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Destabilization and chaos induced by harvesting: insights from one-dimensional discrete-time models

Víctor Jiménez López[1] · Eduardo Liz[2]

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Abstract

One-dimensional discrete-time population models are often used to investigate the potential effects of increasing harvesting on population dynamics, and it is well known that suitable harvesting rates can stabilize fluctuations of population abundance. However, destabilization is also a possible outcome of increasing harvesting even in simple models. We provide a rigorous approach to study when harvesting is stabilizing or destabilizing, considering proportional harvesting and constant quota harvesting, that are usual strategies for the management of exploited populations. We apply our results to some of the most popular discrete-time population models (quadratic, Ricker and Bellows maps). While the usual case is that increasing harvesting is stabilizing, we prove, somehow surprisingly, that increasing values of constant harvesting can destabilize a globally stable positive equilibrium in some cases; moreover, we give a general result which ensures that global stability can be shifted to observable chaotic dynamics by increasing one model parameter, and apply this result to some of the considered harvesting models.

Keywords Discrete-time model · Harvesting · Immigration · Ricker map · Stability · Chaos

Mathematics Subject Classifications 92D25 · 37N25 · 37E05 · 39A30 · 39A33

✉ Eduardo Liz
eliz@dma.uvigo.es

Víctor Jiménez López
vjimenez@um.es

1 Departamento de Matemáticas, Universidad de Murcia, Campus de Espinardo, 30100 Murcia, Spain

2 Departamento de Matemática Aplicada II, Universidade de Vigo, 36310 Vigo, Spain

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1 Introduction

One-dimensional discrete-time equations are suitable models to describe the growth of populations whose generations do not overlap, and they have been the basis of many fishery models (Clark 1990; Quinn and Deriso 1999). The potential complex behaviour of these simple models has been popularized by the papers of R. M. May and others, and the main message is that increasing growth rates lead to oscillatory behaviour and chaotic dynamics (May 1976). Paradigmatic examples are the quadratic map $f(x) = x(1 + r(1 - x/K))$ and the Ricker map $f(x) = xe^{r(1-x/K)}$, where in both cases r represents the growth rate and K the carrying capacity. In mathematical terms, K is the positive equilibrium (that can be normalized to 1) and r is the parameter that controls the dynamics: large values of r lead to complex behaviour.

Since harvesting can reduce the growth rate, a common conclusion is that harvesting tends to stabilize a population that is oscillatory without exploitation. A graphic analysis using unimodal maps can be already found in Ricker's influential paper, who writes (Ricker 1954, p. 620, item 11): *"Another result of exploitation is to reduce the amplitude and complexity of any reproduction-curve oscillations that may be in progress; sufficiently intensive exploitation eliminates such oscillation entirely."*

The most popular harvesting policies are proportional (or constant effort) harvesting, that we will refer to as **PH**, and constant quota harvesting (**CH**). Applying **PH** to a population governed by the normalized quadratic map $f(x) = rx(1 - x)$ leads to the simple model

$$x_{n+1} = (1 - \gamma)rx_n(1 - x_n), \quad n = 0, 1, 2, \ldots,$$

where $\gamma \in (0, 1)$ is the harvesting effort. If the population is unstable in the absence of harvesting ($r > 3$), then it is clear that increasing γ has a stabilizing effect, and the harvested population becomes stable when $(1 - \gamma)r \leq 3$, that is, for $\gamma \geq (r - 3)/r$. Actually, it is not difficult to prove that the same conclusion holds for the Ricker map and for the rest of unimodal maps usually employed in population dynamics. This simple remark has been noticed by several authors (Ruxton 1993; Doebeli 1995), but it is not directly applicable to all population models.

Sometimes, it is meant that increasing harvesting is stabilizing if the equilibrium becomes stable for sufficiently large harvesting rates. In this direction, it is clear that if we consider a family $f_\gamma = (1 - \gamma)f$, where f is a smooth map with $f(0) = 0$, a unique critical point $f(c)$, which is a local maximum, and a unique positive equilibrium $p > c$, with $f'(p) < -1$, then a simple graphical analysis shows that a sufficiently large value of γ takes the equilibrium to c, and thus it becomes asymptotically stable, as claimed by Ricker (see, e.g., Goh (1977); Liz (2010b) for rigorous statements).

The above comments do not imply that increasing harvesting cannot destabilize a stable equilibrium. Actually, one can easily imagine unimodal maps for which **PH** has the potential to destabilize a stable equilibrium. For example, see the figure of Stone (1993), BOX 1 and Fig. 46 of Sharkovsky et al. (1997), which are very similar. These two examples have been suggested in two different but related contexts. Stone's example tries to explain how the so-called period-doubling cascade can be reversed in a one-parameter family of maps, and it is directly linked with ecological models.

The example by Sharkovsky et al. (1997) was first introduced by Kolyada (1989), and has an analytic—though complicated—expression. Kolyada's example proves that there exist unimodal maps with negative Schwarzian derivative for which increasing the value of λ in the family $f_\lambda = \lambda f$ can lead from chaos to a globally attracting equilibrium, thus exhibiting the opposite effect than in the quadratic family.

Of course, it is well known that a similar behaviour cannot happen in concave maps (see Chapter 8 in Sharkovsky et al. (1997)). Even for maps with an inflection point, sometimes it is easy to show that **PH** cannot destabilize an asymptotically stable equilibrium, just by looking at the stability criteria (see, e.g., Examples 1 and 2 of Liz and Buedo-Fernández (2019), that we will mention later in the paper).

For higher order discrete models, it is known that **PH** can destabilize a positive equilibrium; as far as we know, the first who showed this fact were Goh and Agnew (1978), using a two dimensional Clark model

$$x_{n+1} = (1-\gamma)(\alpha x_n + f(x_{n-1})), \quad n = 0, 1, 2, \ldots,$$

where $\alpha \in (0, 1)$ is a survival coefficient and f is the recruitment function. This equation can be derived from a stage-structured model with two classes (juveniles and adults); a thorough analysis about how harvesting can destabilize the equilibrium in this equation can be found in Liz and Pilarczyk (2012). Actually, empirical evidence to show that harvesting can increase variability in the abundance of exploited populations is directly linked to age structure (Anderson et al. 2008; Hsieh et al. 2006).

It is worth noticing that increasing the order of the difference equation is not necessary to show this effect. Using **PH** in Thieme's expression (Thieme 2003, Sect. 9.2) for the Ricker difference equation, we arrive at

$$x_{n+1} = (1-\gamma)\left(\alpha x_n + (1-\alpha)x_n e^{r(1-x_n)}\right), \quad n = 0, 1, 2, \ldots. \tag{1.1}$$

This model assumes that α is an adult's probability of surviving one year, including the reproductive season. Models of this type were already introduced by Clark (Clark 1990, Sect. 7.5), so we will refer to (1.1) as the Ricker–Clark model with **PH**. Equation (1.1) has been studied by Liz and Ruiz-Herrera (2012), who proved that increasing γ can destabilize an attracting positive equilibrium. Moreover, it has been proved there that increasing harvesting can take a population from a stable periodic attractor to chaos.

For the Ricker and Bellows maps with constant harvesting, Schreiber (2001) emphasizes that *"increasing harvesting rates can stabilize chaotic dynamics by shifting the dynamics from chaotic semistability to persisting at a linearly stable fixed point."* However, he also showed (see, e.g., Fig. 5b in Schreiber (2001)) that increasing harvesting rates has the potential to lead a Ricker model from a stable 4-periodic orbit to chaotic semistability and essential extinction.

The above examples show that a suitable rigorous approach is necessary to understand whether or not increasing harvesting can be destabilizing and to find sufficient conditions under which chaos can be an outcome of increasing harvesting rates. Moreover, from an applied point of view, it is important to report phenomena that are visible;

in this direction, most of the existing results of chaos in the ecological literature do not guarantee that this chaos is observable; see, e.g., (Thunberg 2001, Sect. 1.1), where the key difference between visible and invisible chaos is discussed. Our main results in this paper cover these two problems:

On the one hand, we use the characterization of period-doubling bifurcations given by (Sharkovsky et al. 1997, Theorem 8.1) to get a criterion to decide whether or not increasing harvesting rates can destabilize an attracting equilibrium either with a strategy of proportional harvesting (Proposition 3.1) or constant harvesting (Proposition 4.1). In this way we characterize the possible stability switches for usual population models; perhaps the most interesting result in this direction is Theorem 4.2, which establishes the exact range of parameter values for which constant harvesting or constant immigration can destabilize a global attractor for the map $f(x) = ax/(1+x^m)$, $a > 1$, $m > 1$, introduced by Maynard Smith and Slatkin (1973) and usually known as the Bellows map (Bellows 1981) or the generalized Beverton–Holt map (Schreiber 2001). As far as we know, this result is new and it provides the first example for which **CH** induces a period-doubling bifurcation in an attracting equilibrium in one of the usual unimodal maps employed in discrete-time population dynamics (this situation is impossible for the Ricker map). Moreover, we clarify some previous discussions on the effects of constant immigration in discrete-time population models (McCallum 1992; Stone 1993; Doebeli 1995; Solé et al. 1999; Stone and Hart 1999). We will come back to this question in Sect. 5.

On the other hand, Theorem B.1 provides a method to establish the existence of observable chaos in a one-parameter family of one-dimensional maps. The generality of the statement of this theorem allows us to get results of observable chaos both for proportional and constant harvesting. In conjunction with our criteria for destabilization, we are able to present two examples in which harvesting takes a population with a globally stable positive equilibrium to observable chaos. In this way, we complement some results by Schreiber (2001) and Liz and Ruiz-Herrera (2012).

The paper is organized as follows: Sect. 2 is devoted to briefly introduce our framework. Sections 3 and 4 contain the main results for proportional and constant harvesting, respectively. Then we provide further remarks and discussions in Sect. 5. The theoretical results are proved in two appendices: Appendix A is devoted to stability switches, and Appendix B to chaos.

2 Discrete-time harvesting models

Let us consider a first-order difference equation

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, \ldots, \tag{2.1}$$

where $f : I \to I$ is a map defined in an arbitrary interval of the real line I.

We recall some notions and properties for their use in the paper. If $f \in C(I)$ and $x \in I$, then the *orbit* of x is the set $\{f^n(x)\}_{n=0}^{\infty}$. A point p is said to be *periodic of period* $r \geq 1$ or r*-periodic* if $f^r(p) = p$ and $f^i(p) \neq p$ for any $1 \leq i < r$. A 1-periodic point is also called a *fixed* (or an *equilibrium*) *point*. An interval $J \subset I$

satisfying $f(J) \subset J$ is called *invariant*. If f is differentiable and $f'(c) = 0$, then c is called a *critical point* of f. If p is r-periodic for f and $|(f^r)'(p)| \leq 1$, then we say that it is *stable*, and it is *unstable* otherwise.

Since we are interested in population models, in many cases we assume that $I \subseteq [0, \infty)$ and f has a unique positive equilibrium p; moreover, $f(x) > x$ for $x < p$ and $f(x) < x$ for $x > p$. In the framework of population dynamics, x_n represents the population after n reproductive periods, starting at an initial population size x_0.

We also assume that f is smooth enough to apply Theorem A.1 stated in Appendix A. Condition (b) in this theorem concerns the Schwarzian derivative. We recall that the Schwarzian derivative of a $\mathcal{C}^3$ map f is defined by the expression

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2,$$

whenever $f'(x) \neq 0$.

It is worth noticing that, under typical negative Schwarzian derivative assumptions, stability and *asymptotical stability*, meaning by this the attraction of all nearby orbits, amount to the same thing.

In most usual discrete-time models of population dynamics, condition (b) holds. For example, it is well known that the Ricker map $f(x) = xe^{r(1-x)}$ satisfies $Sf(x) < 0$ for all $x \neq 1/r$, the same as the quadratic map $f(x) = rx(1-x)$, for all $x \neq 1/2$, and the Bellows map $f(x) = ax/(1+x^m)$ for $x \neq (m-1)^{-1/m}$, whenever $m \geq 2$. We will use these unimodal maps for applications of the main results, because they are well known by ecologists and they cover many interesting situations.

We will also consider the Ricker–Clark map $f(x) = \alpha x + (1-\alpha)xe^{r(1-x)}$, with $\alpha \in (0, 1)$. The negativity of the Schwarzian derivative of the Ricker map is inherited by the Ricker–Clark map in the points where a flip bifurcation can occur (Liz and Franco 2010, Proposition 1). Thus, we will not take care of condition (b), and we will focus on (c), which will be the responsible for stabilizing or destabilizing effects when a flip bifurcation occurs. Notice that, in the considered population models, flip bifurcations characterize the way by which a stable positive equilibrium can be either stabilized or destabilized as one of the model parameters changes.

We focus our study on the two more common harvesting strategies (see, e.g., Goh (1977); Quinn and Deriso (1999); Deroba and Bence (2008) for more details):

- Proportional harvesting (**PH**), which consists of removing a constant proportion of the total population every period. Mathematically, it leads to equation

$$x_{n+1} = (1-\gamma)f(x_n), \quad n = 0, 1, 2, \ldots, \tag{2.2}$$

  where $\gamma > 0$ represents the harvesting effort.
- Constant quota harvesting (**CH**), in which a constant amount of population is harvested every year. This leads to equation

$$x_{n+1} = \max\{f(x_n) - \gamma, 0\}, \quad n = 0, 1, 2, \ldots, \tag{2.3}$$

  where γ is the constant quota.

3 Proportional harvesting

In this section, we consider the proportional harvesting modeled by equation (2.2). An application of Theorem A.1 provides the following result, whose proof is given in Appendix A.

Proposition 3.1 *Let p be the positive equilibrium of* (2.1). *Then :*

(a) If p is stable then it is destabilized by **PH** *as γ is increased if there is $\gamma > 0$ such that* (2.2) *has an equilibrium x_γ with $(1-\gamma)f'(x_\gamma) = -1$ and $(1-\gamma)x_\gamma f''(x_\gamma) > 2$.*

(b) If p is unstable then it is stabilized by **PH** *as γ is increased if there is $\gamma > 0$ such that* (2.2) *has an equilibrium x_γ with $(1-\gamma)f'(x_\gamma) = -1$ and $(1-\gamma)x_\gamma f''(x_\gamma) < 2$.*

A first consequence of Proposition 3.1 is that the sign of $f''(x)$ is not enough by itself to determine the possible destabilizing effect of proportional harvesting, although it is clear that **PH** cannot destabilize a stable equilibrium of (2.1) if the production function f is concave in the interior of I, like in the quadratic case. We recall that for concave maps (and in particular for quadratic maps), a property of monotonicity of bifurcations of cycles holds. Roughly speaking, this means that the dynamics of the map $f_\gamma(x) = (1-\gamma)f(x)$ becomes simpler as γ increases (Sharkovsky et al. 1997, Chapter 8).

As far as we know, the first example of a unimodal map with negative Schwarzian derivative for which monotonicity of bifurcations of cycles does not hold for the family $f_\gamma(x) = (1-\gamma)f(x)$ was provided by Kolyada (1989). Since the definition of f in this example is quite complicated, we include here an example of a decreasing map with negative Schwarzian derivative for which increasing γ is destabilizing. The following map is based on an example given by (Jiménez López and Parreño 2016, p. 370):

$$f(x) = \frac{1}{(1-2a)(a+(1-a)x) + 2a(a+(1-a)x)^2}, \tag{3.1}$$

with $a = -0.001$. Function f is a small perturbation of the map $h(x) = 1/x$, which is globally periodic ($h^2(x) = x$ for all $x \neq 0$). It is easy to check that f maps the interval $[1/3, f(1/3)]$, $f(1/3) \approx 3.00201$, into itself, and the same is true for $f_\gamma(x) = (1-\gamma)f(x)$ and $[1/3, f_\gamma(1/3)]$ whenever $\gamma \leq 0.5$. Moreover, f is decreasing, convex, and $Sf(x) < 0$ for all x. The unique fixed point of f is $p = 1$, and $-1 < f'(1) < 0$; thus, it is a global attractor by the Allwright–Singer theorem (Allwright 1978; Singer 1978) (see Appendix B). However, for $\gamma = 0.5$, the fixed point $x_\gamma \approx 0.707046$ of $f_\gamma(x)$ is unstable. We can verify the existence of a value $\gamma^* \approx 0.499914$ such that the fixed point $x_{\gamma^*} \approx 0.707107$ of f_{γ^*} satisfies $(1-\gamma^*)f'(x_{\gamma^*}) = -1$ and $(1-\gamma^*)x_{\gamma^*} f''(x_{\gamma^*}) > 2$, in agreement with Proposition 3.1.

The map f in (3.1) is an academic example, but in the rest of the section we consider maps usually employed as production functions in discrete-time models from population dynamics.

Although it is easy to apply Proposition 3.1 to the Ricker and the Bellows models, in these particular cases we can give more information using some known results.

Indeed, it has been proved by Liz (2018) that the unique positive equilibrium p of the gamma-Ricker model with **PH**

$$x_{n+1} = (1-\gamma)\beta x_n^{\alpha} e^{-r x_n}, \quad n = 0, 1, 2, \ldots, \tag{3.2}$$

with $\beta > 0$, $r > 0$, $0 < \alpha \le 1$, and $0 < \gamma < 1$, is globally asymptotically stable in $(0, \infty)$ if and only if

$$\beta(1-\gamma) \le e^{\alpha+1}\left(\frac{\alpha+1}{r}\right)^{1-\alpha}. \tag{3.3}$$

Moreover, p is unstable if (3.3) does not hold. Hence, it is obvious that increasing harvesting is stabilizing. The Ricker map corresponds to the particular case $\alpha = 1$ in (3.2).

A similar result has been recently proved by Liz and Buedo-Fernández (2019) for the following generalization of the Bellows model with proportional harvesting:

$$x_{n+1} = \frac{(1-\gamma)a x_n^{\alpha}}{1 + x_n^m}, \quad n = 0, 1, 2, \ldots, \tag{3.4}$$

where $a, m > 0$, $0 < \alpha \le 1$, and $0 < \gamma < 1$. In this case, the sharp global stability condition is

$$m \le 1 + \alpha, \text{ or } \quad m > 1 + \alpha \text{ and } a(1-\gamma) \le \frac{m}{m-1-\alpha}\left(\frac{1+\alpha}{m-1-\alpha}\right)^{(1-\alpha)/m}. \tag{3.5}$$

The Bellows model corresponds to the particular case $\alpha = 1$ in (3.4). In this case, condition $a(1-\gamma) > 1$ is required to ensure that the positive equilibrium exists.

Hence, destabilization of the positive equilibrium is not a possible outcome of increasing harvesting by **PH** in the previous models. However, there are other models for which the global stability condition is qualitatively different, and actually destabilization by **PH** is possible. As an example, we apply Proposition 3.1 to the Ricker–Clark model with **PH** (1.1), which has been studied by Liz and Ruiz-Herrera (2012) (see also Yakubu et al. (2011)).

In this case, $f(x) = \alpha x + (1-\alpha) x e^{r(1-x)}$ is an increasing or bimodal map. For each $\gamma < 1 - ((1-\alpha)e^r + \alpha)^{-1}$, Eq. (1.1) has a unique positive equilibrium

$$x_\gamma = 1 - \frac{1}{r}\ln\left(\frac{1-\alpha(1-\gamma)}{(1-\gamma)(1-\alpha)}\right). \tag{3.6}$$

The system of equations $x_\gamma = (1-\gamma) f(x_\gamma)$, $(1-\gamma) f'(x_\gamma) = -1$ leads to

$$r x_\gamma = \frac{2}{1-\alpha(1-\gamma)}. \tag{3.7}$$

Using the previous equation, we get that

$$(1-\gamma) x_\gamma f''(x_\gamma) > 2 \Longleftrightarrow \alpha(1-\gamma) > \frac{1}{3}.$$

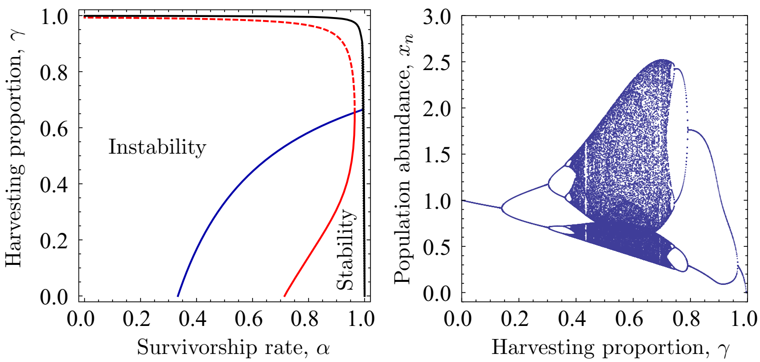

Fig. 1 Left: stability diagram of the **PH** model (1.1), with $r = 7$ in the parameter plane (α, γ). The solid blue line $\alpha(1-\gamma) = 1/3$ divides the region where a positive equilibrium exists (below the extinction boundary delimitated by the black curve) into two parts: above the line, flip bifurcations lead to stabilization (red dashed line); below it, flip bifurcations lead to destabilization (red solid line). In this case, destabilization as γ is increased occurs for $5/7 \approx 0.714 < \alpha < e^4/(2+e^4) \approx 0.964$. Right: bifurcation diagram for $r = 7$, $\alpha = 0.8$, for which Theorem B.1 guarantees abundance of observable chaos for some values of γ (see the text)

Therefore, in the light of Proposition 3.1, the equilibrium $p = 1$ of the Ricker–Clark model can be destabilized if $\alpha(1-\gamma) > 1/3$, and can be stabilized if $\alpha(1-\gamma) < 1/3$. In particular, since $x_\gamma < 1$ for all $\gamma > 0$, destabilization of the equilibrium $p = 1$ in the Ricker–Clark model by **PH** is only possible if $r > 3$. We plot a stability diagram in Fig. 1 for $r = 7$, in terms of parameters α and γ.

It is not difficult to give an explicit range of values for which destabilization by **PH** occurs in the Ricker–Clark model: Eqs. (3.6) and (3.7) lead to $\alpha = 1 - 2/r$ for $\gamma = 0$, and $\alpha = e^{r-3}/(2+e^{r-3})$ when $\alpha(1-\gamma) = 1/3$. Thus, destabilization by **PH** occurs if $1 - 2/r < \alpha < e^{r-3}/(2+e^{r-3})$. This result was already obtained with a more complicated proof (Liz and Ruiz-Herrera 2012, Theorem 2). We emphasize that Proposition 3.1 provides a systematic approach to face this problem.

As an application of Theorem B.1, we show that global attraction may evolve to abundance of observable chaos, via proportional harvesting, in the Ricker–Clark model.

For $r = 7$ and $\alpha = 0.8$, f has, besides its unique positive equilibrium $p = 1$, two critical points:

$$c_1 = \frac{1 - W_0(\alpha e^{1-r}/(\alpha - 1))}{r} \approx 0.144288$$

and

$$c_2 = \frac{1 - W_{-1}(\alpha e^{1-r}/(\alpha - 1))}{r} \approx 1.06898,$$

where the so-called Lambert W functions $W_0 : [-1/e, \infty) \to [-1, \infty)$ and $W_{-1} : [-1/e, 0) \to (-\infty, -1]$ are the two branches of the inverse of $x \mapsto xe^x$. A routine calculation shows that $Sf(x) < 0$ for any $x \notin \{c_1, c_2\}$, $f''(c_2) > 0$ and the interval $I = [f(c_2), f^2(c_2)]$ is invariant for f. Since the restriction of f to I is (lower-) unimodal, and $f'(1) = -0.4$, the Allwright-Singer theorem implies that 1 is globally asymptotically stable for $f|_I$. In fact, 1 attracts the orbits of all points $x > 0$, because they eventually fall into I.

We now apply proportional harvesting to f to get the maps $f_\gamma(x) = (1-\gamma)f(x)$, $\gamma \in [0, \gamma_1]$, $\gamma_1 \approx 0.673626$, and consider their (monotone or unimodal) restrictions to the f_γ-invariant intervals $I_\gamma = [a(\gamma), b(\gamma)] = [f_\gamma(c_2), f_\gamma^2(c_2)]$, with equilibrium $p(\gamma) = x_\gamma$ as in (3.6). Unimodality (at most) holds, in fact, whenever $0 \le \gamma < 1 - c_1/f(c_2) \approx 0.853826$, but we have chosen γ_1 to get $f_{\gamma_1}^3(c_2) = p(\gamma_1) = q \approx 0.653349$, when $f'_{\gamma_1}(q) \approx -2.37932$ implies that the strong Misiurewicz condition (see Appendix B) is satisfied. Then $h(f_0|_{I_0}) = h(f|_I) = 0$ and $h(f_{\gamma_1}|_{I_{\gamma_1}}) > 0$. Finally, with the notation of Theorem B.1,

$$\begin{aligned}\frac{\partial F_3}{\partial \gamma}(c_2, \gamma_1) &\approx -1.64288\\ &\neq -0.592379 \approx p'(\gamma_1) = \frac{-f(q)}{1 - f'_{\gamma_1}(q)} = \frac{(\partial F/\partial \gamma)(q, \gamma_1)}{1 - f'_{\gamma_1}(q)}\end{aligned}$$

and Theorem B.1 guarantees abundance of chaos observable for the family $f_\gamma(x)$, in particular near γ_1.

4 Constant harvesting and constant immigration

In this section, we consider the constant harvesting scheme (2.3). Applying Theorem A.1 to the family $f_\gamma(x) = f(x) - \gamma$, we easily get the following result:

Proposition 4.1 *Let p be the positive equilibrium of (2.1). Then :*

(a) If p is stable then it is destabilized by **CH** *as γ is increased if there is $\gamma > 0$ such that (2.3) has an equilibrium x_γ with $f'(x_\gamma) = -1$ and $f''(x_\gamma) > 0$.*
(b) If p is unstable then it is stabilized by **CH** *as γ is increased if there is $\gamma > 0$ such that (2.3) has an equilibrium x_γ with $f'(x_\gamma) = -1$ and $f''(x_\gamma) < 0$.*

In order to relate our results with discussions in the literature about the effects of immigration in discrete-time models (McCallum 1992; Stone 1993; Doebeli 1995; Stone and Hart 1999), we also consider the model with constant immigration

$$x_{n+1} = f(x_n) + \delta, \quad n = 0, 1, 2, \ldots, \tag{4.1}$$

where $\delta > 0$ is the constant immigration in each reproductive period. Obviously, (4.1) coincides with (2.3) for $\gamma < 0$. Thus, the corresponding version of Proposition 4.1 establishes that the positive equilibrium of (2.1) can be destabilized as δ is increased in (4.1) if $f'(x_\delta) = -1$ and $f''(x_\delta) < 0$ occur for an equilibrium x_δ of (4.1). One is tempted to believe (see, e.g., Doebeli 1995, p. 86) that if constant harvesting is stabilizing for a given model, then constant migration should be destabilizing. Nevertheless, we show below that this is far from being true in general.

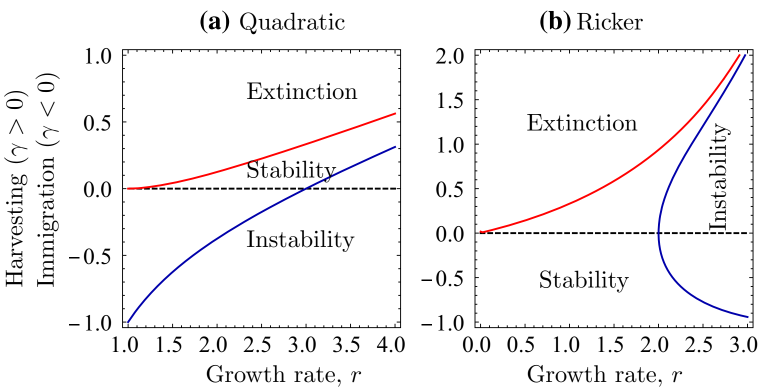

Fig. 2 Stability diagrams of the **CH** ($\gamma > 0$) and immigration ($\gamma < 0$) models (2.3). **a** For the quadratic map, increasing harvesting is stabilizing for $r > 3$ and increasing immigration is destabilizing for $r < 3$. Observe that, strictly speaking, the picture only makes sense when $r/4 - \gamma \leq 1$, otherwise f_γ does not map $[0, 1]$ into itself. **b** For the Ricker map, both increasing harvesting and immigration tend to stabilize the positive equilibrium for $r > 2$. Of course, overharvesting leads to extinction in both models

We recall that increasing γ produces two positive equilibria. Their roles are very different: the smallest one is called the Allee threshold, determines a population level below which extinction occurs, and it is always unstable; the largest one is the equilibrium about which population can persists indefinitely (sometimes it is called the carrying capacity). We are interested in the latter. For more details on Allee effects induced by constant harvesting and their consequences, see, e.g., Sinha and Parthasarathy (1996), Schreiber (2001), and Liz (2010a).

4.1 The quadratic and Ricker maps

We begin with the simple case of the quadratic map $f(x) = rx(1 - x)$. We assume $1 < r \leq 4$ to ensure that there is a positive equilibrium and f maps $[0, 1]$ into itself. In this case, it is clear that constant harvesting stabilizes the equilibrium if $3 < r \leq 4$ and constant immigration destabilizes it if $1 < r \leq 3$ (see Fig. 2a).

However, we will see that the situation is different for the Ricker map and much subtler for the Bellows model.

Consider Eq. (2.3) with the Ricker map $f(x) = xe^{r(1-x)}$. For $\gamma > 0$, it stands for a harvesting model, while for $\gamma < 0$ represents constant immigration. Observe, first of all, that f'' vanishes exactly at $2/r$, hence (when $r > 2$) $f'(x) = f'_\gamma(x) = -1$ has exactly two solutions, one to the left, the other one to the right of $2/r$.

The largest positive equilibrium x_γ (when it exists) is defined by equation

$$x_\gamma e^{r(1-x_\gamma)} = x_\gamma + \gamma.$$

Hence,

$$e^{r(1-x_\gamma)} = \frac{x_\gamma + \gamma}{x_\gamma}.$$

The last equality implies that $e^{r(1-x_\gamma)} > 1$ if $\gamma > 0$ and $e^{r(1-x_\gamma)} < 1$ if $\gamma < 0$. Next,

$$f'(x_\gamma) = -1 \Longleftrightarrow e^{r(1-x_\gamma)} = \frac{1}{rx_\gamma - 1},$$

which implies that $rx_\gamma < 2$ if $\gamma > 0$ and $rx_\gamma > 2$ if $\gamma < 0$.

Finally, since $f''(x_\gamma) = r(rx_\gamma - 2)e^{r(1-x_\gamma)}$, we get that $f''(x_\gamma) < 0$ if $f'(x_\gamma) = -1$ with $\gamma > 0$, and $f''(x_\gamma) > 0$ if $f'(x_\gamma) = -1$ with $\gamma < 0$.

Thus, increasing either constant harvesting or constant migration rates in the Ricker model can never destabilize the positive equilibrium. Actually, both have a stabilizing effect if $r > 2$. We show the stability and extinction regions in Fig. 2b. To avoid further complication of the figures, we do not show the region of essential extinction (the interested reader is referred to Schreiber (2001)).

We notice that a similar diagram to Fig. 2b was obtained numerically by Solé et al. (1999), who considered the stabilizing effect of proportional and constant harvest/migration in the Ricker model but without analytical results; actually their diagram for proportional harvesting seems to be wrong.

4.2 The Bellows map

Now we focus on the Bellows map with constant harvesting or immigration, that is,

$$x_{n+1} = \max\left\{\frac{ax_n}{1 + x_n^m} - \gamma, 0\right\}, \quad n = 0, 1, 2, \ldots, \tag{4.2}$$

where $\gamma > 0$ corresponds to harvesting and $\gamma < 0$ to immigration.

This is a very interesting case; although it is known that increasing immigration can be either stabilizing or destabilizing (some examples are given in Stone and Hart (1999)), sufficient conditions under which one or the other occur are not available. For constant harvesting, it is clear that a sufficiently large harvesting quota is stabilizing: let $c = (m-1)^{-1/m}$ be the unique critical point of $f(x) = ax/(1 + x^m)$; then, for $\gamma = f(c) - c$, $x_\gamma = c$ is an asymptotically stable equilibrium of (4.2) because $f'(c) = 0$. However, as far as we know, it is an open problem whether or not **CH** can destabilize a stable equilibrium in (4.2). Our next result gives a positive answer to this question.

We recall (see, e.g., Liz and Buedo-Fernández 2019, Proposition 2) that the positive equilibrium of (2.1) with the Bellows map $f(x) = ax/(1 + x^m)$ is asymptotically stable if and only if $m \le 2$ or $m > 2$ and $1 < a \le m/(m-2)$.

Theorem 4.2 *For equation (4.2) with $a > 1$ and $m > 2$, we have:*

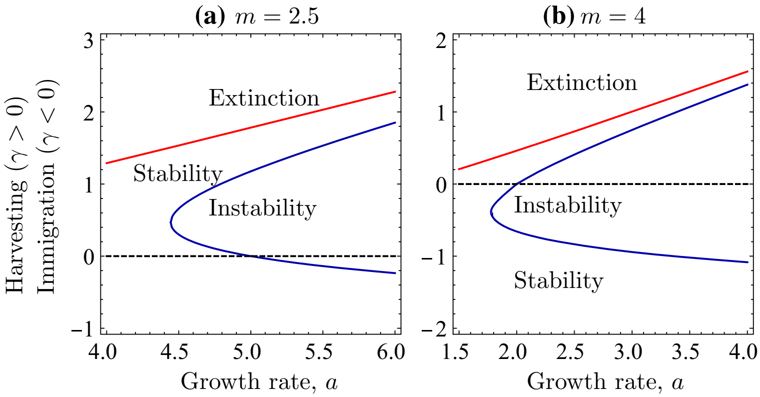

Fig. 3 Stability diagrams of (4.2) with harvesting ($\gamma > 0$) and immigration ($\gamma < 0$). **a** For $m = 2.5$, increasing harvesting is destabilizing for $40/9 \approx 4.44 < a \leq 5$. **b** For $m = 4$, increasing immigration is destabilizing for $16/9 \approx 1.77 < a \leq 2$

(a) Increasing harvesting is destabilizing if and only if

$$m < 3 \quad \text{and} \quad \frac{4m}{(m-1)^2} < a \leq \frac{m}{m-2}.$$

(b) Increasing immigration is destabilizing if and only if

$$m > 3 \quad \text{and} \quad \frac{4m}{(m-1)^2} < a \leq \frac{m}{m-2}.$$

The proof of Theorem 4.2 can be found in Appendix A.

Figure 3 illustrates the destabilizing effects of increasing harvesting ($m = 2.5$, $\gamma > 0$) or immigration ($m = 4$, $\gamma < 0$) in (4.2). Figure 4a shows a bifurcation diagram for increasing harvesting setting $m = 2.5$, $a = 4.75$ in (4.2).

Remark 4.3 The following remarks are in order:

- We have restricted the statement of Theorem 4.2 to $m > 2$ to ensure that the Schwarzian derivative is negative. However, our analysis and simulations show that increasing harvesting is also destabilizing for (4.2) if $1 < m \leq 2$ and $a > 4m/(m-1)^2$. See Fig. 4b.
- It is worth mentioning that the case $m = 3$ is not included in the statement of Theorem 4.2. For $m = 3$, the qualitative behaviour of (4.2) is essentially the same as that of the Ricker map with **CH**, and therefore destabilization is not possible. The stability diagram is similar to Fig. 2b.

As an application of Theorem B.1, we show that global attraction may evolve to abundance of observable chaos, via constant harvesting in the Bellows model.

Consider the Bellows map $f(x) = ax/(1 + x^m)$ with $a = 20$ and $m = 2.1$. Recall that f has negative Schwarzian derivative except at its only critical point $c = (m - 1)^{1/m} \approx 0.955629$, with $f''(c) < 0$. Since $a < m/(m-2)$, its positive equilibrium

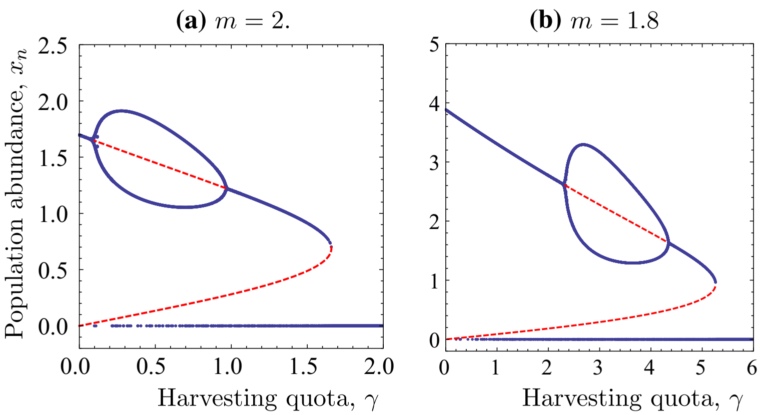

Fig. 4 Bifurcation diagrams of (4.2) with harvesting ($\gamma > 0$), showing the destabilizing effect of increasing the harvesting quota: **a** $m = 2.5$, $a = 4.75$; **b** $m = 1.8$, $a = 12.5$

$p = (a-1)^{1/m} \approx 4.06378$ is asymptotically stable and, in fact, attracts all positive orbits of the map.

Let $p_0 \approx 0.811919$. Then $\gamma_0 = f(p_0) - p_0 = f(c) - c \approx 9.05572$ and $f(x) - x$ is strictly increasing in $[0, p_0]$. Let $\gamma = \gamma(p) = f(p) - p$, $\gamma : [0, p_0] \to [0, \gamma_0]$, and let $p = p(\gamma)$ denote its inverse. If $f(f(c) + p - f(p)) \geq f(p)$, then $f_\gamma(x) = f(x) - \gamma$ leaves invariant the interval $[p, f_\gamma(c)]$ and p is a fixed point of f_γ (and so is c when $\gamma = \gamma_0$). If $p_1 \approx 0.0991346$ and $p_2 \approx 0.653378$, when we accordingly write $\gamma_1 = \gamma(p_1) \approx 1.86821$ and $\gamma_2 = \gamma(p_2) \approx 8.62022$, it can be checked that this exactly happens when $p \in [0, p_1] \cup [p_2, p_0]$ or, equivalently, when $\gamma \in [0, \gamma_1] \cup [\gamma_2, \gamma_0]$. Then $f^2_{\gamma_1}(c) = p_1$, $f^2_{\gamma_2}(c) = p_2$, with $(f_{\gamma_1})'(p_1) \approx 19.5227$ and $(f_{\gamma_2})'(p_2) \approx 5.53959$. Also,

$$\begin{aligned}
&\frac{\partial F_2}{\partial \gamma}(c, \gamma_1) \approx -0.740373 \\
&\neq 0.0539879 \approx \frac{-1}{1 - f'_{\gamma_1}(p_1)} = \frac{(\partial F/\partial \gamma)(p_1, \gamma_1)}{1 - f'_{\gamma_1}(p_1)}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial F_2}{\partial \gamma}(c, \gamma_2) = -1.6668 \\
&\neq 0.220284 = \frac{-1}{1 - f'_{\gamma_2}(p_2)} = \frac{(\partial F/\partial \gamma)(p_2, \gamma_2)}{1 - f'_{\gamma_2}(p_2)}.
\end{aligned}$$

Then we have abundance of observable chaos at the parameter intervals $[0, \gamma_1]$ and $[\gamma_2, \gamma_0]$, in particular near γ_1 and γ_2. See Fig. 5.

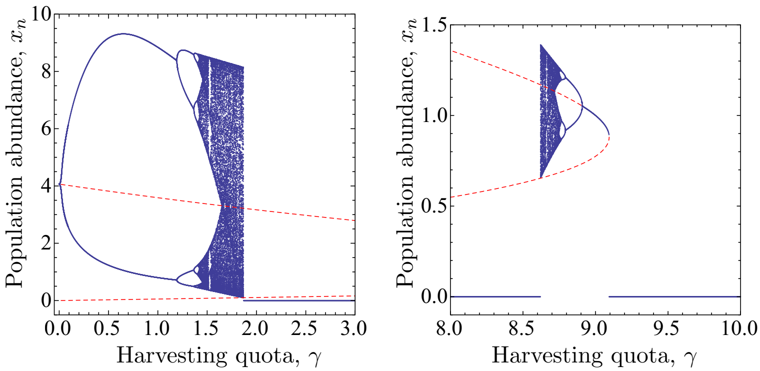

Fig. 5 Bifurcation diagram of (4.2) with harvesting ($\gamma > 0$), showing how increasing the harvesting quota can lead from a globally stable equilibrium for the unharvested population ($\gamma = 0$) to observable chaos and then from chaos to global stability again. Parameter values are $m = 2.1$, $a = 20$. Theorem B.1 ensures existence of observable chaos for γ near $\gamma_1 \approx 1.868$ and $\gamma_2 \approx 8.62$. Essential extinction occurs for $\gamma \in (\gamma_1, \gamma_2)$

4.3 The Ricker–Clark model

In this subsection, we study the destabilizing effects of **CH** in the Ricker–Clark model, that is:

$$x_{n+1} = \max\left\{\alpha x_n + (1-\alpha)x_n e^{r(1-x_n)} - \gamma, 0\right\}, \quad n = 0, 1, 2, \ldots, \tag{4.3}$$

Equation (4.3) has been considered in (Liz 2010a), but that paper focus on extinction. However, some numerical results show that destabilization by **CH** is possible (Liz 2010a, Fig. 5 (*b*), p. 215). Using Proposition 4.1, we give the exact range of values for which the positive equilibrium of the Ricker–Clark model is destabilized by increasing a constant harvesting quota.

Proposition 4.4 *The positive equilibrium of the Ricker–Clark model is destabilized by increasing* γ *in* (4.3) *if and only if* $r > 2$ *and*

$$\frac{r-2}{r} < \alpha < \frac{-1+e^{r-2}}{1+e^{r-2}}. \tag{4.4}$$

The proof of Proposition 4.4 can be found in Appendix A. We illustrate the result of Proposition 4.4 in Fig. 6, with $r = 3$. As we did in Fig. 2, we do not show the region of essential extinction to avoid distracting from the stability region. For a diagram focused on survival and extinction switches in (4.3), we refer to (Liz 2010a, Fig. 4).

5 Discussion

The possibility of increasing dynamical complexity in response to increasing harvesting in simple population models has been an object of many studies because it is

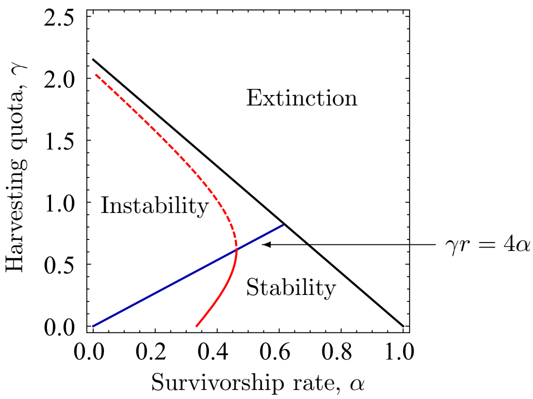

Fig. 6 Stability diagram of the **CH** model (4.3), with $r = 3$ in the parameter plane (α, γ). The solid blue line $\gamma r = 4\alpha$ divides the region where a positive equilibrium exists into two parts: above the line, flip bifurcations lead to stabilization (red dashed line); below it, flip bifurcations lead to destabilization (red solid line). In this case, destabilization occurs for $1/3 < \alpha < (e-1)/(e+1) \approx 0.46$

considered an undesirable effect of increasing harvesting pressure in different contexts, such as fishery management and pest control.

While destabilizing effects of harvesting have been related to variable fishing pressure (Jonzén et al. 2003; Anderson et al. 2008) and age-truncation effects (Hsieh et al. 2006; Anderson et al. 2008; Liz and Pilarczyk 2012), the possibility of inducing fluctuations in a stable population by a combination of overcompensatory effects and high harvesting rates has been rarely detected in the usual single-species models for semelparous populations. On the contrary, stabilizing effects of harvesting have been reported for proportional harvesting (Ricker 1954; Goh 1977; Liz 2010b) and for constant quota harvesting (McCallum 1992; Solé et al. 1999; Schreiber 2001). Although possible destabilizing effects of increasing constant harvesting or immigration have been suggested (Doebeli 1995; Stone and Hart 1999), a rigorous study providing a criterion under which destabilization occurs in a particular model was not available.

Our main contributions in this paper are the following: first, we rigorously prove that stabilization or destabilization switches depend on the harvesting strategy and the considered model. In particular, although destabilization by increasing a constant harvesting quota is not possible for the quadratic and the Ricker maps, we prove that it is possible for the Bellows map, and provide the exact conditions on the parameters for which this effect is observed (Theorem 4.2). This result was an unexpected outcome of our approach. Second, we provide sufficient conditions for a shift from a globally stable positive equilibrium to chaotic dynamics induced by constant quota harvesting. In contrast with most available results in this direction, Theorem B.1 can be easily applied to get concrete values of the model parameters for which observable chaos is generated by increasing harvesting rates.

Our study sheds some new light on the potential stabilizing or destabilizing effects of immigration. This topic has been treated by many authors, but mostly based on numerical observations. McCallum (1992) and Stone (1993) showed that adding a fixed number of individuals to a population every year (for example, by immigration) has the potential to simplify the dynamics. The stabilizing effects of immigration were later studied by Doebeli (1995) and Stone and Hart (1999), among others (see Solé et al. (1999) and its references). However, some conclusions were controversial. For example, Doebeli (1995) argued that the relative position of the equilibrium to the inflection point in the original map is enough to know whether adding constant migration has a stabilizing or a destabilizing effect. This oversimplification has been criticized (and refuted) by Stone and Hart (1999). However, Stone and Hart only justify that sufficiently large values of migration always induce a stable fixed point, but they do not provide a method to study if increasing migration can destabilize a stable equilibrium.

It is worth mentioning that, according to Doebeli's conclusions, adding or removing a constant number of individuals should have opposite effects on stability. However, we proved that a stable fixed point of the Ricker map cannot be destabilized either by constant migration or constant harvesting, a property that can be observed in numerical results made by (Solé et al. 1999, Fig. 2). A more involved example is given by the Bellows map: while numerical bifurcation diagrams in Stone and Hart (1999) suggest that constant migration can either stabilize or destabilize the positive equilibrium, depending on the model parameters, no analytic results are provided. Whether destabilization is possible with constant harvesting in the Bellows model remained as an open question that we solved in this paper.

We have also revisited the possible destabilizing effects of proportional and constant harvesting in simple one-dimensional models for iteroparous populations, providing a more systematic approach to previous studies (Liz 2010a; Yakubu et al. 2011; Liz and Ruiz-Herrera 2012).

We underline that our approach is quite general, and we expect that it can be applied to other harvesting strategies in order to find the exact range of parameter values for which increasing harvest rates leads to stability switches. For example, recent work showed that increasing harvesting by reducing the threshold can be destabilizing in a strategy of proportional threshold harvesting applied to a population growth governed by a Ricker map (Hilker and Liz 2019).

In the context of population management, on the one hand, our results support the view that sustainable exploitation requires harvesting strategies different from the classical constant quota searching to maximize the yield (Lande et al. 1995). In particular, this is especially important for stable populations in which intraspecific competition leads to stock-recruitment functions which can generate chaos when subject to suitable harvesting quotas (see, e.g., Fig. 5). On the other hand, in the line with the discussion in Jonzén et al. (2003), our results also emphasize the importance of a rigorous study of the dynamics of exploited populations because harvesting has the potential to change the behavior of an exploited population in many ways.

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A Appendix: results on stability switches

In this appendix, we prove the results related to the destabilizing effects of harvesting. Our results on stabilization/destabilization are based on the following characterization of flip bifurcations:

Theorem A.1 (Sharkovsky et al. 1997, Theorem 8.2) *Let $f_\gamma : I \to I$ be a family of $\mathcal{C}^3$ maps with smooth dependence on the parameter γ, where I is a real interval. Assume that f_{γ_0} has a fixed point x_{γ_0} and the following conditions hold:*

(a) $f'_{\gamma_0}(x_{\gamma_0}) = -1$,
(b) $(Sf_{\gamma_0})(x_{\gamma_0}) < 0$, *where Sf is the Schwarzian derivative of f, and*
(c) $\dfrac{\partial}{\partial\gamma}\left(f'_\gamma(x)\right) < 0$ *at $\gamma = \gamma_0$ and $x = x_{\gamma_0}$.*

Then, there are $\varepsilon > 0$ and $\delta > 0$ such that

(i) for $\gamma \in (\gamma_0 - \delta, \gamma_0)$, f_γ has exactly one fixed point $x_\gamma \in (x_{\gamma_0} - \varepsilon, x_{\gamma_0} + \varepsilon)$ and x_γ is asymptotically stable;
(ii) for $\gamma \in (\gamma_0, \gamma_0 + \delta)$, there are three fixed points of f^2_γ in $(x_{\gamma_0} - \varepsilon, x_{\gamma_0} + \varepsilon)$. Moreover, the middle point is an unstable fixed point of f_γ, and the other two points form an asymptotically stable cycle of f_γ of period two.

If the inequality in (c) has the opposite sign, then the conclusions remain valid, but the 2-cycle appears as γ decreases.

Proof of Proposition 3.1 The proof follows easily from the application of Theorem A.1 to the family $f_\gamma(x) = (1-\gamma)f(x)$. Denote by $G(x,\gamma) = f_\gamma(x) - x$, so that the equilibria of (2.2) are defined by $G(x,\gamma) = 0$.

Using implicit differentiation, we have:

$$\begin{aligned}\frac{\partial}{\partial\gamma}\left(f'_\gamma(x_\gamma)\right) &= \frac{\partial}{\partial\gamma}\left((1-\gamma)f'(x_\gamma)\right)\\ &= -f'(x_\gamma) - (1-\gamma)f''(x_\gamma)\left(\frac{\partial G/\partial\gamma}{\partial G/\partial x}\right)(x_\gamma)\\ &= -f'(x_\gamma) + (1-\gamma)f''(x_\gamma)\frac{f(x_\gamma)}{(1-\gamma)f'(x_\gamma) - 1}.\end{aligned}$$

If $f'_\gamma(x_\gamma) = -1$, we get $(1-\gamma)f'(x_\gamma) = -1$, and therefore

$$\frac{\partial}{\partial\gamma}\left(f'_\gamma(x_\gamma)\right) = \frac{1}{1-\gamma} - \frac{f''(x_\gamma)x_\gamma}{2}.$$

Thus,

$$\frac{\partial}{\partial\gamma}\left(f'_\gamma(x)\right) < 0 \Longleftrightarrow (1-\gamma)x_\gamma f''(x_\gamma) > 2,$$

from which the result follows. □

Proof of Theorem 4.2 Denote $f(x) = ax/(1 + x^m)$, so that

$$f'(x) = \frac{-a\left((m-1)x^m - 1\right)}{(1+x^m)^2} \quad ; \quad f''(x) = \frac{amx^{m-1}\left(m(x^m-1) - 1 - x^m\right)}{(1+x^m)^3}.$$

If p is the positive fixed point of f and x_γ is the largest positive equilibrium of (4.2), then it is clear that $x_\gamma < p$ if $\gamma > 0$, and $x_\gamma > p$ if $\gamma < 0$.

First we notice that condition $a \le m/(m-2)$ is necessary to ensure that the fixed point p of $f(x)$ is asymptotically stable. Next, condition $a > 4m/(m-1)^2$ is necessary to ensure that there is a value of γ for which $f'(x_\gamma) = -1$ and $x_\gamma = f(x_\gamma) - \gamma$. Indeed, the first equality leads to equation

$$a\left((m-1)x_\gamma^m - 1\right) = \left(1 + x_\gamma^m\right)^2.$$

Denoting $z = 1 + x_\gamma^m$, we get that z must be a zero of the polynomial $q(z) = z^2 - a(m-1)z + am$. Since the discriminant of $q(z)$ is $\Delta = a^2(m-1)^2 - 4am$, we have that $\Delta > 0 \iff a > 4m/(m-1)^2$. In case $\Delta = 0$, we can verify that $f'(x) > -1$ for all $x \ne x_\gamma$, and therefore destabilization by **CH** is not possible.

Hence, under the conditions of Theorem 4.2 (a) and (b), equation $q(z) = 0$ has two positive solutions $z_1 > z_2$, that provide two values γ_1, γ_2 for which $f'(x_{\gamma_i}) = -1$ and $x_{\gamma_i} = f(x_{\gamma_i}) - \gamma_i$, $i = 1, 2$. Moreover, condition $a \le m/(m-2)$ implies that $z_1 > z_2 \ge a$ if $m > 3$, and $z_2 < z_1 \le a$ if $m < 3$, the equality corresponding to the case $a = m/(m-2)$.

If $m < 3$, then $z_2 < z_1 \le a$, which is equivalent to $x_{\gamma_2}^m < x_{\gamma_1}^m \le p$ and then $f(x_{\gamma_2}^m) > x_{\gamma_1}^m \ge p$. Therefore, $0 \le \gamma_1 < \gamma_2$. We prove that $f''(x_{\gamma_1}) > 0$ and $f''(x_{\gamma_2}) < 0$, which implies by Proposition 4.1 that increasing harvesting is first destabilizing and then stabilizing again.

Using the expression of $f''(x)$, we have:

$$\begin{aligned} f''(x_{\gamma_1}) > 0 > f''(x_{\gamma_2}) &\iff m(x_{\gamma_1}^m - 1) - 1 - x_{\gamma_1}^m > 0 > m(x_{\gamma_2}^m - 1) - 1 - x_{\gamma_2}^m \\ &\iff 1 + x_{\gamma_2}^m = z_2 < \frac{2m}{m-1} < 1 + x_{\gamma_1}^m = z_1. \end{aligned}$$

Thus, we only need to prove that $q(2m/(m-1)) < 0$. Indeed,

$$q(2m/(m-1)) = m\left(\frac{4m}{(m-1)^2} - a\right) < 0,$$

because $a > 4m/((m-1)^2)$.

If $m > 3$, then $a \le z_2 < z_1$ and $\gamma_1 < \gamma_2 \le 0$. Repeating the above argument, it follows that $f''(x_{\gamma_2}) < 0$ and $f''(x_{\gamma_1}) > 0$, which implies in this case that increasing immigration is first destabilizing and then stabilizing again. □

Proof of Proposition 4.4 Denote by $f(x) = \alpha x + (1-\alpha)xe^{r(1-x)}$.

The system of equations

$$f'(x_\gamma) = -1 \quad , \quad x_\gamma = f(x_\gamma) - \gamma \tag{A.10}$$

leads to

$$(rx_\gamma - 1)\left((1-\alpha)x_\gamma + \gamma\right) = (1+\alpha)x_\gamma. \tag{A.20}$$

Hence, system (A.10) has a positive solution, which is given by the positive root of the quadratic equation

$$Q(x_\gamma) = r(1-\alpha)x_\gamma^2 + (\gamma r - 2)x_\gamma - \gamma = 0.$$

Since $f''(x_\gamma) > 0$ if and only if $rx_\gamma > 2$, we have, using (A.20), that

$$\begin{aligned} f''(x_\gamma) > 0 &\Longleftrightarrow rx_\gamma > 2 \Longleftrightarrow (1-\alpha)x_\gamma + \gamma < (1+\alpha)x_\gamma \\ &\Longleftrightarrow x_\gamma > \frac{\gamma}{2\alpha} \Longleftrightarrow Q(\gamma/(2\alpha)) < 0 \Longleftrightarrow \gamma r < 4\alpha. \end{aligned}$$

Therefore, Proposition 4.1 ensures that the positive equilibrium $p = 1$ of the Ricker–Clark equation can be destabilized if $\gamma r < 4\alpha$ and can be stabilized if $\gamma r > 4\alpha$.

We next show that for a given $r > 2$, destabilization occurs if and only if $\alpha \in (\alpha_1(r), \alpha_2(r))$, with $\alpha_1(r) = (r-2)/r$ and $\alpha_2(r) = (-1+e^{r-2})/(1+e^{r-2})$.

Notice first that $p = 1$ must be asymptotically stable for the Ricker–Clark equation, that is, $f'(1) \geq -1$. This condition is equivalent to $r > 2$ and $\alpha \geq (r-2)/r$.

The value $\alpha_2(r)$ is determined finding the intersection point between the flip bifurcation curve defined by (A.10) and the line $\gamma r = 4\alpha$ (see Fig. 6).

Using the expression of the positive root of the quadratic polynomial $Q(x)$ and substituting γ by $4\alpha/r$, we easily get $x_\gamma = 2/r$.

Now, using the formulas $rx_\gamma = 2$ and $\gamma = 4\alpha/r$ in the equilibrium equation $x_\gamma = f(x_\gamma) - \gamma$, it follows that $\alpha = (-1+e^{r-2})/(1+e^{r-2})$.

Let us emphasize, to conclude, that (as in Sect. 4.1) $2/r$ is the only zero of f'', allowing two γ's with corresponding x_γ satisfying $f'_\gamma(x_\gamma) = -1$ for every $\alpha \in (\alpha_1(r), \alpha_2(r))$. □

B Appendix: results on chaos

In this appendix, we state and prove our main results related to the existence of observable chaos in discrete-time population models with harvesting.

We follow the notations and definitions stated at the beginning of Sect. 2. The celebrated Allwright-Singer theorem (Allwright 1978; Singer 1978) establishes that if f has a stable fixed point p, $f(x) > x$ (respectively $f(x) < x$) whenever $x < p$ (respectively, $x > p$), and $Sf(x) < 0$ except for at most one critical point (an extremum) of f, then all orbits of f converge to p.

Let $f \in C(I)$ with I compact. The *(topological) entropy* of f, $h(f) \in [0, \infty]$, is defined by

$$h(f) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_n(f, \epsilon),$$

where $s_n(f, \epsilon)$ is the maximal cardinality of the sets E having the property that for all distinct points $x, y \in E$ there is $0 \le i < n$ such that $|f^i(x) - f^i(y)| > \epsilon$. If f consists of finitely many pieces of monotonicity, and c_n is the number of laps of f^n, that is, the minimal cardinality of an interval partition of I such that f^n is monotone on each of the intervals of the partition, then $h(f) = \lim_{n\to\infty}(1/n) \log c_n$ (Misiurewicz and Szlenk 1980). Positive entropy admits several useful characterizations. Namely, if $f \in C(I)$, then the following statements are equivalent (Li et al. 1982; Ruette 2017, Theorem 4.58, p. 84):

- $h(f) > 0$;
- f has a periodic point of period not a nonnegative power of 2;
- there are a point x, a positive integer m and an odd number k such that either $f^{km}(x) \le x < f^m(x)$ or $f^{km}(x) \ge x > f^m(x)$;
- there are subintervals J, K of I with disjoint interiors and positive integers m, l such that $f^m(J) \cap f^l(K) \supset J \cup K$.

On the other hand, if the orbits of all points of I are attracted by periodic orbits (that is, for any x there is a periodic point p such that $\lim_{n\to\infty} |f^n(x) - f^n(p)| = 0$), then $h(f) = 0$ (Ruette 2017, Lemma 5.5, p. 110, and Theorem 5.17, p. 116). A sufficient condition for this to happen (Sharkovsky et al. 1997, pp. 73–74) is that the set of period of periodic points of f is bounded —equivalently, see below, f has type less than 2^∞ in the Sharkovsky order—. Therefore, monotone maps have zero entropy (they can only have periodic points of periods 1 and 2) and so they have the restrictions to invariant compact intervals of maps satisfying the hypotheses of the Allwright-Singer theorem.

Thus, if a map has positive entropy, then it is, in a sense, dynamically "bad-behaved". It is important to stress that this complicated behaviour needs not be "observable" in practice. For instance, for the quadratic map $f(x) = rx(1-x)$, $r \approx 3.83187\ldots$, $I = [0, 1]$, the point $1/2$ is 3-periodic, hence $h(f) > 0$, but the orbits of almost all points (in the sense of Lebesgue measure) are attracted by the 3-point orbit of $1/2$ (Guckenheimer 1979). The following definition takes care of this problem:

Let $f \in C(I)$. We say that $(x, y) \in I^2$ is a *Li-Yorke* pair (for f) if $\limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0$ and $\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0$. We say that f *has observable chaos* if the set of Li-Yorke pairs has (two-dimensional) positive Lebesgue measure.

A sufficient condition (under mild additional assumptions) to guarantee a very strong form of observable chaos is the existence of an absolutely continuous invariant measure (*acip*) for f. By this we mean a Borel probability measure μ such that $\mu(f^{-1}(B)) = \mu(B)$ for any Borel set B and, moreover, satisfying $\mu(B) = 0$ whenever $\lambda(B) = 0$, λ denoting the Lebesgue measure. For instance, if I is compact, $f \in C^3(I)$ and $f''(c) \ne 0$ for any critical point c of f, then the existence of an acip for f implies (besides $h(f) > 0$) the existence of a subinterval J of I, invariant for some iterate f^r of f, such that:

- λ^2-almost every $(x, y) \in (f^i(J))^2$, $0 \le i < r$, is a Li-Yorke pair;
- the orbit of λ-almost every $x \in K = \bigcup_{i=0}^{r-1} J$ is dense in K;
- $\lim_{n\to\infty} \log |(f^n)'(x)|/n > 0$ for λ-almost every $x \in K$;
- $\lim_{n\to\infty} \lambda(f^{rn+i}(A)) = \lambda(f^i(J))$ for any measurable set $A \subset J$ of positive Lebesgue measure.

For more details on this, including many relevant references, see Bruin and Jiménez López (2010); Barrio Blaya and Jiménez López (2012).

Under the previous smoothness and compactness assumptions, a sufficient hypothesis implying the existence of an acip is the so-called *Misiurewicz condition*, that is, the orbits of critical points do not accumulate on the set of critical points and all periodic points are unstable (van Strien 1990). If f has just one critical point c and negative Schwarzian derivate outside c, this condition can be somewhat relaxed (see Lemma B.5); in particular, if there are an integer k and an unstable periodic point p such that $f^k(c) = p$ (when we say that f satisfies the *strong Misiurewicz condition*), then f has an acip.

The following theorem is a kind of mixture of (Misiurewicz 1981a, Theorem 7.9) and (Thieullen et al. 1994, Theorem I.3), and then not substantially new, but some conditions there are not really necessary, so this concrete formulation may be more useful in applications.

Theorem B.1 *Let the family $f_\gamma \in C^3([a(\gamma), b(\gamma)])$, $\gamma \in [\gamma_0, \gamma_1]$, satisfy the following conditions:*

(i) *the maps $a, b : [\gamma_0, \gamma_1] \to \mathbb{R}$ are continuous;*
(ii) *if $M = \{(x, \gamma) : \gamma \in [\gamma_0, \gamma_1], x \in [a(\gamma), b(\gamma)]\}$, then both $F(x, \gamma) = f_\gamma(x)$ and $F'(x, \gamma) = f'_\gamma(x)$ are continuous in M;*
(iii) *every map f_γ has at most one critical point $c(\gamma)$ and $f''_\gamma(c(\gamma)) \neq 0$ when such a point exists;*
(iv) *$Sf_\gamma(x) < 0$ for any $x \neq c(\gamma)$.*

Let $\Lambda \supset \Lambda_S$ be, respectively, the sets of parameters γ such that f_γ has an acip (and then observable chaos) and f_γ satisfies the strong Misiurewicz condition. If one of the maps f_{γ_0}, f_{γ_1} has zero entropy, and the other one positive entropy, then Λ is uncountable and Λ_S is infinite.

Let $\rho \in \Lambda_S$, assume additionally that F is C^3 near $[a(\rho), b(\rho)] \times \{\rho\}$, let $k, r \ge 1$ be such that $q = f_\rho^k(c(\rho))$ is an unstable r-periodic point of f_ρ and write $F_n(x, \gamma) = f_\gamma^n(x)$, $d = c(\rho)$. If

$$\left(1 - (f_\rho^r)'(q)\right) \frac{\partial F_k}{\partial \gamma}(d, \rho) \neq \frac{\partial F_r}{\partial \gamma}(q, \rho), \tag{B.10}$$

then ρ is a Lebesgue density point of Λ (hence the set of parameters γ such that f_γ has observable chaos has positive measure).

Remark B.2 By "F is C^3 near $[a(\rho), b(\rho)] \times \{\rho\}$" we mean that if $\epsilon > 0$ is small enough, then the restriction of F to $M_\epsilon = \{(x, \gamma) : \gamma \in [\rho - \epsilon, \rho + \epsilon], x \in [a(\gamma), b(\gamma)]\}$ can be extended to a C^3-map defined on an open set containing M_ϵ. We say that ρ is a *Lebesgue density point* of Λ if there is a Borel set $\Sigma \subset \Lambda$ such that $\lim_{\epsilon \to 0^+} \lambda(\Sigma \cap [\rho - \epsilon, \rho + \epsilon])/(2\epsilon) = 1$ (or $\lim_{\epsilon \to 0^+} \lambda(\Sigma \cap [\rho - \epsilon, \rho])/\epsilon = 1$ if $\rho = \gamma_1$, and analogously for γ_0).

Before proving Theorem B.1, a number of additional notions and results will be needed. Some of them involve finite and infinite sequences $\alpha = \alpha_1\alpha_2 \cdots$ of symbols L, C, R, with $|\alpha|$ denoting the (finite or infinite) length of α.

The *shift operator* $\mathcal{S}$ is given by $\mathcal{S}(\alpha) = \alpha_2\alpha_3 \cdots$, hence it is well defined except if $|\alpha| = 1$. The *renormalization operator* $\mathcal{R}$ is given by $\mathcal{R}(\alpha) = R\overline{\alpha_1}R\overline{\alpha_2} \cdots$, where we mean $\overline{L} = R, \overline{C} = C$ and $\overline{R} = L$. As usual, we denote by $\mathcal{S}^n$ (whenever it makes sense) and $\mathcal{R}^n$ the n-iterates of these operators. If α (respectively, β) is finite (respectively, finite or infinite), then $\alpha\beta$ is the concatenation of α and β, with $\alpha^n = \alpha \cdots \alpha$ (n times) and $\alpha^\infty = \alpha\alpha \cdots$ (infinitely many times).

We say that α is *admissible* if it is either an infinite sequence of L's and R's, or a (maybe empty) finite sequence of L's and R's, followed (and finished) by C. Observe that if α is admissible, then $\mathcal{S}(\alpha)$ and $\mathcal{R}(\alpha)$ are admissible as well.

We introduce a total order $<$ in the set of admissible sequences as follows. Firstly, $L < C < R$. Now, if $\alpha \neq \beta$ and k is the first index such that $\alpha_k \neq \beta_k$ (such an index does exist because α and β are admissible), then $\alpha < \beta$ if either there is an even number of R's in $\alpha_1 \cdots \alpha_{k-1} = \beta_1 \cdots \beta_{k-1}$ and $\alpha_k < \beta_k$, or there is an odd number of R's in $\alpha_1 \cdots \alpha_{k-1} = \beta_1 \cdots \beta_{k-1}$ and $\alpha_k > \beta_k$. An admissible sequence α is said to be *maximal* if $\mathcal{S}^n(\alpha) \leq \alpha$ for any $n \leq |\alpha| - 1$ (for all n if $|\alpha| = \infty$).

Let $f \in C([a, b])$. We say that f is *unimodal* if there is $a < c < b$ such that both $f|_{[a,c]}$ and $f|_{[c,b]}$ are strictly monotone and c is a turning point of f, when f is called *upper-unimodal* (respectively, *lower-unimodal*) if c is a maximum (respectively, a minimum). If additionally $\{f^2(c), f(c)\} = \{a, b\}$, then we say that f is *strictly unimodal*.

The *kneading invariant* of a unimodal map f, $K(f)$, is an admissible sequence defined as follows. If $f^n(c) \neq c$ for any positive integer n, then $|K(f)| = \infty$ and $K(f)_n$ equals L or R according to whether f is increasing or decreasing on $f^n(c)$ (that is, whether $f^n(c) < c$ or $f^n(c) > c$ when f is upper-unimodal, and whether $f^n(c) > c$ or $f^n(c) < c$ when f is lower-unimodal). If, otherwise, k is the first index such that $f^k(c) = c$, then $|K(f)| = k$ and $K(f)_n$ ($1 \leq n < k$) equals L or R according to whether f is increasing or decreasing on $f^n(c)$, with $K(f)_k = C$. It can be proved, although this will be of no consequence here, that $K(f)$ is always maximal. Observe that if $K(f) = R \cdots$, then there is exactly one fixed point on which f is decreasing. We call it the *essential fixed point* of f. If, moreover, $K(f) = RL \cdots$, and $f^3(c)$ belongs to the interval with endpoints p and $f(c)$, then we say that f is *renormalizable*. If f is renormalizable, then the interval with endpoints $f^2(c)$ and p is invariant for f^2 and the restriction g of f^2 to this interval is unimodal: we call g the *renormalization* of f and write $g = N(f)$. Observe that if f is upper-unimodal then g is lower-unimodal, and conversely. Of course, the renormalization of a map f needs not be renormalizable itself: if the n-iteration of the operator N, N^n, is well defined on f for all n, then we call f *infinitely renormalizable*.

Lemma B.3 *Let $f \in C([a, b])$ be a unimodal map and assume that $K(f) = \mathcal{R}(\alpha)$ for some admissible sequence $\alpha = RL \cdots \neq RL^\infty$. Then f is renormalizable and $K(N(f)) = \alpha$.*

Proof Assume, for instance, that f is upper-unimodal. We have $\beta = K(f) = RLRR\cdots$, so to prove that f is renormalizable we just need to show that $f^3(c) > p$. But this is clear, because otherwise the first index $k \geq 5$ such that $\beta_k = L$ (such an index does exist because $\alpha \neq RL^\infty$) would be odd, contradicting that $\beta = \mathcal{R}(\alpha)$. The second statement of the lemma easily follows from the first one, because if f is renormalizable, then $f^n(x) \geq p$ for any odd integer n and any $x \in [f^2(c), p]$.

The *Sharkovsky order* is a total order $\lhd$ in the set $\mathbb{Z}^+ \cup \{2^\infty\}$ defined as follows:

$$\begin{aligned} 1 \lhd 2 \lhd 4 \lhd \cdots \lhd 2^n \lhd \cdots \lhd 2^\infty \lhd \cdots \lhd \cdots \lhd \\ (2k+1)2^n \lhd \cdots \lhd 7 \cdot 2^n \lhd 5 \cdot 2^n \lhd 3 \cdot 2^n \lhd \cdots \lhd \\ (2k+1)4 \lhd \cdots \lhd 7 \cdot 4 \lhd 5 \cdot 4 \lhd 3 \cdot 4 \lhd \cdots \lhd \\ (2k+1)2 \lhd \cdots \lhd 7 \cdot 2 \lhd 5 \cdot 2 \lhd 3 \cdot 2 \lhd \cdots \lhd \\ 2k+1 \lhd \cdots \lhd 7 \lhd 5 \lhd 3. \end{aligned}$$

We say that $f \in C(I)$ is *of type* $t \in \mathbb{Z}^+ \cup \{2^\infty\}$ (in the Sharkovsky order), and then we write $T(f) = t$, if the set of periods of all periodic points of f is exactly $\{r \in \mathbb{Z}^+ : r \unlhd t\}$. According to the famous Sharkovsky theorem (see, e.g., Ruette 2017, Theorem 3.13, p. 40), every $f \in C(I)$ has a type. Therefore, the type of f is larger than, equal to, or smaller than 2^∞ according to, respectively, f has some periodic point of period not a power of 2 —equivalently, $h(f) > 0$—, f has periodic points of periods all powers of 2, and no other periods, or the set of periods of periodic points of f is bounded.

Let $\omega = \omega_1\omega_2\ldots$ denote the infinite sequence given by $\omega_n = R$ or $\omega_n = L$ according to (after writing $n = r2^m$ for some odd number r and some nonnegative integer m) whether m is even or odd. Clearly,

$$\omega = RLRRRLRLRLRRRLRRRLRRRLRLRLRRRLRL\cdots$$

is characterized by the property $\mathcal{R}(\omega) = \omega$.

Lemma B.4 *Let $f \in C([a, b])$ be a unimodal map. Then the following statements are equivalent:*

(i) $T(f) = 2^\infty$*;*
(ii) *f is infinitely renormalizable;*
(iii) $K(f) = \omega$*.*

Proof We assume, without loss of generality, that f is upper-unimodal.

(i) $\Rightarrow$ (ii): if $f(c) \leq c$ then $T(f) = 1$ and if $f^2(c) \geq c$ then $T(f) \unlhd 2$, contradicting the hypothesis. Finally, if $f^3(c) < p$, with p the essential fixed point of f, then $f^2([f^2(c), c]) \cap f^2([c, p]) \supset [f^2(c), p]$. This, as we mentioned before, means that $T(f) \rhd 2^\infty$, again a contradiction. Therefore, f is renormalizable. But if $T(f) = 2^\infty$, then it is clear that $T(N(f)) = 2^\infty$. Repeating the previous argument, we conclude that f is infinitely renormalizable.

(ii) $\Rightarrow$ (i): If f is renormalizable, then it cannot have periodic points of odd period (except 1), that is, $T(f) \unlhd 6$. Analogously, $T(N(f)) \unlhd 6$ because $N(f)$ is renormalizable, hence $T(f) \unlhd 12$ and, in general, $T(f) \unlhd 3 \cdot 2^n$ for any n, that is, $T(f) \unlhd 2^\infty$. On the other hand, if $T(f) = 2^k$, then $T(N(f)) = 2^{k-1}$ and, by induction, $T(N^k(f)) = 1$. But if $g = N^k(f)$, then, because g is renormalizable, the signs of the numbers $g^2(c) - c$ and $g^2(g^2(c)) - g^2(c)$ differ, hence there is q between $g^2(c)$ and c such that $g^2(q) = q$ (and $g(q) \neq q$), contradicting $T(g) = 1$. Thus $T(f) = 2^\infty$, as we desired to prove.

(ii) $\Rightarrow$ (iii): If $\beta = K(f)$ and $\alpha = K(N(f))$, then $\beta_{2m-1} = R$ and $\alpha_m = \overline{\beta_{2m}}$ for all $m \geq 1$. But $N(f)$ is renormalizable as well, so $\alpha_{2m-1} = R$, that is, $\beta_{2(2m-1)} = L$ for all $m \geq 1$. In fact $N(f)$, as f, is infinitely renormalizable, and we can analogously get $\alpha_{2(2m-1)} = L$ and then $\beta_{4(2m-1)} = R$ for all m. Proceeding in this way we conclude $\beta = \omega$.

(iii) $\Rightarrow$ (ii): This follows immediately from Lemma B.3.

Lemma B.5 *Let $f \in C^3([a, b])$ be unimodal with turning point c and assume $f''(c) \neq 0$ and $Sf(x) < 0$ for any $x \neq c$. If the orbit of c does not accumulate at c and is not attracted by a stable periodic orbit, then f has an acip.*

Proof Assume as usual that f is upper-unimodal. Since the orbit of c is not attracted by any stable periodic orbit, it is easy to check that $f^2(c) < c < f(c)$, and $f^3(c) \geq f^2(c)$. Therefore, $g = f|_{[f^2(c), f(c)]}$ is a well defined strictly unimodal map, and negative Schwarzian derivative, together with Theorem 6.1 in (de Melo and van Strien 1993, p. 145), guarantee that all periodic points of g are unstable, which after adding the hypothesis that the orbit of c does not accumulate at c means the g satisfies the Misiurewicz condition. Therefore, by Misiurewicz (1981a) or van Strien (1990) (here $g''(c) \neq 0$ is needed), g has an acip. This easily implies (just giving zero measure to $[a, b] \backslash [f^2(c), f(c)]$) that f has an acip as well.

Proof of Theorem B.1 To prove the first part of the theorem we assume, without loss of generality, that $h(f_{\gamma_0}) = 0 < h(f_{\gamma_1})$. We claim that there is $\gamma_0 \leq \gamma_0' < \gamma_1$ such that $h(f_{\gamma_0'}) = 0$ (with $T(f_{\gamma_0'}) = 2^\infty$) and $h(f_\gamma) > 0$ for any $\gamma \in (\gamma_0', \gamma_1]$. To prove the claim, let $\psi_{a,b} : [a, b] \to [-1, 1]$ be the affine diffeomorphism given by $\psi_{a,b}(x) = -1 + 2(x - a)/(b - a)$, $a < b$, and write $g_\gamma = \psi_{a(\gamma),b(\gamma)} \circ f_\gamma \circ \psi^{-1}_{a(\gamma),b(\gamma)}$. Then (i) and (ii) just mean that $\gamma \mapsto g_\gamma$ maps continually the interval $[\gamma_0, \gamma_1]$ into the subspace of $C^1([-1, 1])$ (endowed with the C^1-topology) of C^1-selfmaps on $[-1, 1]$ with at most two pieces of monotonicity. But topological entropy is continuous in this space (Misiurewicz 1995), hence there is a parameter $\gamma_0 \leq \gamma_0' < \gamma_1$ such that $h(g_{\gamma_0'}) = 0$ and $h(g_\gamma) > 0$ for any $\gamma \in (\gamma_0', \gamma_1]$. Moreover, the set of C^1-maps of type less than 2^∞ is open in the C^1-topology (Misiurewicz 1981b), so $T(f_{\gamma_0'}) = 2^\infty$. Since $h(f_\gamma) = h(g_\gamma)$ and $T(f_\gamma) = T(g_\gamma)$ for all γ, the claim follows.

As a consequence, all maps f_γ, $\gamma \in [\gamma_0', \gamma_1]$, are unimodal, that is, $c(\gamma)$ exists and is not an endpoint of $[a(\gamma), b(\gamma)]$; moreover, $c(\gamma)$ lies between $f_\gamma^2(c(\gamma))$ and $f_\gamma(c(\gamma))$, that is, $K(f_\gamma) = RL\cdots$. Additionally, (i) and (ii) imply that the sets Γ_{up} and Γ_{low} of parameters γ with f_γ being, respectively, upper- or lower-unimodal, are both open in $[\gamma_0', \gamma_1]$. Therefore, by connectedness, either $\Gamma_{\text{up}} = [\gamma_0', \gamma_1]$ or $\Gamma_{\text{low}} = [\gamma_0', \gamma_1]$. We assume, without loss of generality, that all these maps are upper-unimodal. The continuity conditions (i) and (ii) also imply that the map $c : [\gamma_0', \gamma_1] \to \mathbb{R}$ is continuous, and so are the maps $\tilde{a}(\gamma) = f_\gamma^2(c(\gamma))$, $\tilde{b}(\gamma) = f_\gamma(c(\gamma))$. Let $\varphi_c : [-1, 1] \to [-1, 1]$ be the increasing diffeomorphism defined by $\varphi_c(x) = (x-c)/(1-cx)$, $-1 < c < 1$, and redefine, for any $\gamma \in [\gamma_0', \gamma_1]$, $g_\gamma \in C^3([-1, 1])$ by

$$g_\gamma = \phi_\gamma \circ f_\gamma|_{[\tilde{a}(\gamma), \tilde{b}(\gamma)]} \circ \phi_\gamma^{-1},$$

where we mean $\phi_\gamma = \varphi_{\tilde{c}(\gamma)} \circ \psi_{\tilde{a}(\gamma), \tilde{b}(\gamma)}$, with $\tilde{c}(\gamma) = \psi_{\tilde{a}(\gamma), \tilde{b}(\gamma)}(c(\gamma))$. Once again, $\gamma \mapsto g_\gamma$ maps continually $[\gamma_0', \gamma_1]$ into $C^1([-1, 1])$, the maps g_γ are strictly unimodal and have 0 as its critical point (with $(g_\gamma)''(0) \neq 0$ by (iii)). Notice that the maps $\psi_{a,b}$, φ_c have zero Schwarzian derivative: since the composition of maps with nonpositive Schwarzian derivative has negative Schwarzian derivative provided that one of then has (de Melo and van Strien 1993, p. 144), the maps g_γ have negative Schwarzian derivative by (iv).

Let $\chi = \chi_1 \chi_2 \cdots$ be an arbitrary infinite sequence of 0's and 1's, and let $\alpha = \alpha(\chi) = RLRRL\alpha_1\alpha_2\cdots$, where $\alpha_n = RLRR$ or $\alpha_n = RR$ according to, respectively, $\chi_n = 0$ or $\chi_n = 1$. Also, let $\beta = (RLR)^\infty$. It is easy to check that both α and β are maximal and $\mathcal{R}(\alpha) < \beta$. Fix $m \geq 1$ such that $3 \cdot 2^m \lhd T(g_{\gamma_1}) = T(f_{\gamma_1})$. As shown in Collet and Eckmann (1980) (proof of Theorem II.3.10, p. 92), $\mathcal{R}^m(\beta) < K(g_{\gamma_1})$. Now Proposition II.2.2 in (Collet and Eckmann 1980, p. 74) implies $\mathcal{R}^{m+1}(\alpha) < \mathcal{R}^m(\beta)$, hence $\mathcal{R}^{m+1}(\alpha) < K(g_{\gamma_1})$. Similarly, we have $\omega < \alpha$ and then $\omega = \mathcal{R}^{m+1}(\omega) < \mathcal{R}^{m+1}(\alpha)$. Since $K(g_{\gamma_0'}) = K(f_{\gamma_0'}) = \omega$ by Lemma B.4, and $\mathcal{R}^{m+1}(\alpha)$ is maximal (Collet and Eckmann 1980, Corollary II.2.4, p. 75), we can apply Theorem III.1.1 in (Collet and Eckmann 1980, p. 173) (here the continuity of $\gamma \mapsto g_\gamma$ in the C^1-topology is essential) to find $\gamma_0' < \gamma < \gamma_1$ such that $K(g_\gamma) = \mathcal{R}^{m+1}(\alpha)$.

Let $g = g_\gamma$ and $\tilde{g} = N^{m+1}(g)$, which is well defined by Lemma B.3 (with $K(\tilde{g}) = \alpha$) and assume, for instance, that m is odd, that is, $\tilde{g}$ is upper-unimodal. Observe that 0 is still the critical point of $\tilde{g}$, let p be the essential fixed point of $\tilde{g}$ and let $v = \tilde{g}(0)$, $u = \tilde{g}^2(0)$, when $K(\tilde{g}) = \alpha$ implies $u < 0 < v$. Observe that if U is a sufficiently small neighbourhood of 0 and $y \in U$, then $\tilde{g}(y), \tilde{g}^3(y), \tilde{g}^4(y) > 0$ and $\tilde{g}^2(y), \tilde{g}^5(y) < 0$, so $\tilde{g}^n(0) \notin U$ for any $n \geq 1$ and the $\tilde{g}-$orbit of 0 does not accumulate at 0. In fact, because of the way the renormalization operator is defined, the g-orbit of 0 cannot accumulate at 0 either.

At this point recall that $\alpha = \alpha(\chi)$ and assume that one of the following possibilities holds: (a) χ is not eventually periodic; (b) χ is eventually 1. In we are in case (a), then the $\tilde{g}$-orbit of 0 cannot be attracted by any periodic orbit, and the same thing can be said for $g = g_\gamma$. By Lemma B.5, g_γ has an acip ν, which becomes an acip μ for f_γ by writing $\mu(B) = \nu(\phi_\gamma(B))$ if B is a Borel subset of $[\tilde{a}(\gamma), \tilde{b}(\gamma)]$ and giving

μ-zero measure to $[a(\gamma), b(\gamma)] \setminus [\tilde{a}(\gamma), \tilde{b}(\gamma)]$. In other words, $\gamma \in \Lambda$. But there are uncountably many sequences χ satisfying (a), so Λ is uncountable.

In case (b) we can say more: $\tilde{g}$ satisfies the strong Misiurewicz condition. Indeed, let $q \in (p, v)$ be such that $\tilde{g}(q) = 0$. Then $\tilde{g}^2$ has positive derivative in $(0, q)$ and $\tilde{g}^2(0) = u < 0$, $\tilde{g}^2(q) = v > q$. Therefore, according to the minimum principle (de Melo and van Strien 1993, Lemma 6.1, p. 144) (which establishes that if a map has negative Schwarzian derivative in a compact interval, then the absolute value of its derivative attains its minimum value at some endpoint of the interval), $(\tilde{g}^2)'(p) > 1$, and $\tilde{g}^2(y) < y$ (respectively, $\tilde{g}^2(y) > y$) for any $y \in (0, p)$ (respectively, for any $y \in (p, q)$. Now, realize that the kneading sequence of $\tilde{g}$ guarantees that $\tilde{g}^n(0)$ stays at $(0, q)$ for every n large enough, which is impossible unless 0 is eventually mapped by $\tilde{g}$ to p, that is, the strong Misiurewicz condition is satisfied (when the same happens to g_γ and f_γ). Since there are infinitely many sequences χ satisfying (b), we conclude that Λ_S is infinite, which finishes the proof of the first part of Theorem B.1.

Now we prove the second part of Theorem B.1. Recall, first of all, that f_ρ has an acip and then $h(f_\rho) > 0$, so (by the continuity of entropy) $h(f_\gamma) > 0$ if γ is close to ρ, when the strictly unimodal maps $g_\gamma \in C^3([-1, 1]$ with $g_\gamma(0) = 1$, $g_\gamma(1) = -1$, can be defined as above. Further, the additional hypothesis on F and (iii) imply that $c(\gamma)$ is C^2 near ρ, so each $G_n(y, \gamma) = g_\gamma^n(y)$ is C^2 in an open set containing a small rectangle $[-1, 1] \times [\rho - \epsilon, \rho + \epsilon]$. On the other hand, the conditions $q = p(\rho)$ and $f_\gamma^r(p(\gamma)) = p(\gamma)$, together with $(f_\rho^r)'(q) \neq 1$, define uniquely a continuous (in fact, C^3) map $p(\gamma)$ near ρ. (This works even if $q = \tilde{a}(\rho)$ or $q = \tilde{b}(\rho)$, because the maps f_γ^r can be seen as defined on slightly larger open intervals than $[\tilde{a}(\gamma), \tilde{b}(\gamma)]$.)

We claim that if γ is close enough to ρ, then $g_\gamma(x) > x$ for any $x \in (-1, 0]$. If $g_\rho(-1) > -1$, then the claim follows for $\gamma = \rho$ from the strong Misiurewicz condition, and for all γ near ρ from the continuity of the family. Now assume $g_\rho(-1) = -1$, when $g'_\rho(-1) > 1$ by the strong Misiurewicz condition and then $g'_\gamma(x) > 1$ whenever x is close enough to -1 and γ is close enough to ρ. Then the claim follows from the minimum principle.

Let $z(\gamma) = F_k(c(\gamma), \gamma)$ and realize that (B.10) amounts to say (because $(\partial F_k/\partial x)(d, \rho) = 0$) that $z'(\rho) \neq p'(\rho)$. Write $\phi(x, \gamma) = \phi_\gamma(x)$, $Z(\gamma) = \phi(z(\gamma), \gamma) = G_k(0, \gamma)$, $P(\gamma) = \phi(p(\gamma), \gamma) = G_r(P(\gamma), \gamma)$. Then

$$Z'(\rho) - P'(\rho) = \frac{\partial \phi}{\partial x}(q, \rho)(z'(\rho) - p'(\rho)) \neq 0$$

and all conditions in (Thieullen et al. 1994, Theorem I.3) are satisfied, which ensures that ρ is a Lebesgue density point of a Borel set $\Sigma \subset (\gamma_0', \gamma_1]$ such that g_γ (and then f_γ) has an acip whenever $\gamma \in \Sigma$. This finishes the proof. □

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