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Globally attracting fixed points in higher order discrete population models

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Abstract We address the global stability properties of the positive equilibrium in a general delayed discrete population model. Our results are used to investigate in detail a well-known model for baleen whale populations.

Keywords Clark model · Delayed population model · Age-structured model · Global stability · Global attractor · Permanence

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1 Introduction

The dynamics of single-species population models with nonoverlapping generations has been one of the strong motivations for the impressive development of the theory of discrete dynamical systems. Since the celebrated papers by May ([25,26], among others), a lot of papers and monographs have been written on

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this subject. Models as the Beverton–Holt and Ricker equations [36, Chapter 9], or the well-known quadratic family (also known as the discrete logistic model [30, Sect. 2.3]) were investigated by a large number of authors. These models can be described by a recurrence

$$x_{n+1} = h(x_n), \quad n = 0, 1, \ldots \tag{1}$$

Equation (1) is also referred to as *first-order difference equation* or *one-dimensional dynamical system*. Here, x_n denotes the size of a given population after n years, and h is a stock-recruitment function.

However, as pointed out by several authors (see, e.g., [5,21]), sometimes this recruitment takes place several years after birth. In these situations, the model should include a delay effect, and this leads to study a higher order difference equation

$$x_{n+1} = h(x_n, x_{n-k}), \quad n = 0, 1, \ldots, \tag{2}$$

where $k \geq 1$ is an integer. See also [30, Sect. 2.5] for more discussions on Eq. (2). Notice that a solution of (2) is a real sequence $\{x_n\}_{n \geq -k}$, where $(x_{-k}, \ldots, x_{-1}, x_0)$ is the initial data.

The most famous forms of Eq. (2) are

$$x_{n+1} = x_n f(x_{n-k}), \quad n = 0, 1, \ldots, \tag{3}$$

and

$$x_{n+1} = \alpha x_n + f(x_{n-k}), \quad n = 0, 1, \ldots \tag{4}$$

Equation (3) includes the delayed Pielou [31] and Ricker [30, p. 51] equations (see other models in [21]). On the other hand, Eq. (4) was considered by Clark [5]; the parameter $\alpha \in (0, 1)$ is a survival coefficient, and the term $f(x_{n-k})$ represents recruitment, which takes place with a delay of k years. We notice that Eq. (4) is often referred as to the Clark model, and it represents a simple way of adding explicit age structure to a lumped model (see [3]).

In particular, this is the form of a model for the baleen whale used by the International Whaling Commission. It can be written as

$$x_{n+1} = (1 - \mu)x_n + \mu x_{n-k} \left[1 + q\left(1 - \left(\frac{x_{n-k}}{K}\right)^z\right)\right]_+, \quad n = 0, 1, \ldots, \tag{5}$$

where $[x]_+ = \max\{x, 0\}$. Here, x_n is the size of the population of sexually mature whales at time n, $(1 - \mu)x_n$ is the surviving fraction of whales that contribute to the population after one year $(0 < \mu < 1)$, K is the unharvested equilibrium density, $q = Q/P$ is the rate between the maximum increase Q in the fecundity possible as the population density falls to low levels and the per capita fecundity of females P, and z is a response parameter that measures the severity with which the changes in the population density are registered. For a more detailed explanation on the model and further generalizations, see [3,5,10,28,30].

One of our results establishes that the population modeled by Eq. (5) stabilizes around the positive equilibrium K whenever the relation qz remains below 2, no matter the value of the maturity delay k and the initial size of the population. Moreover, this result cannot be improvable in the sense that, if $qz > 2$, then there exist values of the delay for which the equilibrium becomes unstable.

If one considers the one-dimensional version of model (5)

$$x_{n+1} = x_n \left[1 + q\left(1 - \left(\frac{x_n}{K}\right)^z\right)\right]_+, \quad n = 0, 1, \ldots,$$

then $qz = 2$ is the exact bifurcation point where the equilibrium losses its asymptotic stability and nontrivial periodic cycles appear. The situation for (5) is different. Our approach allows us to prove that the equilibrium is still globally stable if $qz > 2$ and the survival rate after k years $(1-\mu)^k$ is large enough. This fact is another support to Bostford's claim that *addition of age structure is stabilizing* [3].

Other important motivation for the study of Eq. (4) is due to the fact that it is obtained as a discrete version of the delay differential equation

$$x'(t) = -x(t) + h(x(t-\tau)), \quad \tau > 0, \tag{6}$$

which has been used as a model for many biological processes as the Mackey–Glass equation [24], and the Nicholson's blowflies model [13]. See [16, p.78] for more applications. In fact, the Euler discretization of (6) with step τ/k leads to equation

$$x_{n+1} = \alpha x_n + (1-\alpha)h(x_{n-k}), \quad n = 0, 1, \ldots, \tag{7}$$

where $\alpha = 1 - \tau/k \in (0,1)$ (see, e.g., [15,18]). The reader can find some examples of Eq. (7) which are discrete analogues of models of the form (6) in [19, Chapter 4]; see also [8,9,14,18,37].

In this paper we will consider precisely Eq. (7), which is equivalent to (4). The reason is that the equilibrium points of (7) are exactly the fixed points of h, and it is quite natural to relate the dynamics of (7) with that of the first order difference equation (1). In general, the nonlinearities h in difference equations of population dynamics have the following properties

Assumption 1 [36, Assumption 9.4] $h : (0,\infty) \to (0,\infty)$ is continuous, has a unique fixed point $\bar{x}$, and is bounded on $(0,\bar{x}]$. Furthermore, $h(x) > x$ for $x < \bar{x}$, and $h(x) < x$ for $x > \bar{x}$.

In this paper, we consider a more general assumption

Assumption 2 $h : (0,\infty) \to [0,\infty)$ is continuous and has a unique fixed point $\bar{x}$. Furthermore, $h(x) > x$ for $x < \bar{x}$, and $h(x) < x$ for $x > \bar{x}$.

We will assume as well that there exists $h(0^+) = \lim_{x \to 0^+} h(x)$, allowing the case $h(0^+) = \infty$. Notice that, under our assumptions, h may vanish at some points $x > 0$ [this is the case for Eq. (5)], and some of our results are valid for the unbounded case when $h(0^+) = \infty$. If $h(0^+) < \infty$, h can be obviously extended to a continuous function $h : [0, \infty) \to [0, \infty)$; moreover, a typical case is $h(0) = 0$.

Due to biological reasons, only nonnegative initial conditions will be considered; more precisely, when $h(0)$ is not defined, the unique admissible initial conditions are those vectors $(x_{-k}, \ldots, x_{-1}, x_0) \in \mathbf{R}^{k+1}$ such that $x_i > 0, i = -k, \ldots, 0$. On the other hand, if h is defined on $[0, \infty)$, then the set of initial conditions can be extended to

$$S = \{(x_{-k}, \ldots, x_{-1}, x_0) \in \mathbf{R}^{k+1} : x_i \geq 0, i = -k, \ldots, 0,\ x_0 > 0\}.$$

In any case, we will only consider solutions corresponding to admissible initial conditions, which will be called *admissible solutions*.

The study of the stability properties of the equilibrium $\bar{x}$ for Eq. (7) is the main aim of this paper. The local asymptotic stability was discussed in [5] (see also [20] for an explicit formula). Global stability properties are much more difficult to obtain. We say that $\bar{x}$ is a *global attractor* for (7) if all admissible solutions converge to $\bar{x}$ as $n \to \infty$. On the other hand, $\bar{x}$ is called *globally stable* if it is a stable global attractor.

In the literature, one can find many conditions to ensure that $\bar{x}$ is a global attractor for the one-dimensional dynamical system generated by (1). For example, under Assumption 1, $\bar{x}$ is a global attractor for (1) if and only if $\bar{x}$ is the unique fixed point of function $h^2 = h \circ h$ ([36, Theorem 9.6]). In fact, such a result can be obtained from the classical theorems for maps of *type* 1, that is, maps without cycles of period greater than one (see, e.g., [2,34]). Furthermore, in this case, a globally attracting fixed point for (1) is always globally stable (see, e. g., [6]).

It is quite natural that the first idea to study the global stability properties of the equilibrium $\bar{x}$ in Eq. (7) consists in relating Eqs. (1) and (7). This was the approach made independently by Fisher et al. [10,11] and Ivanov [17]. Roughly speaking, their results establish that if $\bar{x}$ is a global attractor for (1) in a closed invariant interval $[a, b]$ containing $\bar{x}$, then $\bar{x}$ attracts all solutions of (7) with initial conditions $x_{-k}, \ldots, x_0 \in (a, b)$. The results in [10] are valid for systems of difference equations, but they only apply to intervals centered at $\bar{x}$. For particular cases, some related results can be found in [18,19] (for decreasing h), and [1,15] (for the limit case $\alpha = 0$).

We emphasize that, for any admissible initial condition $(x_{-k}, \ldots, x_{-1}, x_0)$, a simple induction argument shows that the corresponding solution $\{x_n\}$ of (7) is well defined and satisfies $x_n > 0$ for all $n \geq 0$. This is not necessarily true for Eq. (1) under Assumption 2, since $h(x_n)$ may be zero for some $n \geq 1$ even if $x_0 > 0$. Hence, the results in [10,17] do not always allow to establish the global stability for equation (7). One of the aims of this paper consists in extending these results to a more general situation. We will prove the following result:

Theorem 1 *Assume that h satisfies the conditions of Assumption* 2. *Let* $M = \sup_{0<x<\bar{x}} h(x) < \infty$. *If* $h(x) \neq 0$ *for all* $x \in (0, M]$, *and* $\bar{x}$ *is a global attractor for* (1) *in* $(0, M]$, *then* $\bar{x}$ *is globally stable for Eq.* (7).

Remark 1 When $h(0) = 0$, condition $h(x) \neq 0$ for all $x \in (0, M]$ cannot be weakened. Indeed, according [36, Theorem 9.8], if there exists $N \in (0, M)$ such that $h(0) = h(N) = 0$, then h^2 must have a fixed point in $(0, \bar{x})$, and hence $\bar{x}$ cannot be globally attracting in $(0, M]$.

In order to apply Theorem 1 to particular examples of Eq. (7), one can use some of the various sufficient conditions in the literature to ensure that a continuous map with a unique fixed point is globally stable. Among them, due to its applications in population models, we mention [6,7,12,32]. In Sect. 4, we use some appropriate results of this type to study the baleen whale population model (5).

However, as noticed in [10], this kind of results only apply when the equilibrium point of (1) is attracting. Moreover, they are independent on the delay k in Eq. (7) (such stability conditions which are independent of the delay are usually called *absolute stability* conditions). In order to obtain results on the global stability when $\bar{x}$ is unstable for (1), other approach is necessary. There are some recent results of this type for Eq. (7); see, for example [8,9,14,19] for h decreasing, and [14,37] for unimodal h (that is, h has a unique hump). As far as we know, the general case of h satisfying Assumption 2, including functions with several humps, has not been addressed. The other aim of our paper consists in proving some results in this direction.

We have organized our results as follows: in Sect. 2 we prove the boundedness and persistence of the solutions to (7) when h satisfies the conditions of Assumption 2; then, some lemmas are derived which are very important in next section. Section 3 is devoted to find sufficient conditions to ensure that the equilibrium $\bar{x}$ is a global attractor for (7); in particular, Theorem 1 is proved, and other stability conditions are found which apply when $\bar{x}$ is not a global attractor of (1). Finally, in Sect. 4, we apply our results to model (5), improving earlier results in the literature.

2 Boundedness and persistence

In this section, we prove that, if Assumption 2 holds, then all solutions of (7) are bounded and persistent, that is, for all solutions $\{x_n\}$ with admissible initial conditions, the following inequalities hold:

$$0 < \liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n < \infty. \tag{8}$$

A difference equation satisfying the relations (8) for any admissible solution is sometimes called *permanent* (see [19, Sect. 2.2]).

Theorem 2 *Let Assumption* 2 *holds. Then Eq.* (7) *is permanent.*

Proof We consider two different cases.

Case 1 $h(0^+) < \infty$.

Suppose that $\{x_n\}$ is an unbounded solution of (7). Then there exists an integer sequence $\{n_i\}$, $n_i \to \infty$ as $i \to \infty$, such that $x_{n_i+1} = \max\{x_n : n \le n_{i+1}\}$. Thus $\lim_{i\to\infty} x_{n_i+1} = \infty$ and

$$x_{n_i+1} = \alpha x_{n_i} + (1-\alpha)h(x_{n_i-k}) \le \alpha x_{n_i+1} + (1-\alpha)h(x_{n_i-k}).$$

Rearranging, we get the inequality

$$x_{n_i+1} \le h(x_{n_i-k}). \tag{9}$$

According to the definition of x_{n_i+1} and (9), we get $x_{n_i-k} \in (0, \bar{x}]$ for all i. Therefore, the continuity of h leads to the existence of a number $A > 0$ such that $h(x_{n_i-k}) \le A$ for all i. Combining this with (9), we get $x_{n_i+1} \le A$ for all i, which contradicts the definition of x_{n_i+1}. Thus $\{x_n\}$ is bounded.

Next, if there exists a solution $\{x_n\}$ of (7) such that $\liminf_{n\to\infty} x_n = 0$, then an integer sequence $\{n_i\}$, $n_i \to \infty$, can be chosen such that

$$x_{n_i+1} = \min\{x_n : n \le n_i + 1\}, \quad \lim_{i\to\infty} x_{n_i+1} = 0. \tag{10}$$

Since $x_{n+1} \ge \alpha x_n$ for all $n \ge 1$, we get $\lim_{i\to\infty} x_{n_i} = 0$. An induction argument on the previous inequality yields that $\lim_{i\to\infty} x_{n_i-k} = 0$. Choose an integer i_0 so large that $x_{n_i-k} \in (0, \bar{x})$ for all $i \ge i_0$. Then (7) implies

$$x_{n_i+1} \ge h(x_{n_i-k}) > x_{n_i-k}, \quad i \ge i_0,$$

which contradicts the definition of x_{n_i+1}.

Case 2 $h(0^+) = \infty$.

First, assume that $\liminf_{n\to\infty} x_n = 0$. Define the integer sequence $\{n_i\}$ as in (10). Then $\lim_{i\to\infty} x_{n_i+1} = \lim_{i\to\infty} x_{n_i} = \lim_{i\to\infty} x_{n_i-k} = 0$. From (7), we get

$$0 = (\lim_{i\to\infty} x_{n_i+1} - \alpha \lim_{i\to\infty} x_{n_i}) = (1-\alpha) \lim_{i\to\infty} h(x_{n_i-k}) = \infty,$$

which is impossible. Now, if $\limsup_{n\to\infty} x_n = \infty$, one can define the sequence $\{n_i\}$ as in the proof of boundedness in Case 1. Then (9) holds and $x_{n_i-k} \in (0, \bar{x}]$ for all i. So, letting $i \to \infty$, the inequality (9) yields $\lim_{i\to\infty} x_{n_i-k} = 0$, which is impossible according to the first part of Case 2. □

It is not hard to prove (see [8, p. 754]) that any solution of (7) is also a solution of the higher order difference equation

$$x_{n+1} = F(x_{n-k}) + (1-\alpha) \sum_{i=1}^{k} \alpha^i h(x_{n-k-i}), \quad n \ge 2k, \tag{11}$$

where $F(x) = \alpha^{k+1}x + (1-\alpha)h(x)$ for all $x > 0$. This relation will be useful in the following two technical lemmas, which in turn are crucial to prove our global stability results.

Lemma 1 *Let* $\{x_n\}$ *be a solution of* (7) *that is not attracted to the equilibrium point* $\bar{x}$. *Then there exists a set of positive real numbers* L, L_i, S, S_i, *where* $L_i, S_i \in [L, S]$, $i = 0, 1, \ldots, 2k$, *such that the following relations hold:*

$$L = \alpha L_0 + (1-\alpha)h(L_k), \tag{12}$$

$$S = \alpha S_0 + (1-\alpha)h(S_k), \tag{13}$$

$$L = F(L_k) + (1-\alpha)\sum_{i=1}^{k}\alpha^i h(L_{k+i}), \tag{14}$$

and

$$S = F(S_k) + (1-\alpha)\sum_{i=1}^{k}\alpha^i h(S_{k+i}). \tag{15}$$

Moreover,

$$L \geq h(L_k) \quad and \quad S \leq h(S_k). \tag{16}$$

Proof Using Theorem 2, one can find two positive real numbers; say L, S such that $L = \liminf_{n\to\infty} x_n$ and $S = \limsup_{n\to\infty} x_n$. Following [9], there exist two sequences of integers $\{n_l\}$ and $\{n'_l\}$, with $\lim_{l\to\infty}\ n_l = \lim_{l\to\infty}\ n'_l = \infty$, such that

$$\lim_{l\to\infty} x_{n_l+1} = L,\quad \lim_{l\to\infty}\ x_{n_l-i} = L_i,\quad \lim_{l\to\infty}\ x_{n'_l+1} = S,\quad \lim_{l\to\infty}\ x_{n'_l-i} = S_i,$$

with $L_i, S_i \in [L, S]$ for all $i = 0, 1, \ldots, 2k$. Now, taking the limits of both sides of (7) and (11) through n_l and n'_l, we obtain Eqs. (12)–(15), while (16) is derived from (12) and (13) using the fact that $L_0 \geq L$ and $S_0 \leq S$. □

Lemma 2 *Assume that* $\{x_n\}$ *is a solution of (7) which is not attracted to* $\bar{x}$. *If* L, L_k, S_k *and* S *are defined as before, then*

$$L \leq S_k < \bar{x} < L_k \leq S. \tag{17}$$

Proof From (16) we obtain $L_k \geq h(L_k)$ and $S_k \leq h(S_k)$. Therefore, $L_k \in [\bar{x}, S]$ and $S_k \in [L, \bar{x}]$. If $S_k = \bar{x}$, then (16) implies that $S \leq h(S_k) = \bar{x}$ and hence $S = L_k = \bar{x}$, which yields $L \geq h(L_k) = \bar{x}$. Thus $L = S = \bar{x}$, which is a contradiction. If we assume that $L_k = \bar{x}$, similar arguments imply that $L = S = \bar{x}$. □

3 Global stability

In this section, we obtain some sufficient conditions to ensure that $\bar{x}$ is globally stable or globally attracting for Eq. (7). From now on, we always assume that h satisfies the conditions of Assumption 2, and we denote

$$M = \sup_{0<x<\bar{x}} h(x).$$

Now we are in a position to prove Theorem 1.

Proof of Theorem 1 Assume that $\{x_n\}$ is a solution of (7) which is not attracted to $\bar{x}$. Let m be a positive number defined by $m = \min_{L\leq x\leq M} h(x)$. Then $h(x) \in [m, M]$ for all $x \in [L, M]$. For $x \in [m, L)$, we have $M \geq h(x) > x \geq m$. Thus $h([m, M]) \subseteq [m, M]$. Since $\bar{x}$ is a global attractor for (1) in $(0, M]$, it follows from [14, Corollary 4] that

$$h^i([m, M]) \to \bar{x} \quad \text{as } i \to \infty.$$

Next, from (16) and (17) we get $[L, S] \subseteq h([L, S])$ and $[L, S] \subseteq [m, M]$. Hence, $[L, S] \subseteq h([m, M])$, and simple induction yields

$$[L, S] \subseteq h^i([m, M]) \quad i = 0, 1, \ldots.$$

Letting $i \to \infty$ in the above relation, it follows that $L = S = \bar{x}$. This shows that $\bar{x}$ is a global attractor. Next, since a unique globally attracting fixed point for (1) must be stable, Theorem 3 in [10] implies that $\bar{x}$ is also stable for (7), and hence it is globally stable. □

In our next result, condition $M < \infty$ is not necessary.

Theorem 3 *Assume that either $M = \bar{x}$ or h is monotonically increasing on $(\bar{x}, M)$, then $\bar{x}$ is globally stable for Eq. (7).*

Proof Assume, as before, that $\{x_n\}$ is a solution of (7) which is not attracted to $\bar{x}$, and let L, S be as defined in the statement of Lemma 1. From (16) and (17) it follows that $M \geq S > \bar{x}$, which is impossible if $M = \bar{x}$. If $h(x)$ is increasing on $(\bar{x}, M)$, then (16) and the fact that $M > L_k > \bar{x}$ imply that $L \geq h(L_k) > \bar{x}$, a contradiction to (17). Thus $\bar{x}$ is globally attracting for (7). Finally, since the conditions of the theorem imply that $\bar{x}$ must be stable for (1), we conclude as before that it is globally stable for (7). □

Remark 2 Theorem 3 generalizes [14, Corollary 10], where a similar result is proved for C^1-maps with at most one critical point.

In order to address the global attractivity of $\bar{x}$ for (7) when it is not a globally attracting fixed point of h, we introduce the linear functions u_i, $i = 1, 2$, as follows:

$$u_i(x) = -d_i x + (1 + d_i)\bar{x},$$

where $d_i > 0$ for $i = 1, 2$. Assume that h satisfies

$$h(x) < u_1(x) \ \text{if } 0 \leq x < \bar{x}, \quad \text{and} \quad h(x) > u_2(x) \ \text{if } x > \bar{x}. \tag{18}$$

If $L < S$, then (16), (17) and (18) yield

$$L > u_2(L_k), \quad S < u_1(S_k). \tag{19}$$

Thus,

$$L_k > u_2^{-1}(L), \quad S_k < u_1^{-1}(S). \tag{20}$$

It is also easy to see that

$$\begin{array}{ll} h(L_{k+i}) \geq L_{k+i} \geq L > u_2(L_k) \geq u_2(S) & \text{if } L_{k+i} \leq \bar{x}, \\ h(L_{k+i}) > u_2(L_{k+i}) \geq u_2(S) & \text{if } L_{k+i} > \bar{x}, \end{array} \tag{21}$$

and

$$\begin{array}{ll} h(S_{k+i}) < u_1(S_{k+i}) \leq u_1(L) & \text{if } S_{k+i} < \bar{x}, \\ h(S_{k+i}) \leq S_{k+i} \leq S < u_1(S_k) \leq u_1(L) & \text{if } S_{k+i} \geq \bar{x}. \end{array} \tag{22}$$

We denote by C_B the class of functions h satisfying (18) for some $d_1, d_2 > 0$; notice that C_B contains all continuous functions on $(0, \infty)$ with $h(\bar{x}) = \bar{x}$ except possibly functions with either one of the following properties:

$$h(0^+) = \infty, \ \text{ or } \ \lim_{\Delta \to 0^{+}_{-}} \frac{h(\bar{x} + \Delta) - h(\bar{x})}{\Delta} = -\infty.$$

An application of Theorem 1 shows that $\bar{x}$ is globally stable if $d_1 d_2 \leq 1$ in (18).

Theorem 4 *Let $h \in C_B$ on the interval $(0, M)$. If $d_1 d_2 \leq 1$, then $\bar{x}$ is globally stable for equation (7).*

Proof Condition $d_1 d_2 \leq 1$ implies that $u_1(0) \leq u_2^{-1}(0)$, and therefore $h(x) \neq 0$ for all $x \in (0, M]$. Next, it is easy to show that $h^2(x) > x$ for $x < \bar{x}$, and $h^2(x) < x$ for $x > \bar{x}$. Hence, $\bar{x}$ is a global attractor of (1) in $(0, M]$ and the result follows from Theorem 1. □

Remark 3 The conclusion of Theorem 4 remains valid if $d_1 d_2 < 1$ and the inequalities relating h to u_1 and u_2 in (18) are nonstrict.

Next, to obtain sharper stability conditions depending on the parameters α and k, we define the numbers A_{d_j}, B_{d_j} as follows:

$$A_{d_j} = \alpha^{k+1} - d_j(1 - \alpha^{k+1}), \quad B_{d_j} = \frac{d_j^2(1 - \alpha^k)}{\alpha^k + d_j}, \ \text{ for } j = 1, 2. \tag{23}$$

Theorem 5 *Let $h \in C_B$ on the interval* $(0, M)$. *Then $\bar{x}$ is a global attractor for* (7) *if either one of the following conditions is satisfied:*

$$\alpha^{k+1} \leq d_j(1-\alpha),\ i = 1, 2 \quad \textit{and} \ \ A_{d_1}A_{d_2} \leq 1, \tag{24}$$

$$\alpha^{k+1} \geq d_j(1-\alpha),\ i = 1, 2 \quad \textit{and} \ \ B_{d_1}B_{d_2} \leq 1, \tag{25}$$

$$d_1(1-\alpha) \leq \alpha^{k+1} \leq d_2(1-\alpha) \quad \textit{and} \ \ -A_{d_2}B_{d_1} \leq 1, \tag{26}$$

or

$$d_2(1-\alpha) \leq \alpha^{k+1} \leq d_1(1-\alpha) \quad \textit{and} \ \ -A_{d_1}B_{d_2} \leq 1. \tag{27}$$

Proof We will give a proof when (24) or (25) are satisfied. The other cases can be handled similarly. As usual we assume that (7) has a solution $\{x_n\}$ with $L < S$. Using the relations (19), (21) and (22), we conclude from (14) and (15) that

$$\begin{aligned} L &> \alpha^{k+1}L_k + (1-\alpha)u_2(L_k) + (\alpha - \alpha^{k+1})u_2(S) \\ &= (\alpha^{k+1} - d_2(1-\alpha))L_k - d_2(\alpha-\alpha^{k+1})S + (1-\alpha^{k+1})\bar{x}(1+d_2), \end{aligned} \tag{28}$$

and

$$\begin{aligned} S &< \alpha^{k+1}S_k + (1-\alpha)u_1(S_k) + (\alpha - \alpha^{k+1})u_1(L) \\ &= (\alpha^{k+1} - d_1(1-\alpha))S_k - d_1(\alpha-\alpha^{k+1})L + (1-\alpha^{k+1})\bar{x}(1+d_1). \end{aligned} \tag{29}$$

When (24) holds, then (28) and the fact that $S_k, L_k \in [L, S]$ imply that

$$\begin{aligned} L &> (\alpha^{k+1} - d_2(1-\alpha))S - d_2(\alpha-\alpha^{k+1})S + (1-\alpha^{k+1})\bar{x}(1+d_2) \\ &= A_{d_2}S + (1-\alpha^{k+1})\bar{x}(1+d_2). \end{aligned} \tag{30}$$

Similarly, (24) and (29) yield

$$S < A_{d_1}L + (1-\alpha^{k+1})\bar{x}(1+d_1). \tag{31}$$

Notice that $A_{d_j} = \alpha^{k+1} - d_j(1-\alpha) - d_j(\alpha - \alpha^{k+1}) < 0, j = 1, 2$. Hence, it follows from (30) and (31) that

$$(1 - A_{d_1}A_{d_2})L > (1-\alpha^{k+1})\bar{x}(1 + d_2 + A_{d_2}(1+d_1)),$$

and

$$(1 - A_{d_1}A_{d_2})S < (1-\alpha^{k+1})\bar{x}(1 + d_1 + A_{d_1}(1+d_2)).$$

Simple calculations show that $1 + d_1 + A_{d_1}(1+d_2) = 1 + d_2 + A_{d_2}(1+d_1)$. Therefore,

$$(1 - A_{d_1}A_{d_2})L > (1 - A_{d_1}A_{d_2})S,$$

which is impossible since $A_{d_1} A_{d_2} \leq 1$.

If (25) holds, then, using (19), inequalities (28) and (29) imply that

$$\begin{aligned} L &> (\alpha^{k+1} - d_2(1-\alpha)) u_2^{-1}(L) - d_2(\alpha - \alpha^{k+1}) S + (1-\alpha^{k+1}) \bar{x} (1+d_2) \\ &= (\alpha^{k+1} - d_2(1-\alpha)) \left(-\frac{L}{d_2} + \left(\frac{1+d_2}{d_2} \right) \bar{x} \right) - d_2(\alpha - \alpha^{k+1}) S \\ &\quad + (1-\alpha^{k+1}) \bar{x} (1+d_2), \end{aligned}$$

and

$$\begin{aligned} S &< (\alpha^{k+1} - d_1(1-\alpha)) u_1^{-1}(S) - d_1(\alpha - \alpha^{k+1}) L + (1-\alpha^{k+1}) \bar{x} (1+d_1) \\ &= (\alpha^{k+1} - d_1(1-\alpha)) \left(-\frac{S}{d_1} + \left(\frac{1+d_1}{d_1} \right) \bar{x} \right) - d_1(\alpha - \alpha^{k+1}) L \\ &\quad + (1-\alpha^{k+1}) \bar{x} (1+d_1). \end{aligned}$$

Rearranging, we obtain

$$L > -B_{d_2} S + \bar{x}(1+d_2) \frac{d_2 + (1-d_2)\alpha^k}{\alpha^k + d_2}, \tag{32}$$

and

$$S < -B_{d_1} L + \bar{x}(1+d_1) \frac{d_1 + (1-d_1)\alpha^k}{\alpha^k + d_1}. \tag{33}$$

Combining these inequalities, it follows that

$$(1 - B_{d_1} B_{d_2}) L > \bar{x}(1+d_2) \frac{d_2 + (1-d_2)\alpha^k}{\alpha^k + d_2} - B_{d_2} \bar{x}(1+d_1) \frac{d_1 + (1-d_1)\alpha^k}{\alpha^k + d_1},$$

and

$$(1 - B_{d_1} B_{d_2}) S < \bar{x}(1+d_1) \frac{d_1 + (1-d_1)\alpha^k}{\alpha^k + d_1} - B_{d_1} \bar{x}(1+d_2) \frac{d_2 + (1-d_2)\alpha^k}{\alpha^k + d_2}.$$

Simple calculations show that the right-hand sides of the last two inequalities are equal. Therefore,

$$(1 - B_{d_1} B_{d_2}) S < (1 - B_{d_1} B_{d_2}) L,$$

which is impossible since $B_{d_1} B_{d_2} \leq 1$. □

Remark 4 From the above proof, we see that the conclusion of Theorem 5 is true also if just one of the inequalities relating h to u_1 and u_2 in (18) is not strict.

If both inequalities are nonstrict, then the conclusion of the theorem holds also but with strict inequality signs in either one of the inequalities in (24)–(27).

The following lemma can be proved by direct calculations:

Lemma 3 *Let A_{d_j}, B_{d_j}, $j = 1, 2$, be defined as in* (23). *Then,*

(a) $A_{d_1} A_{d_2} \le 1$ *if and only if* $\alpha^{k+1} \ge \dfrac{d_1 d_2 - 1}{(1+d_1)(1+d_2)}$,

(b) $A_{d_j} \ge -1$ *if and only if* $\alpha^{k+1} \ge \dfrac{d_j - 1}{d_j + 1}$,

(c) $B_{d_j} \le 1$ *if and only if* $\alpha^k \ge \dfrac{d_j^2 - d_j}{d_j^2 + 1}$.

Notice that $B_{d_1} B_{d_2} \le 1$ holds provided that $B_{d_j} \le 1, j = 1, 2$.

The following result is an easy consequence of Theorem 5 and Lemma 3.

Corollary 1 *Let $h \in C_B$ on the interval* $(0, M)$, *and denote* $d = \min\{d_1, d_2\}$, $D = \max\{d_1, d_2\}$. *Then $\bar{x}$ is a global attractor for (7) if either one of the following conditions is satisfied:*

(i) $\dfrac{d_1 d_2 - 1}{(1+d_1)(1+d_2)} \le \alpha^{k+1} \le (1-\alpha)d$.

(ii) $\alpha^{k+1} \ge \max\left\{(1-\alpha)D, \alpha \dfrac{D^2 - D}{D^2 + 1}\right\}$.

When $d_1 = d_2$, Corollary 1 reads as follows:

Corollary 2 *Assume that $h \in C_B$ on the interval* $(0, M)$ *for some* $d_1 = d_2 = d > 0$. *Then $\bar{x}$ is a global attractor for* (7) *if either*

$$\frac{d-1}{d+1} \le \alpha^{k+1} \le (1-\alpha)d, \tag{34}$$

or

$$\alpha^{k+1} \ge \max\left\{(1-\alpha)d,\ \alpha\frac{d^2 - d}{d^2 + 1}\right\}, \tag{35}$$

is satisfied.

A combination of (34) and (35) provides the following result.

Corollary 3 *Assume that $h \in C_B$ on the interval* $(0, M)$ *for some* $d_1 = d_2 = d > 0$. *If*

$$\alpha^{k+1} \ge \max\left\{\frac{d-1}{d+1},\ \alpha\frac{d^2 - d}{d^2 + 1}\right\},$$

then $\bar{x}$ is a global attractor for (7).

Remark 5 In the recent paper [37], the global stability of a more general difference equation

$$x_{n+1} = \alpha x_n + f(n, x_n, \ldots, x_{n-k}), \quad n = 0, 1, \ldots$$

is investigated using a generalized Yorke condition on f. When applying their results to Eq. (7), such a condition requires the existence of a rational function $r(x) = ax/(1+bx)$, $a < 0$, $b \geq 0$, such that $r(x-\bar{x}) \leq (1-\alpha)(h(x)-\bar{x}) \leq 0$ for $x \geq \bar{x}$, and $r(x-\bar{x}) \geq (1-\alpha)(h(x)-\bar{x}) \geq 0$ for $x \in (\bar{x}-1/b, \bar{x})$ (we notice that a similar approach was also used for Eqs. (1), (3) and (6) in [7,22,23], respectively). The form of this condition with $b = 0$ (sublinear case) becomes (18) with $d_1 = d_2 = -a$. In [37] it is suggested that condition

$$\alpha^{k+1} > \frac{d-1}{d+1} \tag{36}$$

should imply that $\bar{x}$ is a global attractor for (7) if $\alpha \in (0, \alpha_k^*]$, where $\alpha_k^* \in (0,1)$ is a solution of the algebraic equation $x^{k+2} + \cdots + x^{2k+1} = 1$. Corollary 3 and Remark 4 allow us to ensure that condition (36) implies the global attractivity of $\bar{x}$ for

$$\alpha \leq \widetilde{\alpha} = \frac{2(2+\sqrt{2})}{4+3\sqrt{2}} = 0.828427\ldots$$

This result supports the above conjecture, since it seems that the minimum of α_k^* is attained for $k = 4$, and $\alpha_4^* = 0.828811\ldots$. Notice the surprising proximity between $\widetilde{\alpha}$ and α_4^*.

On the other hand, the possibility to choose $d_1 \neq d_2$ allows us to manage situations not covered by the results in [37] (see the related example in [7, Sect. 3.8]).

4 Global stability in Eq. (5)

As we mentioned in the introduction, we apply our results to the baleen whale model (5). Here, $\alpha = 1-\mu$ and $h(x) = x[1 + q(1 - (\frac{x}{K})^z)]_+$. Therefore, $\bar{x} = K$, and $h(x) = 0$ if and only if $x = 0$ or $x \geq N^* = K(\frac{1+q}{q})^{1/z}$. Also, h is differentiable on $(0, N^*)$ and

$$h'(x) = 1 + q - q(1+z)(x/K)^z, \quad x \in (0, N^*).$$

In particular, $h'(\bar{x}) = 1 - qz$, and

$$h'(x) = 0 \text{ if and only if } x = x^* = N^* \left(\frac{1}{1+z}\right)^{1/z} = K\left(\frac{1+q}{q(1+z)}\right)^{1/z}. \tag{37}$$

Moreover,

$$h''(x) = \frac{-qz(1+z)}{K}(x/K)^{z-1}, \quad h'''(x) = \frac{-qz(z^2-1)}{K^2}(x/K)^{z-2}. \qquad (38)$$

As far as we know, the only global stability conditions available in the literature for (5) were obtained by using the global stability properties of $\bar{x}$ for the associated first-order difference equation (1) (see [10,19]). Our first result goes in the same direction, and it improves the mentioned references; in particular, we show that the restrictive condition (3.3) in [10, p. 652] is not necessary, and we solve the open problem 4.7.2 in [19, p. 122].

Theorem 6 *Assume that $qz \le 2$. Then $\bar{x}$ is globally stable for Eq.* (5).

Remark 6 We emphasize that Theorem 6 is the best possible result for the global stability of (5) based on the stability of (1). Indeed, $qz > 2$ is equivalent to $h'(\bar{x}) < -1$, which in turn implies that $\bar{x}$ is an unstable fixed point of h.

On the other hand, condition $qz \le 2$ is also the best possible absolute stability condition for (5) (see [9, p. 118]).

In order to prove Theorem 6, we will use Theorem 3 when $\bar{x} \le x^*$, and Theorem 1 otherwise. In view of (37), $\bar{x} \le x^*$ if and only if $qz \le 1$. When $qz > 1$, we have

$$M = h(x^*) = N^* \frac{(1+q)z}{(1+z)^{1+1/z}}, \qquad (39)$$

so that $M < N^*$ if and only if

$$qz < (1+z)^{1+1/z} - z. \qquad (40)$$

We will show in the Appendix that inequality (40) always holds when $qz \le 2$, since the right-hand side is greater than 2 for all $z > 0$. Hence, to use Theorem 1, it remains to show that $\bar{x}$ is a globally attracting fixed point for h in $(0, M]$. For it, we will use some known results from one-dimensional dynamical systems.

One of them is based on the Schwarzian derivative of h, and, more precisely, in the results of Singer [35]. We recall that the Schwarzian derivative, $(Sh)(x)$, of a C^3−map h is defined by

$$(Sh)(x) = \frac{h'''(x)}{h'(x)} - \frac{3}{2}\left(\frac{h''(x)}{h'(x)}\right)^2,$$

whenever $h'(x) \neq 0$. The following proposition can be deduced from Singer's results (see, e.g., [17, Corollary 1] or [22, Proposition 3.3]).

Proposition 1 *Let $h : [a,b] \to [a,b]$ be a C^3 map with a unique fixed point $\bar{x} \in (a,b)$, and with at most one critical point x^* (maximum). If $|h'(\bar{x})| \le 1$, and $(Sh)(x) < 0$ for all $x \neq x^*$, then $\bar{x}$ is the global attractor of* (1) *in (a,b).*

In some cases when Lemma 1 does not apply, we can use the following result, which is an easy consequence of [6, Theorem 3].

Proposition 2 *Let h satisfy Assumption 2, and, in addition,*

(a) *h has a unique critical point x^* such that $h'(x)(x - x^*) < 0$ for all $x \in [0, h(x^*)]$, $x \neq x^*$,*
(b) *$h(0) = 0$ and $h(x) > 0$ for all $x \in [0, h(x^*)]$,*
(c) *$h''(x) < 0$ for $x \in [x^*, \bar{x})$ and $h''(x)$ has at most one sign change,*
(d) *$h'''(x) \geq 0$ for all x such that $h''(x) < 0$.*

If $|h'(\bar{x})| \leq 1$, then $\bar{x}$ is a global attractor of (1) on $(0, h(x^*)]$.

Notice that, in view of (37), (38),

$$(Sh)(x) = \frac{-qz(z+1)}{2K^2(h'(x))^2}(x/K)^{z-2}\left(2(z-1)(q+1) + (qz^2 + 3qz + 2q)(x/K)^z\right),$$

for all $x \in (0, N^*) \setminus \{x^*\}$, so that h has negative Schwarzian derivative for $z \geq 1$.

In general, h does not have a negative Schwarzian derivative on $(0, N^*)$ for all $z > 0$ since

$$\left(2(z-1)(q+1) + (qz^2 + 3qz + 2q)(x/K)^z\right)|_{x=K,\, z=0} = -2 < 0,$$

which means that $(Sh)(\bar{x}) > 0$ for $z \approx 0$. This is the reason why we need Proposition 2.

Now, we are in a position to prove Theorem 6.

Proof of Theorem 6 We consider two cases. If $qz \in (0, 1]$, then we get $h'(x) \geq 0$ on $(0, K)$, which implies that $M = \bar{x}$. Thus, Theorem 3 proves that $\bar{x}$ is globally stable for (5).

Next, assume that $qz \in (1, 2]$. Lemma 4 in Appendix (with $y = z+1$) implies that $(z+1)^{1+1/z} - z > 2$. Hence, $qz \leq 2$ implies that (40) holds and therefore $M < N^*$. Next, it is easy to check that conditions of Proposition 2 hold provided that $z < 1$. For $z \geq 1$, we have shown that $(Sh)(x) < 0$ for all $x \neq x^*$, and therefore Proposition 1 applies with $[a, b] = [0, N^*]$. Thus, for $z > 0$ the equilibrium $\bar{x}$ is a global attractor of (1) in $(0, M]$. Hence, the global stability of $\bar{x}$ follows from Theorem 1. □

When $qz > 2$ or $M > N^*$, then Theorem 1 fails to apply according to Remark 1. In this case, we can apply Theorem 5. In this regard, the concavity of the graph of h implies that h satisfies (18) with $d_1 = |h'(\bar{x})| = qz - 1$ and

$$d_2 = \begin{cases} \dfrac{K}{N^* - K} & \text{if } N^* > 0, \\ \dfrac{h(M) - K}{K - M} & \text{if } \quad N^* > h(x^*). \end{cases}$$

Using the mean value theorem, we notice that $-d_2$ is the slope of the tangent line to a point on the graph of h which lies either between the points (K, K) and $(N^*, 0)$ or between the points (K, K) and $(M, h(M))$. Since $h''(x) < 0$ for all $x \in (K, N^*)$, we conclude that $d_1 < d_2$. Now, using Theorem 5 and its corollaries, we may obtain some delay-dependent global attractivity conditions. For example, assuming that $d_2 = K/(N^* - K)$, we have the following result.

Theorem 7 *The equilibrium K is the global attractor of Eq. (5) if at least one of the following conditions is satisfied:*

(c1) $\left(\frac{q}{1+q}\right)^{1/z} - \frac{1}{qz} \le (1-\mu)^{k+1} \le \mu(qz-1)$,

(c2) $(1-\mu)^{k+1} \ge \max\left\{\mu\frac{q^{1/z}}{(q+1)^{1/z}-q^{1/z}}, (1-\mu)\frac{2q^{2/z}-(q^2+q)^{1/z}}{q^{2/z}+(q^{1/z}-(q+1)^{1/z})^2}\right\}$,

(c3) $\mu(qz-1) \le (1-\mu)^{k+1} \le \mu\frac{q^{1/z}}{(q+1)^{1/z}-q^{1/z}}$ *and*

$$(1-\mu)^{k+1} \ge \max\left\{(1-\mu)\frac{(qz)^2-3qz+2}{(qz)^2-2qz+2}\, ,\ -1+2\left(\frac{q}{1+q}\right)^{1/z}\right\}.$$

Proof Since

$$d_1 = qz - 1 < d_2 = \frac{K}{N^*-K} = \frac{q^{1/z}}{(1+q)^{1/z}-q^{1/z}}\,,$$

conditions (c1) and (c2) follow directly from Corollary 1. For (c3), use (26) in Theorem 5, and Lemma 3. □

Remark 7 The stability conditions in Theorem 7 seem to be new for the baleen whale model (5). They provide new insight on the influence of the involved parameters in the global stability properties of the positive equilibrium $\bar{x} = K$.

5 Discussion

Higher order difference equations are the most appropriate theoretical setting to study discrete population models for which recruitment takes place several years after birth; moreover, they seem to be a good way to represent simple age-structured populations. Although some work has been recently done on this type of models, even the dynamically simplest case of a globally attracting fixed point is far from being well studied.

The reason is that, while for many one-dimensional models a simple graphic analysis allows to derive some basic properties as the boundedness and persistence of the solutions and the attracting properties of the positive equilibrium, this method is impossible to apply for a delay–difference equation. In this paper, we have made some progress in this research. First, Theorem 2 is, up to our knowledge, the first analytic proof of permanence for the solutions of system (7) under rather general assumptions. In particular, extinction is not possible for populations modeled by (7) if Assumption 2 holds. This setting covers not only the usual monotone or one-humped shape of the recruitment, but also more general nonlinearities as the two-humped model for the growth of bobwhite quail populations proposed in [29]. See also related discussions in [3,6,32]. It is also worth pointing out that we allow the map h to take the value zero, which is especially useful to address models with truncated nonlinearity, as in the recent work [4] (where one-dimensional models of this type are discussed).

Regarding the global stability, as noticed by Levin and May [21], even in the most simple higher order models, it is very difficult to show analytically that all solutions converge to the equilibrium. In this way, Theorem 1 becomes very useful, since it reduces the study of the global stability in a delayed model to that of a related first-order equation. This provides global delay-independent stability results for Clark-type models based on known stability criteria for one-dimensional equations. For example, see the list of seven population models worked out by Cull in [7].

Theorems 3 and 4 provide easily verifiable conditions for the global stability in a great variety of models. For example, Theorem 3 ensures that the equilibrium is globally stable under Assumption 2 if h is differentiable and $h'(\bar{x}) > 0$. We show the applicability of Theorem 4 in Clark model with some typical stock-recruitment functions. First, for the Ricker model considered in [3,28]

$$x_{n+1} = \alpha x_n + (1-\alpha) x_{n-k} \mathrm{e}^{r(1-x_{n-k})}, \quad n = 0, 1, \ldots, \tag{41}$$

it can be easily checked that all assumptions of Theorem 4 hold with $d_1 = d_2 = 1$ if $r \leq 2$. Thus, the equilibrium $\bar{x} = 1$ in (42) is globally stable if $r \leq 2$. One can check that this is the best delay-independent global stability condition for this equation, since $\bar{x}$ becomes unstable for any $r > 2$ if α is small enough (the exact value can be explicitly computed from [20]). A completely analogous argument shows that the equilibrium $\bar{x} = ((r-1)/b)^{1/3}$ in the model

$$x_{n+1} = \alpha x_n + (1-\alpha) \frac{r x_{n-k}}{1 + b x_{n-k}^3}, \quad n = 0, 1, \ldots, \ (r > 1, b > 0), \tag{42}$$

is globally stable for $r \leq 3$. In this case, the stock-recruitment function was proposed (for one-dimensional models) by Maynard Smith [27].

All these results provide global stability conditions depending only on the stock-recruitment function. As suggested in previous works (see, e.g., [3]), the addition of age structure is stabilizing. Thus, it is interesting to state some global stability criteria for Eq. (7) depending also on the delay k and the surviving parameter α. We have addressed this task in Theorem 5 and their corollaries. The obtained stability conditions are given in terms of the survival rate after $(k+1)$ years, which involves both parameters α, k.

We have thoroughly applied our results to the main motivating model for Eq. (7), namely, the classical fishing model (5), showing (Theorem 6) that condition $qz \leq 2$ is necessary and sufficient for the absolute (delay-independent) global stability of the equilibrium. We emphasize that the recruitment term in Eq. (5) is rather complicated (as mentioned in [30]). Probably this is the reason why earlier results by Fisher [10] were not improved up to now. We have also stated some stability conditions involving the parameters μ and k. In particular, condition (c2) in Theorem 7 shows that the equilibrium point K is globally stable when $qz > 2$ if the survival rate after k years is close to 1.

To finish, we think that Eq. (7) deserves more attention from the mathematical point of view. In this regard, we suggest some possible directions.

- It was proved that, when the nondelayed model (1) is globally stable, then the positive equilibrium of (7) is not only globally attracting, but also globally stable. It would be interesting to study whether is true or not that a globally attracting fixed point of (7) is necessarily globally stable. Notice that this is in general false for a higher order difference equation even in the second-order case $x_{n+1} = F(x_n, x_{n-1})$, with F continuous. For an example, see [33].
- Another interesting open problem consists in studying for which discrete population models the local asymptotic stability of the positive equilibrium implies its global stability. For some population models in the form $x_{n+1} = x_n F(x_{n-k})$, this was conjectured to be true by Levin and May in [21], and this conjecture was recently revisited in [23,37]. See the related open problems for Eq. (7) in [9,14], and recent results for the nondelayed case in [7].
- It would be of much interest as well to investigate the periodic structure of the Clark model when the equilibrium losses its global stability, generalizing in this way the research initiated in [1,15] for the limiting case $\alpha = 0$ in (7).

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Appendix

Lemma 4 $y^y > (y+1)^{y-1}$ *for all* $y > 1$.

Proof Set $R(y) = y \ln y - (y-1)\ln(y+1)$ for all $y > 1$. Then

$$R'(y) = \frac{2}{y+1} + \ln \frac{y}{y+1}, \quad R''(y) = \frac{1-y}{(1-y)^2}, \quad \text{for all } y > 1.$$

Thus $R''(y) < 0$ for all $y > 1$, which implies that

$$R'(y) > \lim_{y \to \infty} R'(y) = 0, \quad \text{for all } y > 1.$$

Then R increases on $(1, \infty)$. Since $R(1) = 0$, it follows that $R(y) > 0$ for all $y > 1$, that is,

$$y \ln y > (y - 1) \ln(y + 1),$$

which is equivalent to the required inequality. $\square$

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