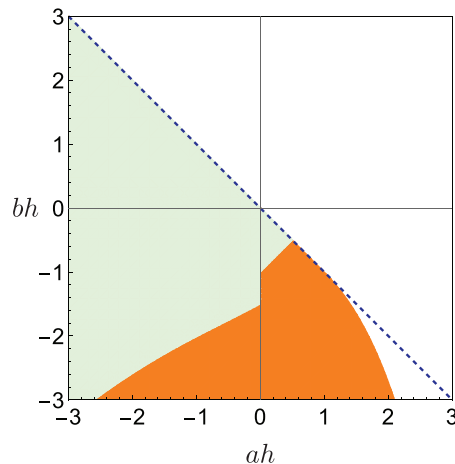


FIGURE 4. The total shaded region corresponds to the stability region of the autonomous equation (2.1) represented on the left of Figure 1. The region shaded in green corresponds to the parameter values given by conditions (S1)–(S4) in Theorem 3.1.

(S3) $b < 0$, $a = 0$, and $-bh < 3/2$.

(S4) $b < 0$, $a < 0$, and $\frac{a}{b}e^{ah} > \ln\left(\frac{b^2 + ab}{b^2 + a^2}\right)$.

We represent the asymptotic stability conditions from the statement of Theorem 3.1 in the parameter plane (ah, bh) in Figure 4.

The three last global stability conditions correspond to parameter values satisfying **(H1)**. Thus, under those conditions, the utility profile predicted by (3.1) would be asymptotically periodic. In Figure 5 we plot a numerical solution of (3.1), with $a = -1$, $b = -1$, $h = 3\pi/2$, and $f(t) = 1 - \sin(t)$. Notice that condition (S4) in the statement of Theorem 3.1 clearly holds. The solution shows a rapid convergence to a 2π -periodic solution. This periodic utility profile can be seen as a sequence of repeated U-shaped profiles, where happiness intensity goes up and down with time. Although this situation seems acceptable for a utility profile, in general one does not expect such a regular behavior for happiness in an individual's life.

During some time, our working hypothesis was that equation (3.1) should have a unique periodic solution for each periodic function f and each pair of parameters $a \neq b$. However, this conjecture was disproved in [30], by finding two different periodic solutions in a particular case of (3.1) with $a = 0$, $b = -1$, $h = 3\pi/2$, and a continuous 2π -periodic function f . This result opened the door to look for complex behavior in (3.1). Actually, the main results in [34] allow to prove that the differential equation with maxima

$$u'(t) = 0.32u(t) - \max_{s \in [t-h, t]} u(s) + 1 - \sin(t), \quad (3.2)$$

with $h = 3\pi/2$, has infinite periodic solutions and exhibits chaos. Since **(H1)** holds for $a = 0.32$, $b = -1$, (3.2) provides an example of mathematical model for happiness based on the peak-end rule that predicts a chaotic utility profile.

The way to arrive at the result about chaos reveals other interesting properties of the solutions of (3.1) for a certain class of periodic functions f , and allows the construction of a return map which keeps essential dynamical information of (3.1) and has independent interest. Here we outline the main ideas, referring to [34] for technical details.

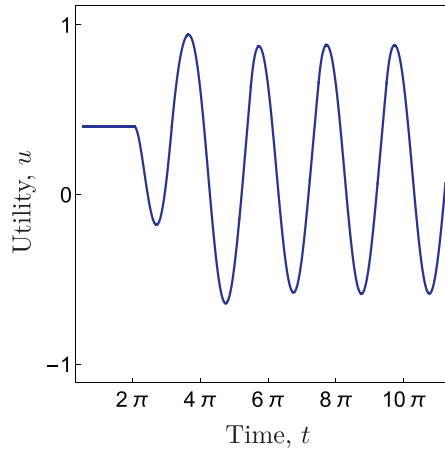


FIGURE 5. Numerical plot of a typical asymptotically periodic solution of equation (3.1) under condition (S4) in Theorem 3.1. We chose $a = -1$, $b = -1$, $h = 3\pi/2$, $f(t) = 1 - \sin(t)$, and constant initial condition $\phi \equiv 0.4$ on $[q - h, q]$, with $q \approx 6.5$.

3.1. Construction of the return map

We begin with a pair of definitions.

Definition 3.2. We will say that a continuous T -periodic function $f : \mathbb{R} \rightarrow \mathbb{R}$ has *sine-like shape* if there exist τ_0, τ_1 such that $0 < \tau_1 - \tau_0 < T$, f is strictly decreasing on $[\tau_0, \tau_1]$ and strictly increasing on $[\tau_1, \tau_0 + T]$.

For example, $f(t) = \sin(t)$ has sine-like shape, with $T = 2\pi$, $\tau_0 = \pi/2$, and $\tau_1 = 3\pi/2$.

Definition 3.3. Let $u : [t_0, \infty) \rightarrow \mathbb{R}$ be a solution of (3.1). If there exists a point $\nu > t_0 + h$ such that $u(\nu) = \max \{u(s) : s \in [\nu - h, \nu + \varepsilon]\}$ for some $\varepsilon > 0$, then we say that $u(\nu)$ is a *good peak* of u .

The following result from [34] provides sufficient conditions for the existence of good peaks.

Proposition 3.4. [34, Thm. 3] *If f has sine-like shape and one of the stability conditions in (H2) holds, then all solutions of (3.1) are bounded. Moreover, for each solution $u : [t_0, \infty) \rightarrow \mathbb{R}$ of (3.1) there exist $\nu > t_0 + h$ and $\varepsilon > 0$ such that $u(\nu) = \max \{u(s) : s \in [\nu - h, \nu + \varepsilon]\}$, that is, $u(\nu)$ is a good peak of u .*

In the following, we assume that (H2) holds and f has sine-like shape, which is the case of equation (3.2). Repeated applications of Proposition 3.4 allow to affirm that, for each solution $u : [t_0, \infty) \rightarrow \mathbb{R}$ of (3.1) there is an infinite sequence of good peaks.

Next, we define the T -periodic function $f^* : \mathbb{R} \rightarrow \mathbb{R}$ associated to equation (3.1) by

$$f^*(t) = \frac{-1}{a+b} f(t), \quad t \in \mathbb{R}.$$

The importance of f^* comes from the following result ([34], Lem. 4):

Proposition 3.5. *If f has sine-like shape in the sense of Definition 3.2 and $u(\nu)$ is a good peak of a solution u of (3.1), then $u(\nu) = f^*(\nu)$. Moreover, $\nu = \bar{\nu} + kT$, with $\bar{\nu} \in (\tau_0, \tau_1)$ and $k \in \mathbb{N}$.*

Proposition 3.5 establishes that the solutions of (3.1) reach their good peaks at intersection points with the decreasing branches of f^* .

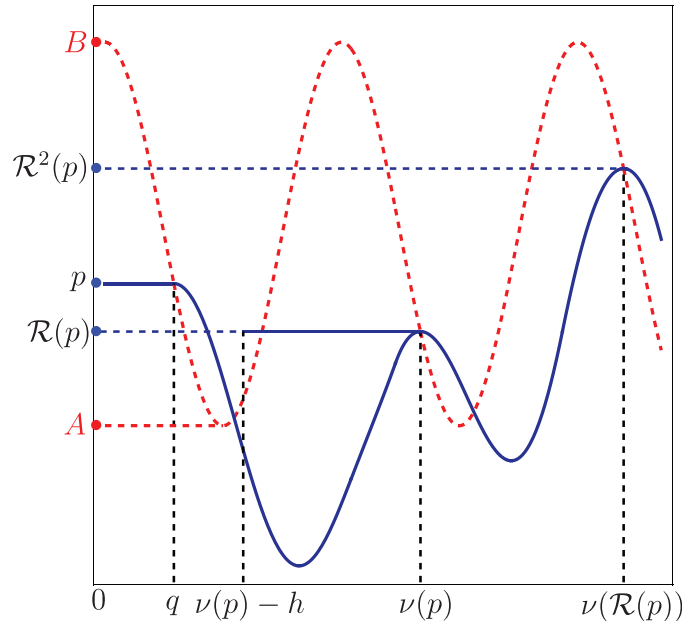


FIGURE 6. Sketch of the construction of two iterations of the return map \mathcal{R} . The solid blue curve represents the solution u_p of (3.1) with initial condition $u_p(t) = p$, $t \in [q - h, q]$, and the red dashed curve represents the map $f^* = (-1/(a + b))f$. See the text for details.

We are now in a position to define the return map associated to (3.1). Define

$$A = \min\{f^*(x) : x \in \mathbb{R}\} \quad ; \quad B = \max\{f^*(x) : x \in \mathbb{R}\}.$$

Then we define the map $\mathcal{R} : [A, B] \rightarrow [A, B]$ in the following form: Let $p \in [A, B]$; then there is a unique $q \in [0, T]$ such that $p = f^*(q)$ and f^* is nonincreasing on some vicinity of q . Let $u = u_p : [q, +\infty) \rightarrow \mathbb{R}$ be the solution of (3.1) with initial condition $u_p(t) = p$, $t \in [q - h, q]$. By Proposition 3.4, there exists a first point $\nu = \nu(p) > q$ such that $u_p(\nu)$ is a good peak of u_p . Moreover, by Proposition 3.5, $u_p(\nu) = f^*(\nu) \in (A, B)$. We define $\mathcal{R}(p) = u_p(\nu)$. We notice that the successive iterations of \mathcal{R} provide the sequence of good peaks of u_p for each $p \in [A, B]$. This claim is a simple consequence of the periodic character of f and the definition of a good peak (Def. 3.3). See Figure 6.

In Figure 7, we provide a numerical plot of the return map \mathcal{R} associated to (3.2), which is a piece-wise continuous function defined in the interval $[A, B]$, with $A = 0$, $B = 2/0.68 \approx 2.94$.

3.2. Some properties of the return map

The map \mathcal{R} provides a lot of information about the dynamics of (3.2). For example, there are two fixed points $p_1 \approx 1.037$ and $p_2 \approx 1.65$, and both provide periodic solutions of (3.2). Since the first one is in the first interval of continuity of \mathcal{R} , its associated periodic solution has period $T = 2\pi$, while p_2 corresponds to a periodic solution of period $2T = 4\pi$ (see [34], Cor. 18). A numerical plot of these periodic solutions is shown in Figure 8.

In general, the return map is discontinuous. For equation (3.2), there are two points of discontinuity $q_1 \approx 1.2$, $q_2 \approx 2.6$, characterized by the properties $\mathcal{R}(q_1^-) = \mathcal{R}(q_2^-) = 0$, $\mathcal{R}(q_1) = \mathcal{R}(q_2) = \mathcal{R}(0)$ (see [34], Thm. 9). A graphic idea of the mechanism behind the discontinuity is shown in Figure 9.

Analytical conditions for the continuity and differentiability properties of \mathcal{R} are provided in [34]. In particular, it is possible to give estimates for the derivative of \mathcal{R} at a point p and determine the monotonicity intervals of \mathcal{R} (see [34], Thm. 17). The notion of U-shaped solution also plays a key role in the analysis of the derivative of \mathcal{R} ;

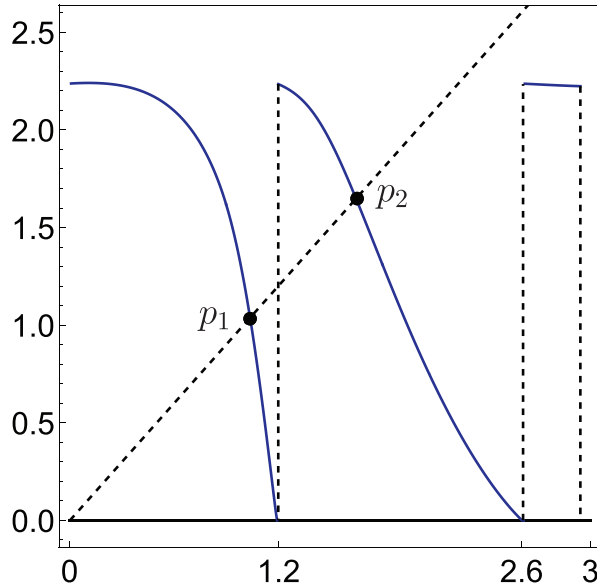


FIGURE 7. Numerical plot of the return map \mathcal{R} associated to (3.2), showing the discontinuity points $q_1 \approx 1.2$, $q_2 \approx 2.6$, and the fixed points $p_1 \approx 1.037$, $p_2 \approx 1.65$.

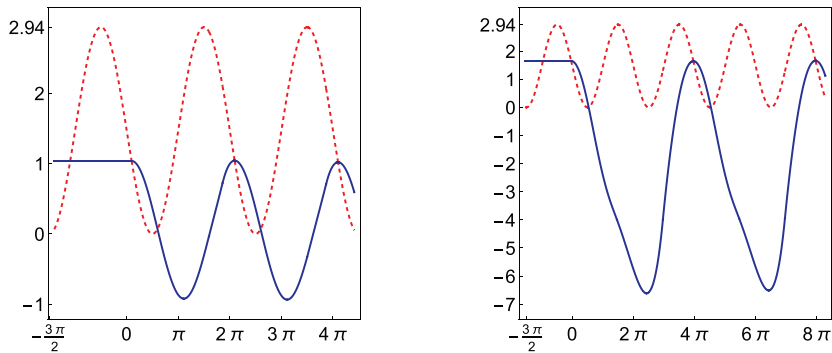


FIGURE 8. Periodic solutions of (3.2) of periods 2π (left) and 4π (right). The 2π -periodic solution corresponds to the initial condition $p_1 \approx 1.037$, and the 4π -periodic solution corresponds to $p_2 \approx 1.65$ (see Fig. 7). As before, the red dashed curve represents the map $f^* = (-1/(a+b))f$.

in particular, sufficient conditions on the periodic function f and the involved parameters a , b , h are established to ensure that a given solution of (3.1) is U-shaped on an interval between two peaks (see [34], Lem. 14).

Finally, the existence of chaotic solutions in (3.2) is proved in Theorem 21 of [34] constructing a semiconjugation between the restriction of \mathcal{R} to a closed subset of $[A, B]$ and a suitable shift on a space of sequences of three symbols.

4. DISCUSSION

Nowadays, we are often asked to evaluate how satisfied we are with many experiences: a meal in a restaurant, a stay in a hotel, the publication process of a paper, an on-line purchase, and so on. And these evaluations guide future decisions, sometimes not only ours. Thus, it is not surprising that the evaluation of experiences undergo an extraordinary boom. In this context, the works of Daniel Kahneman, an outstanding figure in the

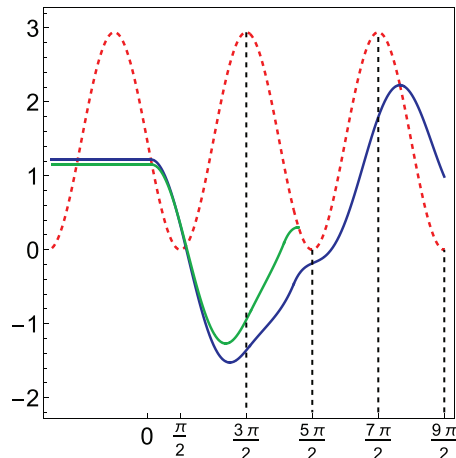


FIGURE 9. Illustration of the first discontinuity point of \mathcal{R} . The solution of (3.1) with initial condition 1.15 (colored in green) hits the graph of f^* in the interval $[3\pi/2, 5\pi/2]$, and the solution with initial condition 1.22 (colored in blue) hits the graph of f^* in the interval $[7\pi/2, 9\pi/2]$. For the discontinuity point $q_1 \approx 1.2$, $\mathcal{R}(q_1^-) = 0$, $\mathcal{R}(q_1) = \mathcal{R}(0) \approx 2.23$.

study of behavioral economics and hedonic psychology, play a prominent role. In this paper, we followed the moment-based approach to the evaluation of happiness [24]; according to this theory, the experienced utility of an episode can be derived from real-time measures of the pleasure and pain that the subject experienced during that episode¹. Since these evaluations consist of two types of utility concepts: instant utility and remembered utility, a dynamic model of happiness based on this approach must be defined by a delay differential equation, rather than an ordinary differential equation (as in [32]). Moreover, the application of the peak-end rule leads to a class of delay-differential equations called differential equations with maxima.

Our model provides some interesting insights for happiness studies. First, the simplest model (2.1) sheds some new light on the midlife crisis, which can be represented by U-shape profiles (or happiness U-curves) [7–9]. Numerous international surveys of life satisfaction showed this recurrent shape for age-happiness over the lifespan in countries around the world, and it has been suggested that there may be an underlying pattern in life satisfaction that is independent of a particular personal situation [37]. According to Blanchflower [8], the U-shape also does not depend on what question is asked or how the responses are coded. In this direction, the U-shaped happiness curve can be viewed as a general behavior for happiness over the life cycle that does not depend on external factors. This is in agreement with our results, that show this profile for the utility function when the effect of external stimuli are not considered in the model (see Thm. 2.4). As mentioned before, Weiss *et al.* [36] go one step further claiming that human well-being’s curved shape is not uniquely human, since they found evidence for a midlife crisis in great apes.

On the other hand, the consideration of periodic exogenous stimuli reveals the possibility of a rich dynamics for the solutions to equation (3.1), ranging from a globally attracting periodic solution to chaos. Our main aim in Section 3 was to write the main ideas and results from [34], keeping the exposition at a simple level. The interested reader can see the technical details and rigorous proofs in [34].

Our results can be viewed as a first attempt to construct dynamic models of happiness based on the peak-end rule. This opens further directions for research; for example, it would be interesting the formulation and analysis of discrete models, control optimization problems, or stochastic models based on the peak-end rule.

¹Fredrickson and Kahneman [19] mention a sentence due to Milan Kundera that illustrates quite well the philosophy behind the evaluation by moments: “*memory does not make films, it makes photographs.*”

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