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Mackey–Glass type delay differential equations near the boundary of absolute stability

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Abstract

For an equation $x'(t) = -x(t) + \zeta f(x(t-h))$, $x \in \mathbb{R}$, $f'(0) = -1$, $\zeta > 0$, with C^3-nonlinearity f which has a negative Schwarzian derivative and satisfies $xf(x) < 0$ for $x \neq 0$, we prove the convergence of all solutions to zero when both $\zeta - 1 > 0$ and $h(\zeta - 1)^{1/8}$ are less than some constant (independent on h, ζ). This result gives additional insight to the conjecture about the equivalence between local and global asymptotical stabilities in the Mackey–Glass type delay differential equations.

Keywords: Delay differential equations; Global asymptotic stability; Schwarz derivative

1. Introduction and main results

In this note, we consider the delay differential equation

$$x'(t) = -x(t) + \zeta f\bigl(x(t-h)\bigr), \quad x \in \mathbb{R},\ \zeta > 0, \tag{1}$$

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where $f \in C^3(\mathbb{R}, \mathbb{R})$ satisfies the following three basic properties (H):

(H1) $xf(x) < 0$ for $x \neq 0$ and $f'(0) = -1$.
(H2) f is bounded below and there exists at most one point $x^* \in \mathbb{R}$ such that $f'(x^*) = 0$. Moreover, in this case x^* is a local extremum.
(H3) $(Sf)(x) < 0$ for all $x \neq x^*$, where $Sf = f'''(f')^{-1} - 3/2(f'')^2(f')^{-2}$ is the Schwarz derivative of f.

We call such a delay equation the Mackey–Glass type equation.

The main purpose of this work is to give an additional insight to the following conjecture:

(C) "*Local asymptotic stability of the equilibrium* $e(t) \equiv 0$ *of Eq.* (1) *implies global asymptotic stability, that is, all solutions of* (1) *converge to zero when* t *tends to infinity*."

This conjecture was first suggested by Smith (see [5,13]) for Nicholson's equation, while the above form (C) has been proposed in [11]. Moreover, the celebrated Wright conjecture [7,9–12,15] can be viewed as a limit case of (C). It should be noted here that the asymptotic stability of the linearized equation

$$x'(t) = -x(t) - \zeta x(t-h), \quad x \in \mathbb{R}, \tag{2}$$

is well studied (see [6] and Proposition 1 below), while there are only few results about the global stability of (1) (e.g., see [5,11] for more references).

To formulate a criterion of asymptotical stability for Eq. (2), we define new parameters $\mu = 1/\zeta \geqslant 0$, $\nu = \exp(-h)/\zeta \geqslant 0$.

Proposition 1 [6]. *Suppose that* $\mu \geqslant 1$, *or* $\mu < 1$ *and*

$$\nu > \nu_1(\mu) = \mu \exp\left(\frac{-\mu \arccos(-\mu)}{\sqrt{1-\mu^2}}\right). \tag{3}$$

Then Eq. (2) *is uniformly exponentially stable.*

Next, the following global stability result was proved in [5]:

Proposition 2. *Assume that* f *satisfies hypotheses* (H). *If* $\mu \geqslant 1$, *or* $\mu < 1$ *and*

$$\nu \geqslant \nu_2(\mu) = \frac{\mu - \mu^2}{1 + \mu^2} \tag{4}$$

then the steady state $e(t) \equiv 0$ *attracts all solutions* $x(t)$ *of Eq.* (1): $x(t) \to 0$ *as* $t \to +\infty$.

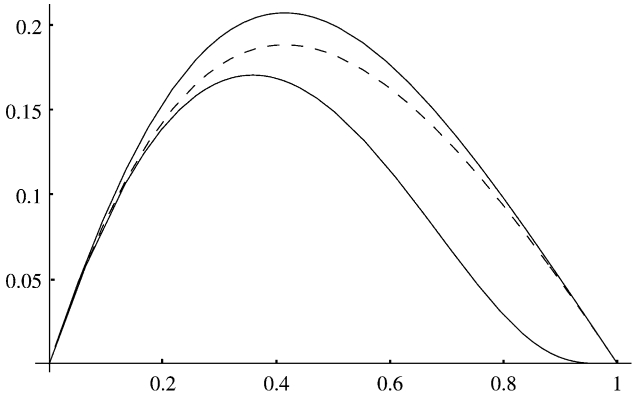

Fig. 1. Domains of global and local stability.

Remark 3. To our best knowledge, the global stability condition (4) (formulated for the Mackey–Glass type Eq. (1)) seems to be the best result ever reported in the literature.

The two solid lines in Fig. 1 represent the boundaries of local and global stability regions described in Propositions 1 and 2: for $\mu \in (0, 1)$, they are determined by the functions $\nu = \nu_1(\mu)$ and $\nu = \nu_2(\mu)$ (where $\nu_2(\mu) > \nu_1(\mu)$).

From Fig. 1, we observe that there is a rather good agreement between the solid curves for sufficiently large ζ (e.g., for $\zeta > 5$ that corresponds to $\mu < 0.2$), while considerable discrepancy occurs for values ζ close to $\zeta = 1$. This difference in the behavior of these curves reaches its maximum at the point $(\mu, \nu) = (1, 0)$, where the boundary of the local stability domain given by (3) (for $\mu \leqslant 1$) with C^∞-smoothness is continued by its other part $\nu = 0$ (for $\mu \geqslant 1$). Indeed, at the same point $(\mu, \nu) = (1, 0)$ the tangent line of the global stability curve undergoes an abrupt change. Hence, surprisingly, in order to construct a counter-example to (C), we should work out parameters μ, ν close to $(\mu, \nu) = (1, 0)$.

Moreover, there is another fact motivating the reconsideration of (C). To see this, we first state the following result from [8]:

Proposition 4. *Let $\mu > 0$ and $0 < \nu < \nu_3(\mu) = \ln[(1 + \mu)/(1 + \mu^2)]$. Then there exists a periodic function $\tau : \mathbb{R} \to [0, h]$ such that the trivial solution to*

$$x'(t) = -x(t) + \zeta f\big(x\big(t - \tau(t)\big)\big), \quad x \in \mathbb{R},\ \zeta > 0, \tag{5}$$

is unstable. On the other hand, if $\nu > \nu_3(\mu)$, then the steady state $e(t) \equiv 0$ of the equation

$$x'(t) = -x(t) + \xi(t) f\big(x\big(t - \tau(t)\big)\big), \quad x \in \mathbb{R}, \tag{6}$$

is uniformly exponentially stable for every continuous function $\tau : \mathbb{R} \to [0, h]$ and for every $\xi \in L^\infty(\mathbb{R}, \mathbb{R}_+)$ with $\operatorname{ess\,sup}_{t \in \mathbb{R}} \xi(t) \leqslant \zeta$.

Remark 5. The graph of the function $\nu_3(\mu)$ is depicted in Fig. 1 by a dashed line (notice that $\nu_3(\mu) = \ln(1 + (\mu - \mu^2)/(1 + \mu^2)) \geqslant (\mu - \mu^2)/(1 + \mu^2) = \nu_2(\mu)$).

Clearly, in view of the similarity of (1) and (5), Proposition 4 provides another reason to reconsider the global asymptotical stability of (1) for $\nu > \nu_1(\mu)$ (at least in the vicinity of $\mu = 1$).

Therefore, it is important to explain the difference in the behavior of solid curves pictured in Fig. 1. We will show below that this difference is only due to the insufficiently sharp form of the stability conditions given in Proposition 2. Indeed, let $\mathcal{D} \subset \mathbb{R}^2_+$ be the set of all parameters μ, ν for which Eq. (1) is globally asymptotically stable, and define $\Gamma : \mathbb{R}_+ \to [0, 0.25]$ by $\Gamma(\mu) = \inf\{\nu \geqslant 0: \{\mu\} \times (\nu, +\infty) \subset \mathcal{D}\}$. The next theorem represents the main result of the present note, and states that functions ν_1 and Γ have the same slope at $\mu = 1$.

Theorem 6. *There exist* $\epsilon = \epsilon_f > 0$, $K = K_f > 0$ *such that Eq.* (1) *is globally stable whenever* $0 \leqslant \zeta - 1 \leqslant \epsilon$ *and*

$$0 \leqslant h < K(\zeta - 1)^{-1/8}. \tag{7}$$

As a consequence, Γ *is differentiable at* $\mu = 1$, *and* $\Gamma'(1) = 0$.

Remark 7.

(a) Notice that $\Gamma(\mu) \equiv 0$ for $\mu \geqslant 1$ and $0 < \Gamma(\mu) < \nu_2(\mu)$ if $\mu \in (0, 1)$. Conjecture (C) states that $\Gamma(\mu) = \nu_1(\mu)$; however, we are now even unable to prove the continuity of Γ over the interval $(0, 1)$, although Γ is lower semi-continuous thanks to the robustness of global attractivity.

(b) It should be noted that, in a small neighbourhood of $(\mu, \nu) = (1, 0)$, Eq. (1) can be viewed as a singularly perturbed equation [6, Section 12.7]

$$\varepsilon x'(t) = -x(t) + \zeta f\bigl(x(t-1)\bigr), \quad \varepsilon = h^{-1}.$$

It is known [6, Theorem 7.2] that assumptions (H) imply the existence of $\delta > 0$ such that, for every $(\mu, \nu) \in \{(\mu, \nu): 1 - \delta < \mu < 1,\ 0 < \nu < \nu_1(\mu)\}$, Eq. (1) has a unique slowly oscillating periodic solution with period $T(h, \zeta) = 2h + 2 + O(h^{-1} + |\zeta - 1|)$.

(c) It can be proved that the set $\mathcal{D}$ is open (see [7,14]). If, moreover, one can show that $\mathcal{D}$ is closed in the metric space $\{(\mu, \nu) \in (0, +\infty)^2: \nu > \nu_1(\mu)$ for $\mu \in (0, 1]\}$, the global stability conjecture will be established (compare with [7, p. 65]). However, we do not even know if $\mathcal{D}$ is simply connected (or connected).

Theorem 6 will be obtained as an easy consequence of several asymptotic estimations, one of which is stated below:

Theorem 8. *Let $v(t,h)$ be the fundamental solution of the linear delay differential equation*

$$x'(t) = -x(t) - x(t-h). \tag{8}$$

Then, for every $\alpha > 2$, there exist $h_0 = h_0(\alpha) > 0$, $c = c(\alpha) > 0$ such that

$$\left|v(t,h)\right| \leqslant ch \exp\left(-\frac{\pi^2 t}{\alpha h^3}\right), \quad t \geqslant 0, \tag{9}$$

for all $h \geqslant h_0$.

Remark 9.

(a) By definition, $v(\cdot,h) : [-h,+\infty) \to \mathbb{R}$ is the solution of Eq. (8) satisfying $v(0,h) = 1$ and $v(s,h) = 0$ for all $s \in [-h,0)$.
(b) It is not difficult to show (see also Remark 14) that the factor h^{-3} from the exponent in the right-hand side of (9) is the best possible (asymptotically). However, we can not say the same about h before the exponential (for example, we do not know if h could be replaced by $\ln h$).
(c) We can take $c(\alpha) = b\alpha(\alpha-2)^{-1}$, where $b > 0$ does not depend on α.

Finally, we will also need the following simple statement, which is an immediate consequence of Proposition 2 and the well-known results about period-doubling bifurcation for one-dimensional dynamical systems defined by functions with negative Schwarzian (e.g., see [2, p. 92]):

Theorem 10. *There exist $\epsilon_1 = \epsilon_1(f) > 0$, $K_1 = K_1(f) > 0$ such that every bounded solution $x : \mathbb{R} \to \mathbb{R}$ of Eq.* (1) *satisfies the inequality*

$$\sup_{t\in\mathbb{R}} \left|x(t)\right| \leqslant K_1(\zeta - 1)^{1/2} \tag{10}$$

for $0 \leqslant \zeta - 1 \leqslant \epsilon_1$.

The paper is organized as follows. The proof of Theorem 8, which is the most difficult ingredient of our note, can be found in the second section. In Section 3 we prove Theorem 10 and our main result (Theorem 6), and in the last section we discuss some other aspects of the global stability conjecture (C).

2. Proof of Theorem 8

We will use the following representation of the fundamental solution

$$v(t,h) = \lim_{T\to+\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{\exp((c+is)t)}{p(c+is,h)}\, ds, \tag{11}$$

where $p(z) = p(z,h) = z + 1 + \exp(-zh)$ is the characteristic quasipolynomial associated with Eq. (8) and $c > \max\{\Re\lambda\colon\ p(\lambda,h) = 0\}$ (see [6, Section 1.5]). First we get an asymptotic estimate for $|p(z,h)|$ along the vertical lines defined by $\lambda(s) = a + is$, $s \in \mathbb{R}$:

Lemma 11. *Let $\alpha > 2$ and define $\beta = (2\alpha + 1)/(\alpha - 2) > 0$. There exists $h_1 = h_1(\alpha) > 0$ such that*

$$\bigl|p\bigl(\lambda(s)\bigr)\bigr| \geqslant \frac{\pi^2}{\beta h^2} \tag{12}$$

for all $s \in [0, 2\pi/h]$, $a \in [-\pi^2/(\alpha h^3), 0]$, $h \geqslant h_1$.

Proof. We prove the lemma by contradiction. Let us suppose that there exist $h_k \to +\infty$, $s_k \in [0, 2\pi/h_k]$ and $a_k \in [-\pi^2/(\alpha h_k^3), 0]$ such that

$$\bigl|p(a_k + is_k)\bigr| < \pi^2/\bigl(\beta h_k^2\bigr). \tag{13}$$

Without loss of generality, we can assume that $s_k h_k \to \phi \in [0, 2\pi]$ and $a_k h_k^3 \to \psi \in [-\pi^2/\alpha, 0]$ as $k \to \infty$. Since

$$\lim_{k\to\infty} s_k = \lim_{k\to\infty} a_k = \lim_{k\to\infty} a_k h_k = 0,$$

we obtain from (13) that $\lim_{k\to\infty} |p(a_k + is_k)| = |1 + \exp(-i\phi)| = 0$. Hence $\phi = \pi$ and $\epsilon_k = s_k h_k - \pi \to 0$ when $k \to \infty$.

Now, it is easy to see that the inequality (13) implies

$$\frac{\pi^2}{\beta h_k} > \bigl|\pi + \epsilon_k + \exp(-a_k h_k) h_k \sin\epsilon_k\bigr|$$

and

$$\frac{\pi^2}{\beta h_k^2} > \bigl|a_k + 1 - \exp(-a_k h_k)\cos\epsilon_k\bigr|.$$

The first of these inequalities is possible for all k only if $h_k\epsilon_k \to -\pi$ as $k \to \infty$. The second inequality can be written as

$$\pi^2/\beta > \bigl|a_k h_k^2 + h_k^2\bigl(1 - \exp(-a_k h_k)\bigr) + h_k^2(1 - \cos\epsilon_k)\exp(-a_k h_k)\bigr|,$$

and takes the following limit form (when $k \to \infty$):

$$\pi^2/\beta \geqslant |\psi + \pi^2/2| \geqslant \pi^2/2 - \pi^2/\alpha = \frac{(\alpha - 2)\pi^2}{2\alpha},$$

a contradiction, proving Lemma 11. □

Lemma 12. *For $\alpha > 2$, there exists $h_2 = h_2(\alpha) > 0$ such that for every $h > h_2$, $s \geqslant 2\pi/h$, $a = -\pi^2/(\alpha h^3)$ we have*

$$\max\{s - 3, 0\} < \sqrt{(1 + a)^2 + s^2} - \exp(-ah) \leqslant \bigl|p\bigl(\lambda(s)\bigr)\bigr| < s + 3. \tag{14}$$

Proof. We have, for $s > 0$ and sufficiently large $h > 0$, that

$$\begin{aligned}\bigl|p\bigl(\lambda(s)\bigr)\bigr| &= \bigl|a + is + 1 + \exp(-ah)\exp(-ish)\bigr| \\ &\leqslant 1 + |a| + s + \exp(-ah) < 3 + s.\end{aligned}$$

On the other hand, by the triangular inequality,

$$\begin{aligned}\bigl|p\bigl(\lambda(s)\bigr)\bigr| &= \bigl|a + is + 1 + \exp(-ah)\exp(-ish)\bigr| \\ &\geqslant |a + is + 1| - \bigl|\exp(-ah)\exp(-ish)\bigr| \\ &= \sqrt{(1+a)^2 + s^2} - \exp(-ah),\end{aligned}$$

the last part being positive for $s \geqslant 2\pi/h$ and h large enough. □

Corollary 13. *We have, for each $\alpha > 2$ and all $h > h_1(\alpha)$, that $\sigma(h) = \max\{\Re\lambda\colon\ p(\lambda, h) = 0\} < -\pi^2/(\alpha h^3)$.*

Proof. It is well known that Eq. (8) is uniformly stable for every $h > 0$ (see, e.g., [6, p. 154]), so that $\sigma(h) \leqslant 0$. It suffices now to apply Lemmas 11 and 12 to complete the proof. □

Remark 14. In fact, we claim that $\sigma(h) \sim -\pi^2/(2h^3)$ for $h \gg 1$.

Indeed, we will establish below that the roots $\lambda(h) = a(h) \pm ib(h)$, $a(h) < 0$, $b(h) > 0$ of $p(\lambda, h) = 0$ have the following asymptotic representations for $h \gg 1$:

$$\lambda_k(h) \sim -\pi^2(1+2k)^2/(2h^3) \pm \pi(1+2k)i/h, \quad k \in \{0, 1, 2, \ldots\}. \tag{15}$$

Moreover, it is easy to prove that, for $h > 1$, there exists a unique pair of conjugate roots $\lambda(h)$ such that $|\Im(\lambda(h))|h \leqslant \pi$. Thus, from (15) we have that, for large h, $\sigma(h) = \Re(\lambda(h)) \sim -\pi^2/(2h^3)$, proving our claim.

To establish (15), we observe that, due to the implicit function theorem, $\lambda(h)$ depends smoothly on the positive parameter $h \geqslant 1$. Therefore, rewriting the characteristic equation in the form

$$a(h) + 1 + \exp\bigl(-a(h)h\bigr)\cos\bigl(b(h)h\bigr) = 0, \tag{16}$$

$$b(h) = \exp\bigl(-a(h)h\bigr)\sin\bigl(b(h)h\bigr), \tag{17}$$

and analyzing Eq. (17), we see that there exists $k \in \{0, 1, 2, \ldots\}$ such that $b(h)h \in [2\pi k, \pi + 2\pi k]$ for all $h \geqslant 1$ (notice that the characteristic equation has no real roots for $h \geqslant 1$). This means that $\lim_{h\to\infty} b(h) = 0$, so that, by (17), $b(h)h \to 2\pi k$ or $b(h)h \to \pi + 2\pi k$ when $h \to \infty$. We claim that $b(h)h \to \pi + 2\pi k$. Indeed $b(h)h \to 2\pi k$ and Eq. (16) imply that $a(h) < -1$ for large h. This estimate allows us to conclude, again due to (16), that $\lim_{h\to+\infty} a(h) = -\infty$ so that

$$\lim_{h\to+\infty} h = \lim_{h\to+\infty} \bigl|a(h)\bigr|^{-1} \ln\bigl[\bigl|a(h) + 1\bigr| \big/ \cos\bigl(b(h)h\bigr)\bigr] = 0,$$

a contradiction. Thus $b(h)h = \pi(1+2k) + e(h)$, where $e(h) \to 0$ as $h \to \infty$. Using this representation of $b(h)h$, we rewrite Eqs. (16) and (17) as

$$a(h) + 1 = \exp\bigl(-a(h)h\bigr)\cos\bigl(e(h)\bigr), \tag{18}$$

$$\pi(1+2k) + e(h) = -\exp\bigl(-a(h)h\bigr)h\sin\bigl(e(h)\bigr). \tag{19}$$

Now, Eq. (18) implies that $c(h) = a(h)h \to 0$ for $h \to \infty$. Therefore, by (19), we get $e(h)h = -\pi(1+2k) + o(1/h)$. Finally, Eq. (18) gives $c(h)(1+o(1)) = -(e^2(h)/2)(1+o(1))$, and therefore

$$a(h) = \frac{c(h)}{h} \sim \frac{-e^2(h)}{2h} \sim \frac{-\pi^2(1+2k)^2}{2h^3}.$$

Lemma 15. *For each $\alpha > 2$, there exist $h_0 = h_0(\alpha) > 0$ and $K_2 = K_2(\alpha) > 0$ such that, for every $h > h_0$, we have*

$$\left|\lim_{T\to+\infty}\int_{-T}^{T}\frac{e^{ist}\,ds}{p(-\pi^2/(\alpha h^3) + is)}\right| \leqslant K_2 h. \tag{20}$$

Proof. First notice that the value of the integral is a real number, so that we have to consider only the real part of the integrand $e^{ist}/q(s)$. Since this real part is an even function, it suffices to prove that

$$|I_1| = \left|\int_0^{2\pi/h}\Re\bigl[e^{ist}/q(s)\bigr]\,ds\right| \leqslant K_3 h,$$

$$|I_2| = \left|\int_{2\pi/h}^{1}\Re\bigl[e^{ist}/q(s)\bigr]\,ds\right| \leqslant K_4 h,$$

and

$$|I_3| = \left|\int_1^{+\infty}\Re\bigl[e^{ist}/q(s)\bigr]\,ds\right| \leqslant K_5 h$$

for some $K_3, K_4, K_5 > 0$ and sufficiently large h.

Now, by Lemma 11, we have that, for $h \geqslant h_1$,

$$|I_1| \leqslant \int_0^{2\pi/h}\bigl|q(s)\bigr|^{-1}\,ds \leqslant (2\pi/h)\bigl(\pi^2/(\beta h^2)\bigr)^{-1} = 2\beta h/\pi = K_3 h,$$

where $\beta = (2\alpha+1)/(\alpha-2)$.

Next, by Lemma 12,

$$|I_2| \leqslant \int_{2\pi/h}^{1} \left|q(s)\right|^{-1} ds \leqslant \int_{2\pi/h}^{1} \frac{ds}{\left|\sqrt{a^2(h)+s^2} - b(h)\right|},$$

where $a(h) = 1 - \pi^2/(\alpha h^3)$ and $b(h) = \exp(\pi^2/(\alpha h^2))$. For sufficiently large h and $s \in [2\pi/h, 1)$, we have

$$\sqrt{a^2(h)+s^2} - b(h) > 0, \qquad 1/a(h) < 1 + \pi/h,$$
$$\frac{2\pi}{a(h)h} > \frac{\pi}{h}, \qquad \sqrt{1+(s+\pi/h)^2} + b(h)/a(h) \leqslant 3,$$

so that

$$\begin{aligned}
|I_2| &\leqslant \int_{2\pi/h}^{1} \frac{ds}{\sqrt{a^2(h)+s^2} - b(h)} = \int_{2\pi/(a(h)h)}^{1/a(h)} \frac{ds}{\sqrt{1+s^2} - b(h)/a(h)} \\
&\leqslant \int_{\pi/h}^{1+\pi/h} \frac{ds}{\sqrt{1+s^2} - b(h)/a(h)} \leqslant \int_{0}^{1} \frac{ds}{\sqrt{1+(s+\pi/h)^2} - b(h)/a(h)} \\
&= \int_{0}^{1} \frac{\sqrt{1+(s+\pi/h)^2} + b(h)/a(h)}{1+(s+\pi/h)^2 - b^2(h)/a^2(h)}\, ds \\
&\leqslant \int_{0}^{1} \frac{3\, ds}{(s+\pi/h)^2 + (1 - b^2(h)/a^2(h))} = R(h).
\end{aligned}$$

Now, since

$$\lim_{h\to+\infty} R(h)h^{-1} = \frac{3\sqrt{\alpha/2}}{\pi} \int_{0}^{\infty} \frac{du}{(u+\sqrt{\alpha/2})^2 - 1} = K_6 \in \mathbb{R}_+,$$

we obtain that $|I_2| \leqslant (K_6+1)h = K_4 h$ for sufficiently large h.

Next,

$$\begin{aligned}
I_3 = {} & \int_{1}^{+\infty} \frac{\cos(s(t+h))\exp(\pi^2/(\alpha h^2)) + \cos(st)(-\pi^2/(\alpha h^3)+1)}{|q(s)|^2}\, ds \\
& + \int_{1}^{+\infty} \frac{s\sin(st)}{|q(s)|^2}\, ds = I_4 + I_5.
\end{aligned}$$

Now, for large h,

$$|I_4| \leqslant 3 \int_1^{+\infty} \left|q(s)\right|^{-2} ds \leqslant 3 \int_1^{+\infty} \left(\sqrt{s^2+0.75}-1.25\right)^{-2} ds \leqslant K_7 \in \mathbb{R},$$

so that we only have to evaluate I_5. We obtain that

$$I_5 = \int_1^{+\infty} \frac{(s-|q(s)|)\sin(st)}{|q(s)|^2}\, ds + \int_1^{+\infty} \frac{\sin(st)}{|q(s)|}\, ds = I_6 + I_7,$$

where, in virtue of (14),

$$|I_6| \leqslant 3 \int_1^{+\infty} \left|q(s)\right|^{-2} ds \leqslant K_7.$$

Finally, using Lemma 12 again, we get

$$\begin{aligned}
\left| I_7 - \int_1^{+\infty} \frac{\sin(st)}{s}\, ds \right| &\leqslant \int_1^{+\infty} \frac{|s-|q(s)||\,|\sin(st)|}{s|q(s)|}\, ds \\
&\leqslant 3 \int_1^{+\infty} \left|sq(s)\right|^{-1} ds \leqslant 3 \int_1^{+\infty} \frac{1}{s\left|\sqrt{a^2(h)+s^2}-b(h)\right|}\, ds \\
&\leqslant 3 \int_1^{+\infty} \frac{ds}{\left|\sqrt{a^2(h)+s^2}-b(h)\right|^2} \leqslant 3 \int_1^{+\infty} \frac{ds}{\left(\sqrt{s^2+0.75}-1.25\right)^2} \leqslant K_7,
\end{aligned}$$

and since, for all $t \geqslant 0$, $h > 0$,

$$\left| \int_1^{+\infty} \frac{\sin(st)}{s}\, ds \right| = \left| \int_t^{+\infty} \frac{\sin(u)}{u}\, du \right| \leqslant \sup_{x \geqslant 0} \left| \int_x^{+\infty} \frac{\sin(u)}{u}\, du \right| = K_8,$$

we have the necessary estimate $|I_5| \leqslant K_9$. $\square$

Proof of Theorem 8. Now we can end the proof of Theorem 8 noting that, by (11),

$$\begin{aligned}
\left|v(t,h)\right| &\leqslant \frac{\exp(-\pi^2 t/(\alpha h^3))}{2\pi} \left| \lim_{T\to+\infty} \int_{-T}^{T} \frac{\exp(ist)}{p(-\pi^2/(\alpha h^3)+is,h)}\, ds \right| \\
&\leqslant \frac{K_2}{2\pi} h \exp\left(-\pi^2 t/(\alpha h^3)\right) = ch \exp\left(-\pi^2 t/(\alpha h^3)\right). \qquad \square
\end{aligned}$$

3. Proof of Theorems 6 and 10

In order to prove Theorem 10, we will need the following result:

Proposition 16. *Assume that f satisfies hypotheses* (H1), (H2) *and set $f_\zeta = \zeta f$. Then we have*:

(1) *The set $A_\zeta = \bigcap_{j=0}^{+\infty} f_\zeta^j(\mathbb{R})$ is global attractor of the map f_ζ; in particular, $A_\zeta = [a_\zeta, b_\zeta]$ and $f_\zeta(A_\zeta) = A_\zeta$.*
(2) *Every bounded solution $x:\mathbb{R} \to \mathbb{R}$ to Eq.* (1) *satisfies $a_\zeta \leqslant x(t) \leqslant b_\zeta$ for all $t \in \mathbb{R}$.*
(3) *If $A_1 = \{0\}$, then $\lim_{\zeta\to 1} a_\zeta = \lim_{\zeta \to 1} b_\zeta = 0$.*
(4) *If $f_\zeta'(x) < 0$ for all $x \in A_\zeta$, then $f_\zeta(a_\zeta) = b_\zeta$ and $f_\zeta(b_\zeta) = a_\zeta$.*

Proof. For (1) and (3), see [7, Sections 2.4, 2.5]; (4) is an immediate consequence of (1), and, finally, (2) was established in [5]. □

Proof of Theorem 10. First, note that hypotheses (H1), (H2) imply the existence of some δ-neighbourhood U of $x = 0$, where f is strictly decreasing: $f_\zeta'(x) < 0$, $x \in U$, $\zeta > 0$. Next we claim that $A_1 = \{0\}$. Indeed, since $(f^2)'(0) = 1$ and $(f^2)''(0) = 0$ we obtain, in view of the negativity of Sf^2, that $(f^2)'''(0) < 0$. Therefore zero is an asymptotically stable point for f^2 (see, for example, [3, p. 25]), and hence for f. By [5, Proposition 7], these facts guarantee the global stability of the fixed point $x = 0$ to f, that is, $A_1 = \{0\}$.

Next, by Proposition 16(3), there exists $\sigma > 0$ such that $A_\zeta \subset U$ for $0 < \zeta - 1 < \sigma$. Since f is decreasing on U we have, in view of Proposition 16(4), that $f_\zeta(a_\zeta) = b_\zeta$, $f_\zeta(b_\zeta) = a_\zeta$.

Now, by [2, Corollary 12.8], there exists a subset $U_\beta \subset U$ and a smooth function $\xi : U_\beta \to [1, +\infty)$, $\xi(0) = 1$, $\xi'(0) = 0$, $\xi''(p) > 0$ for all $p \in U_\beta$, such that $f^2_{\xi(p)}(p) = p$ and $f_{\xi(p)}(p) \neq p$. Thus, for $\zeta \to 1^+$, we have that $\zeta = \xi(p_1) = \xi(p_2)$ for some $p_1, p_2 \in U_\beta$, $p_1 < 0 < p_2$. Moreover, the negativity of Sf_ζ^2 and monotonicity of f_ζ^2 inside U imply that p_1 and p_2 are the unique nonzero fixed points for f_ζ^2 (in particular, $f_\zeta(p_1) = p_2$). Hence, $p_1 = a_\zeta$, $p_2 = b_\zeta$, and, by Proposition 16(2), every bounded solution $x:\mathbb{R} \to \mathbb{R}$ to Eq. (1) satisfies the inequality

$$p_1 \leqslant x(t) \leqslant p_2, \quad t \in \mathbb{R}.$$

Finally, using the relations $\xi(p_i) = \zeta$, $i = 1, 2$, and the properties of ξ, we obtain (10) for $\zeta - 1 > 0$ sufficiently small. □

Proof of Theorem 6. Let $z:\mathbb{R} \to \mathbb{R}$ be a bounded solution to Eq. (1). Then $z(t)$ satisfies the following linear equation

$$x'(t) = -x(t) - x(t-h) + a(t), \tag{21}$$

where $a(t) = \zeta f(z(t-h)) + z(t-h)$. Take now $\epsilon_1 \in (0,1)$, $K_1 > 0$ as indicated in Theorem 10. Then for $0 < \zeta - 1 \leqslant \epsilon_1$, we have that

$$\begin{aligned}\left|a(t)\right| &\leqslant K_1(\zeta-1)^{1/2} \max_{|y|\leqslant K_1(\zeta-1)^{1/2}} \left|1+\zeta f'(y)\right| \\ &\leqslant K_1(\zeta-1) \max_{|y|\leqslant K_1(\zeta-1)^{1/2}} \left[(\zeta-1)^{1/2} + \zeta K_1 \left|f''(y)\right|\right] \leqslant \widetilde{K}(\zeta-1),\end{aligned}$$

where $\widetilde{K} = K_1(1 + 2K_1 \max_{|y|\leqslant K_1} |f''(y)|)$.

Since Eq. (21) is asymptotically stable and $a(t)$ is bounded and continuous, it has a unique bounded solution $x(t) = z(t)$ defined for all $x \in \mathbb{R}$. Moreover,

$$z(t) = \int_{-\infty}^{t} v(t-s,h)a(s)\,ds,$$

so that, using Theorem 8 for an arbitrarily fixed $\alpha > 2$, we get

$$\begin{aligned}\left|z(t)\right| &\leqslant \widetilde{K}(\zeta-1) \int_{-\infty}^{t} \left|v(t-s,h)\right| ds \\ &\leqslant \widetilde{K}(\zeta-1) \int_{-\infty}^{t} ch \exp\left(-\frac{\pi^2(t-s)}{\alpha h^3}\right) ds \\ &= \frac{\widetilde{K}c\alpha}{\pi^2}(\zeta-1)h^4 < (1/2)K_1(\zeta-1)^{1/2},\end{aligned}$$

for $h \geqslant h_0$ whenever $h(\zeta-1)^{1/8} < K = \sqrt{\pi}(K_1/(2\widetilde{K}c\alpha))^{1/4}$. By repeating the same argument, we can prove that $|z(t)| < (1/2)^n K_1(\zeta-1)^{1/2}$ for all $t \in \mathbb{R}$ and $n \in \mathbb{N}$. Thus $z(t) \equiv 0$.

Without loss of generality, we can assume that $h_0 < K(\zeta-1)^{-1/8}$ for all $\zeta \in [1, 1+\epsilon_1]$. Hence we have shown above that Eq. (1) is globally stable for $h_0 \leqslant h < K(\zeta-1)^{-1/8}$ and $0 \leqslant \zeta - 1 \leqslant \epsilon_1$. Finally, Proposition 2 permits us to find $\epsilon_2 > 0$ such that $0 \leqslant \zeta - 1 \leqslant \epsilon_2$ implies that (1) is globally stable for $0 \leqslant h < h_0$. Thus inequality (7) is proved choosing $\epsilon = \min\{\epsilon_1, \epsilon_2\}$.

Now, (7) implies that, for $\delta > 0$ sufficiently small, $0 \leqslant \Gamma(1-\delta) \leqslant F(1-\delta)$, where $F(\mu) = \mu \exp\{-K((1/\mu)-1)^{-1/8}\}$.

Since $\lim_{\delta\to 0^+} F(1-\delta)/\delta = F'(1^-) = 0$, we can conclude that $\Gamma'(1) = 0$. $\square$

4. Remarks and discussion

It is easy to see that the study of global asymptotical stability of the unique positive equilibrium to the following well-known (e.g., see [1,4,5,9,13]) delay equations (with positive ζ, a, x)

$$x'(t) = -x(t) + \frac{\zeta a^n}{a^n + x^n(t-h)}, \quad n > 1 \quad \text{(Mackey–Glass)}, \tag{22}$$

$$x'(t) = -x(t) + \zeta \exp\bigl(-ax(t-h)\bigr) \quad \text{(Lasota–Wazewska)} \tag{23}$$

can be reduced, via a simple change (a translation) of variable, to the investigation of global attractivity of the trivial solution to Eq. (1). In some cases (e.g., when ζ is close to 1), the same observation is also valid for the equations

$$x'(t) = -x(t) + \frac{\zeta a^n x(t-h)}{a^n + x^n(t-h)}, \quad n > 1 \quad \text{(Mackey–Glass)}, \tag{24}$$

$$x'(t) = -x(t) + \zeta x(t-h) \exp\bigl(-ax(t-h)\bigr) \quad \text{(Nicholson)}. \tag{25}$$

As mentioned before (see Propositions 1, 2 and [5]), in the domain $(h, \zeta) \in \mathbb{R}^2_+$, the decay dominance condition $1 \geqslant \zeta$ (or $\mu \geqslant 1$) determines all parameters for which Eq. (1) (and, in particular, (22)–(25)) is absolutely stable (we recall here that "absolute stability" means "delay independent stability").

Now let $\zeta > 1$ and denote by $h_c(\zeta)$ the global stability delay threshold: $h_c(\zeta)$ is the maximal positive number for which the inequality $h < h_c(\zeta)$ implies convergence of all solutions to the equilibrium. By the above comments, it is natural to expect that $h_c(\zeta) \to +\infty$ as $\zeta \to 1+$ (while the folklore statement: "*Small delays are harmless*" implies that $h_c > 0$). Indeed, by Proposition 2, $h_c(\zeta) \geqslant \ln(\zeta + \zeta^{-1}) - \ln(\zeta - 1) \sim -\ln(\zeta - 1)$, so that for every $h > 0$ the Mackey–Glass delay differential equation can be stabilized by choosing $\zeta > 1$ sufficiently close to 1. This means that even large delays are harmless near the boundary of absolute stability. Moreover, Theorem 6 has improved the above logarithmic estimation of $h_c(\zeta)$ near $\zeta = 1$ saying that we have there, for some $K > 0$, $\epsilon > 0$,

$$h_c(\zeta) \geqslant K(\zeta - 1)^{-1/8} \quad \text{if } 0 < \zeta - 1 < \epsilon. \tag{26}$$

Now, let us indicate briefly some aspects of the considered problem which could be studied in the future.

First, it seems that the exponent $-1/8$ in (26) can be significantly improved (up to $-1/2$, if the global stability conjecture were true). Unfortunately, our method (when we establish some estimates for the global attractivity domain (Theorem 10) and then use the contractivity argument inside this domain (Theorem 8)) does not allow this improvement at all. The best estimate within our approach is $-1/6$, and it could be reached if we were able to show that h before the exponential in (9) is not necessary (or at least could be replaced with $\ln h$, see also Remark 9).

Second, it will be very interesting to obtain some K, ϵ in (26) explicitly. Also, in the statement of Theorem 6, both constants depend on the nonlinearity f: we hope that this dependence can be discarded with a different approach.

Finally, we note that at the moment of the acceptance of this paper we already proved that the inequality $\nu > \nu_3(\mu)$, $\mu \in (0, 1)$ (see Proposition 4) is also sufficient for the global stability in (1) (see [12]).

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