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Periodic points and stability in Clark's delayed recruitment model

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Abstract

We provide further insight on the dynamics of Clark's delayed recruitment equation, depending on the relevant parameters involved in the model. We pay special attention to the stability and bifurcations from the positive equilibrium, and to the existence and attraction properties of nontrivial cycles. A detailed analysis is worked out for a three-dimensional example.

Keywords: Clark's model; Delayed discrete population model; Stability; Attracting cycle; Ricker oscillation; Bifurcation analysis

1. Introduction

Clark's equation is a simple discrete-time mathematical model to represent the evolution of a population in which the number of adults each year is calculated as the sum of the survival adults in the previous year and the recruitment, which is in general a nonlinear function of the size of population of adults a number of k years before. See, e.g., [4,5]. In general, it is written as

$$x_{n+1} = \alpha x_n + f(x_{n-k}), \tag{1}$$

where $\alpha \in [0, 1)$ is a survival rate, and $f : (0, \infty) \to [0, \infty)$ is the recruitment function. Here, we will use a slightly different form of Clark's equation, already employed by Fisher [12], and suggested by Botsford [4] to explain an apparent contradiction between two data tables. Namely, we will consider equation

$$x_{n+1} = \alpha x_n + (1 - \alpha)h(x_{n-k}). \tag{2}$$

Notice that, for a fixed $\alpha \in [0, 1)$, Eqs. (1) and (2) are equivalent, taking $h = (1 - \alpha)^{-1} f$. We believe that Eq. (2) is more convenient to investigate the variation in the dynamics of the solutions as the parameter α varies on $[0, 1)$. We will try to justify this assertion. One of the key points in Clark's model is how the stock-recruitment relationship should be. Form (2) represents well two of the postulates indicated in [4], namely: (a) high fecundity at low stock sizes (hence, the recruitment should be larger for decreasing α), and (b) increasing pre-recruitment mortality at high stock

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sizes due to compensatory density-dependent effects (thus, the recruitment rate should decrease as α approaches 1). From the mathematical point of view, form (2) has as an advantage that the equilibria are the same for all values of α, and they coincide with the fixed points of function h. Another argument was shown in [11] (see also [1,17]): Eq. (1) is exactly the form of the Euler discretization with step $\tau/k = 1 - \alpha$ of the well known delay differential equation used in mathematical biology

$$x'(t) = -x(t) + h(x(t-\tau)), \quad \tau > 0.$$

Other important point is the shape of the recruitment function h. In most of examples, h is chosen strictly decreasing or unimodal, with a local maximum. Some properties of (2) were derived in [11] under more general assumptions. Due to biological reasons, only nonnegative initial conditions will be considered; in particular, if $h(0)$ is not defined and $\alpha > 0$, then the admissible initial conditions are those vectors $(x_{-k}, \ldots, x_{-1}, x_0) \in \mathbb{R}^{k+1}$ such that $x_i > 0$, $i = -k, \ldots, 0$. In any case, we will only consider solutions corresponding to admissible initial conditions, which will be called *admissible solutions*.

Throughout the paper we assume that $h : (0, \infty) \to [0, \infty)$ is continuous, has a unique fixed point p, and satisfies $h(x) > x$ for $x < p$ and $h(x) < x$ for $x > p$.

The following result on permanence in Eq. (2) can be easily derived from the proof of Theorem 2 in [11]:

Theorem 1 (*El-Morshedy and Liz [11]*). *Assume that either h is bounded on $(0, p]$ or $\lim_{x\to 0} h(x) = \infty$. Then Eq.* (2) *is permanent*, *that is*,

$$0 < \liminf_{n\to\infty} x_n \leqslant \limsup_{n\to\infty} x_n < \infty$$

for all admissible solutions (x_n) of (2).

We say that p is a *global attractor* for (2) if all admissible solutions converge to p as $n \to \infty$. On the other hand, p is called *globally stable* if it is a stable global attractor.

Theorem 2 (*El-Morshedy and Liz [11, Theorem 3]*). *In the conditions of Theorem* 1, *define $M = \sup\{h(x) : 0 < x < p\}$. If either $M = p$ or h is monotonically increasing on (p, M)*, *then p is globally stable for equation* (2).

For example, if h is a unimodal C^1 function, Theorem 2 shows that p is globally stable if $h'(p) \geqslant 0$ (see also [14]).

The issue of global stability in (2) was widely addressed in the recent literature (see, e.g., [8,10,11,14,17,30] and references therein). Our aim in this paper is the analysis of some properties of the solutions of (2) when p is not globally attracting. In view of Theorem 2, we will assume that $h'(p) < 0$.

Some particular cases of (2) will be of special interest in our discussions. The first one is the one-dimensional difference equation obtained from (2) by setting $k = 0$

$$x_{n+1} = \alpha x_n + (1-\alpha) h(x_n). \tag{3}$$

We note that Eq. (3) is interesting by itself. For example, for $h(x) = \beta x/(1 + x^r)$, $\beta > 1$, $r > 0$, this equation was proposed by Milton and Bélair [23] to describe the growth of bobwhite quail populations.

We will also consider the limit case $\alpha = 0$ in Eqs. (2) and (3), respectively, that is,

$$x_{n+1} = h(x_{n-k}); \tag{4}$$

$$x_{n+1} = h(x_n). \tag{5}$$

Since we will refer to Eqs. (2)–(5) quite often within the text, we include Table 1 to make easier their identification.

Note that p is the unique equilibrium of all equations in Table 1.

We have organized our exposition as follows: the second section is devoted to analyze the influence of the parameters in the local stability of the equilibrium. Although this study was already initiated by Clark himself and continued by other authors, we think that some new interesting comments may be provided here. In Section 3, we get some results on the existence of nontrivial periodic solutions for Eqs. (2) and (3). Under some additional assumptions, we may describe the global attractors of (2) for small α. Finally, in Section 4, a more detailed analysis is worked out for a three dimensional example.

Table 1
Main equations cited in the text

Label	Description	Equation
(2)	Clark's delayed model	$x_{n+1} = \alpha x_n + (1-\alpha)h(x_{n-k})$
(3)	Clark's model with $k=0$	$x_{n+1} = \alpha x_n + (1-\alpha)h(x_n)$
(4)	Clark's model with $\alpha=0$	$x_{n+1} = h(x_{n-k})$
(5)	First order difference equation	$x_{n+1} = h(x_n)$

2. Local stability

One of the important problems in the dynamics consists in finding the values of the involved parameters for which the equilibrium point is locally stable. For Eq. (2), this study was initiated by Clark [5]. It is well known that p is locally asymptotically stable if all roots of the characteristic equation

$$\lambda^{k+1} - \alpha\lambda^k = (1-\alpha)h'(p), \tag{6}$$

associated to the linearized equation at $x = p$, lie in the unit circle in the complex plane. Fortunately, necessary and sufficient conditions for this to happen are known (see, e.g., [18,24]). We will use the parametric equations given in [24]. Let us consider two significant parameters involved in the model, namely the survival rate α, and the derivative of h at p (connected with the recruitment rate). Assuming $h'(p) < 0$, let us denote $c = -h'(p) > 0$. We have the following result.

Proposition 3. *The equilibrium p in* (2) *is locally asymptotically stable if the pair* (c, α) *is above the parametric curve defined by*

$$c = \frac{\sin(\theta)}{\sin(k\theta) - \sin((k+1)\theta)}; \quad \alpha = \frac{\sin((k+1)\theta)}{\sin(k\theta)},$$

$\theta \in (0, \pi/(k+1))$.

We represent the border of the asymptotic stability region for $k = 0, 1, 2$, and 10 in Fig. 1.

Next we briefly discuss the stability properties of the equilibrium depending on the three parameters α, $c = -h'(p)$, and the delay k, in view of Proposition 3 and Fig. 1:

- *For fixed k and c, increasing α is stabilizing*: In particular, for $\alpha = 0$, Eq. (4) is locally asymptotically stable if $c < 1$. This condition implies the local asymptotic stability in (2) regardless the value of $k \geqslant 0$ and $\alpha \in [0, 1)$. Sometimes this delay-independent stability property is called *absolute stability*. On the other hand, when α approaches 1, Eq. (2) tends to be stable for all values of c and k.
- *For fixed α and c, adding delay is destabilizing*: In fact, the greatest stability region occurs for $k = 0$, when (2) has the form (3). In this case, the equilibrium loses its asymptotic stability for $\alpha = (c-1)/(c+1)$ (see the first curve from below in Fig. 1). As k tends to infinity, the equilibrium becomes unstable for all values outside the absolute stability region.
- Finally, *for fixed k and α, increasing c is destabilizing.*

Mertz and Myers [22] investigated Eq. (1), and suggested that "*increasing α decreases stability*". In view of the above remarks, we note that the introduction of the term $(1-\alpha)$ in the recruitment function leads to a different conclusion.

We also emphasize that the stability of p always depends on α. As stated in [22], if one considers Eq. (1) with $k = 1$, the equilibrium point is asymptotically stable when $0 < -f'(p) < 1$ regardless the value of α. But, in this case, the dependence of α is hidden in the equilibrium point itself, which is calculated as the solution of $(1-\alpha)x = f(x)$, and thus it should be denoted by p_α rather than p. We note that the border of the stability region for Eq. (2) in terms of α and c when $k = 1$ is defined by the curve $\alpha = (c-1)/c$, and it is shown in Fig. 1.

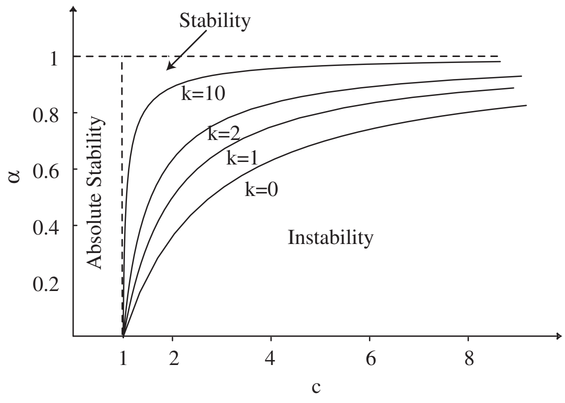

Fig. 1. Border of the local stability region in (2) for different values of k.

3. Periodic solutions

In general, the loss of the asymptotic stability of the equilibrium in a discrete model gives place to nontrivial cycles, that is, periodic solutions with minimal period $p \geqslant 2$. For example, for the usual population models governed by the first order difference equation (5), the equilibrium is locally asymptotically stable for $0 > h'(p) > -1$, and nontrivial 2-periodic points appear as $h'(p)$ crosses the critical value -1 in a supercritical period-doubling bifurcation, see [7,15]. Moreover, for the most common models, it was observed that the equilibrium p is globally stable if $|h'(p)| \leqslant 1$. Thus, in this case, the local asymptotic stability of p implies its global stability (see [19], and also [6,20] for recent results). The same property was conjectured to be true for (2) (see, e.g., [10,11,14,30]). An interesting result is that in general p is globally stable in the region of absolute stability.

Theorem 4 (*El-Morshedy and Liz [11, Theorem 1]*). *In the conditions of Theorem* 1, *if* $h(x) \neq 0$ *for all* $x \in (0, M]$ *and p is a global attractor for* (5), *then p is globally stable for Eq*. (2).

We will not discuss here the global stability; we would like to get some new information about what happens beyond the local stability region. Several authors have addressed this study, mainly from the numerical point of view (see, for instance, [4,22]). We show that some interesting facts may be derived analytically, at least for relatively simple models.

3.1. Periodic solutions of Eq. (3)

We start with the analysis of the one-parametric family of first order difference equations (3), which can be written in the form

$$x_{n+1} = h_\alpha(x_n),$$

with $h_\alpha(x) = \alpha x + (1-\alpha)h(x)$.

It is straightforward to check that p is asymptotically stable if either $c < 1$ or $c \geqslant 1$ and $\alpha > (c-1)/(c+1)$, where $c = -h'(p)$ (see Fig. 1). Moreover, when $\alpha = (c-1)/(c+1)$ we have $h'_\alpha(p) = -1$. In order to describe the period-doubling bifurcation, let us fix a value of $c = -h'(p) > 1$ and define

$$\lambda = \frac{\alpha_0 - \alpha}{1+c}, \quad \text{where } \alpha_0 = \frac{c-1}{c+1}.$$

Then the family of scalar maps $\{h_\alpha(x)\}$ is equivalent to

$$F(\lambda, x) = h_{\alpha_0}(x) + \lambda(1+c)(h(x) - x).$$

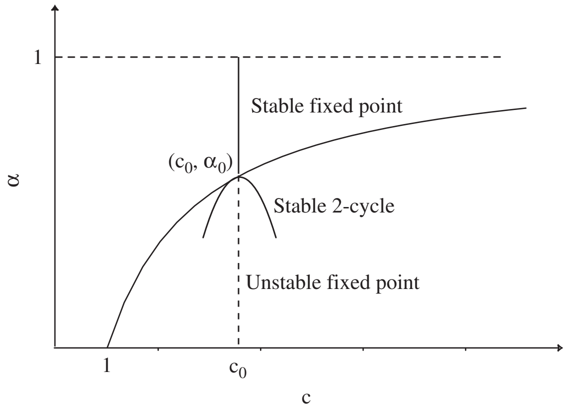

Fig. 2. Period-doubling bifurcation for Eq. (3).

We will assume that h is a C^3 function. It is easily seen that:

1. $F(0, x) = h_{\alpha_0}(x)$;
2. $F(\lambda, p) = p$;
3. $(\partial F/\partial x)(\lambda, p) = -(1+\lambda)$;
4. $h'_{\alpha_0}(p) = -1$;
5. $(h^2_{\alpha_0})'''(p) = 2(Sh_{\alpha_0})(p)$,

where, in the last item, $(Sh)(x) = (h'''(x)/h'(x)) - (3/2)(h''(x)/h'(x))^2$ is the Schwarzian derivative of h.

Note also that $\lambda < 0$ for $\alpha > \alpha_0$, and $\lambda > 0$ for $\alpha < \alpha_0$. Hence, a direct application of Theorem 3.21 in [15] provides the following result.

Theorem 5. *Assume that h is a C^3 function, and $c = -h'(p) > 1$. Denote $\alpha_0 = (c-1)/(c+1)$. Then, if $(Sh_{\alpha_0})(p) < 0$, there exists a neighborhood of (α_0, p) such that*

1. *for $\alpha < \alpha_0$ there exists a unique 2-cycle $\{a_\alpha, b_\alpha\}$ of minimal period 2 of Eq. (3). Moreover, this 2-cycle is asymptotically stable*;
2. *for $\alpha > \alpha_0$ Eq. (3) has no cycle of minimal period 2.*

Theorem 5 provides the existence of a local branch of attracting 2-cycles of Eq. (3) for each value of $c = -h'(p) > 1$ starting at the point $\alpha_0 = (c-1)/(c+1)$, where p losses its asymptotic stability. (see Fig. 2).

In fact, one can easily prove that, if h meets the conditions of Theorem 1 and is bounded in $(0, p)$, then h_α satisfies the same conditions, and there is at least one nontrivial 2-cycle $\{a_\alpha, b_\alpha\}$ of Eq. (3) for all values of α for which p is unstable. However, in general this 2-cycle will not be stable for all values of α.

Remark 1. Let us note that if h has an attracting or repelling 2-cycle $\{a, b\}$, then there is also a 2-cycle $\{a_\alpha, b_\alpha\}$ of (3) such that $a_\alpha \to a$ and $b_\alpha \to b$ as $\alpha \to 0$ (see, e.g, [26, Corollary 7.1]). Moreover, if h is a C^r map and $h'(a)h'(b) \notin \{-1, 0, 1\}$, then h_α is locally conjugate to h for α close to 0, due to the structural stability [26, Theorem 7.4].

3.2. Periodicity in Eq. (2)

Next we try to obtain some information about the periodic solutions of (2) from those of (3).

For a Ricker type nonlinearity $h(x) = \gamma x e^{-qx}$, $\gamma, q > 0$, the periods of the oscillations in Eq. (2) were discussed by Mertz and Myers [22] on the basis of the characteristic equation (6). In particular, they suggest the existence of periodic solutions of minimal period 2 in Eq. (2) when k is even due to the presence of roots of the characteristic equation on the negative real axis. Some references showing the empirical evidence of these 2-cycles are included in their discussion.

It turns out that it is very easy to prove the existence of 2-cycles based on our discussion for Eq. (3) in Section 3.1, since there is a correspondence between 2-cycles of (2) and (3) when k is even.

Proposition 6. *Eq.* (2) *has a cycle of minimal period* 2 *if and only if k is even and* (3) *has a cycle of minimal period* 2.

Proof. First we assume that k is even and $\{a_\alpha, b_\alpha\}$ is a 2-cycle of (3), that is, $h_\alpha(a_\alpha) = b_\alpha$, $h_\alpha(b_\alpha) = a_\alpha$, $a_\alpha \neq b_\alpha$. Then the initial vector $(a_\alpha, b_\alpha, a_\alpha, b_\alpha, \ldots, a_\alpha) \in \mathbb{R}^{k+1}$ defines a cycle of period 2 for Eq. (2). Conversely, if $\{a_\alpha, b_\alpha\}$ is 2-cycle of (2) with k even then $h_\alpha(a_\alpha) = b_\alpha$, $h_\alpha(b_\alpha) = a_\alpha$, $a_\alpha \neq b_\alpha$, and hence it is also a 2-cycle of (3).

Finally, we show that, for odd k, Eq. (2) does not have cycles of minimal period 2. Indeed, if Eq. (2) has a period 2 solution when k is odd, then this solution is a period 2 solution of the equation

$$x_{n+1} = \alpha x_n + (1-\alpha)h(x_{n-1}),$$

and hence

$$x_2 = \alpha x_1 + (1-\alpha)h(x_0); \quad x_3 = \alpha x_2 + (1-\alpha)h(x_1).$$

Since $x_2 = x_0$ and $x_3 = x_1$, the above equations yield

$$x_0 = \alpha x_1 + (1-\alpha)h(x_0), \tag{7}$$

$$x_1 = \alpha x_0 + (1-\alpha)h(x_1). \tag{8}$$

If we assume that $x_0 > x_1$, then (7) and (8) yield respectively that $x_0 < p$ and $x_1 > p$, a contradiction. A similar contradiction is obtained if we assume that $x_0 < x_1$. This finishes the proof. □

Remark 2. We showed that 2-cycles of (3) appear in a period-doubling bifurcation as α crosses the curve $c = (1+\alpha)/(1-\alpha)$, $c > 1$. Note that $\lambda = -1$ is a root of the characteristic equation (6) if and only if k is even and $c = (1+\alpha)/(1-\alpha)$.

The first part of the proof in Proposition 6 can be derived from a more general result.

Proposition 7. *If* $r|k$, *then* (2) *has an r-periodic solution if and only if* (3) *has an r-periodic solution.*

Proof. It is a straightforward consequence of Theorem 7 in [9]. □

Here, as usual, $r|k$ means that r divides k.

As mentioned in [22], Ricker was the first to notice long-period endogenous population fluctuations in his simulations of an age-structured fish stock [25]. He estimated that the natural period of these oscillations (called *Ricker oscillations* in [22]) is twice the median time from oviposition to oviposition.

For $\alpha = 0$, the cycles of period $2(k+1)$ in (4) are linked in [22] to the fact that all solutions of (6) are $(k+1)$ roots of the unit for $c = 1$. Using the results of an der Heiden and Liang [1], it is easy to derive the existence of these oscillations from the existence of 2-periodic points of h; as noticed above, such periodic points do exist for $c > 1$ under rather general assumptions on h (in particular, for the Ricker function).

Indeed, from Theorem 1.2 in [1], it follows that if (5) has cycles with minimal periods 1 and 2, then (4) has cycles with minimal periods 1 and r, where $r \in S_{k+1}(2)$, and

$$S_{k+1}(2) = \{2r \,:\, p \in \mathbb{N},\ r|(k+1) \quad \text{and} \quad ((k+1)/r, 2) \text{ is coprime}\}.$$

Recall that the pair of natural numbers (m, n) is called *coprime* if 1 is the only common divisor of m and n.

In particular, for $c > 1$, Eq. (4) has always cycles of period $2(k+1)$, while it has 2-cycles if and only if k is even. Moreover, if $k > 1$ and $k+1$ is prime, then $S_{k+1}(2) = \{2, 2(k+1)\}$.

For $\alpha > 0$, the existence of periodic solutions of (2) with period approximately twice the mean of the age distribution as it affects recruitment was suggested by Botsford [4].

The following result shows that under some additional conditions on h, Eq. (2) has cycles of period $2(k+1)$ for α small enough. Furthermore, the dynamics of (2) is remarkably simple. By a *hyperbolic attracting* (respectively, *hyperbolic* *repelling*) point of a map f we mean a fixed point x such that f is differentiable at x and $|f'(x)|<1$ (respectively, $|f'(x)|>1$). When we say that an r-cycle of f is hyperbolic attracting or repelling, then we mean that the points of the cycle are, respectively, hyperbolic attracting or repelling fixed points of f^r.

Theorem 8. *Assume that h is C^1 function having a hyperbolic repelling fixed point p, a hyperbolic attracting 2-cycle $\{q_1, q_2\}$ and no other periodic points. Also assume that either h is unbounded at $(0, p]$ and $h^2([p, q_2])$ is well-defined, or h is bounded at $(0, p]$ and both $h^4(l)$ and $h^4(s)$ are well defined (here $l = \lim\inf_{x\to 0} h(x)$, $s = \lim\sup_{x\to 0} h(x)$). If $\alpha \geqslant 0$ is small enough, then Eq. (2) has exactly as many cycles (with the same periods and local dynamics) as Eq. (4) (in particular, some attracting $2(k+1)$-cycle) and these cycles attract all solutions of (2).*

For convenience of the reader, we include the rather tedious proof of Theorem 8 in Appendix. Examples can be provided showing that the awkward "well-defined" hypothesis is essential.

Remark 3. Conditions of Theorem 8 hold, for example, if h is strictly decreasing, has a unique fixed point p such that $|h'(p)|>1$, and has negative Schwarzian derivative everywhere (see, e.g., [21]).

4. An example

It would be interesting to understand the variation in the dynamics of (2) as the parameter α ranges between 0 and 1. When $k=0$, this task is a matter of bifurcation analysis in a family of one-dimensional discrete dynamical systems. For general k, Eq. (2) generates a dynamical system in $\mathbb{R}^{k+1}$, which is much more difficult to manage. See, for example, the analysis made in [13] for the 2-dimensional case ($k=1$).

In this section we consider an example for $k=2$ with the hope that our conclusions may be useful in more general situations.

Maybe the simplest model for (2) corresponds to a strictly decreasing function h. See some examples in [10,14]. A typical function is

$$h(x) = \gamma e^{-qx}, \quad \gamma, q > 0, \tag{9}$$

which appears in some models in hematopoiesis. We notice that also the delayed Ricker-type difference equation

$$y_{n+1} = y_n e^{-qy_{n-k}}, \quad q > 0,$$

(see, e.g., [19]) can be written in the form

$$x_{n+1} - x_n = qe^{-x_{n-k}},$$

after the change of variables $x_n = -\log(y_n)$. This is another motivation to consider this form of h. Since we are interested in taking α as the bifurcation parameter, we will fix $q=1$ in (9), and choose a value of γ in such a way that $c=-h'(p)>1$ and the calculus of the fixed point is easy. Taking $\gamma = 2e^2$, we have $h(x)=2e^{2-x}$, so our equation is

$$x_{n+1} = \alpha x_n + (1-\alpha)2e^{2-x_{n-2}}, \tag{10}$$

whose unique positive equilibrium is $p=2$.

4.1. The one-dimensional model

First we analyze the one parametric family of maps $\{h_\alpha\}$, or, equivalently, equation

$$x_{n+1} = \alpha x_n + 2(1-\alpha)e^{2-x_n} = h_\alpha(x_n). \tag{11}$$

Function h_α has at most one critical point at $m_\alpha = 2 - \log(\alpha/(2-2\alpha))$, which is a local minimum. In fact, h_α is decreasing for $x \in (0, m_\alpha)$, and h_α is increasing for $x > m_\alpha$. In particular, h_α is strictly increasing if $m_\alpha < 0$, that is, for $\alpha > \alpha_* = 2e^2/(1+2e^2) \approx 0.936621$. Thus, the equilibrium is a global attractor of (11) for $\alpha \in (\alpha_*, 1)$ and the convergence is monotonic. Next, if $m_\alpha \leqslant 2$ (i.e., $\alpha \geqslant 2/3$), then $p=2$ is globally stable with eventually monotone convergence.

By direct computation, one can check that $(Sh_\alpha)(x) < 0$ for all $x \neq m_\alpha$. This property plays a key role in our next result, which shows that the local situation described in Theorem 5 holds in fact globally.

Theorem 9. *Eq.* (11) *has a globally stable equilibrium if* $\alpha \in [\frac{1}{3}, 1)$, *and a unique* 2-*cycle if* $\alpha \in [0, \frac{1}{3})$.

Proof. In view of our discussion above, we can assume that $\alpha < \frac{2}{3}$. We distinguish two cases.

(a) We first assume that $h_\alpha(0) > m_\alpha$ (which happens for $\alpha > \alpha_5 = 2e^2/(2e^2 + 2^{2e^2}) \approx 5.6437 \times 10^{-6}$).

In this case, $g_\alpha = h_\alpha \circ h_\alpha$ has exactly three critical points x_1, m_α, x_2, where $h_\alpha(x_1) = h_\alpha(x_2) = m_\alpha$. Moreover, $x_1 < p < m_\alpha < x_2$, g_α is strictly increasing on $(x_1, m_\alpha) \cup (x_2, \infty)$, and g_α is strictly decreasing on $(0, x_1) \cup (m_\alpha, x_2)$.

If $\alpha \geqslant \frac{1}{3}$ then p is asymptotically stable, and hence [27, Theorem 2.1, p. 47] implies that there exists a neighborhood $U = (p - \varepsilon, p + \varepsilon)$ of p such that $g_\alpha(x) > x$ for $x \in (p - \varepsilon, p)$. On the other hand, in view of [29, Theorem 9.6], p is globally stable if and only if g_α has no fixed points different from p. Assume, for the sake of contradiction, that g_α has a fixed point different from p. Then, there must exist $c \in [x_1, p)$ such that $g_\alpha(c) = c$ (we may choose the point c closest to p satisfying this condition). Since $d = h_\alpha(c) \in (p, m_\alpha]$, it follows that c, p, d are three consecutive fixed points of g_α, and g_α does not have any critical point on (c, d). Since $(Sh_\alpha)(x) < 0$ for all $x \neq m_\alpha$, [28, Lemma 2.6] implies that $g'_\alpha(p) > 1$, a contradiction.

If $\alpha < \frac{1}{3}$ then p is unstable and hence there is at least a nontrivial 2-cycle of h_α. Since $|h'_\alpha(p)| > 1$, if we assume that this 2-cycle is not unique, then there must exist two points $c_1, c_2 \in (x_1, p)$ such that $g'_\alpha(c_1) = g'_\alpha(c_2) = 1$. This contradicts the result in [28, Proposition 2.4].

(b) If $h_\alpha(0) \leqslant m_\alpha$ then h_α is strictly decreasing in the interval $I_0 = [0, h_\alpha(0)]$, which is invariant and attracting for (11). Since $(Sh_\alpha)(x) < 0$ for all $x \in I_0$, and $|h'_\alpha(p)| > 1$, the arguments used in the previous case prove that there is a unique 2-cycle of (11). □

Our next result shows that the unique 2-cycle of (11) attracts all solutions except those starting from 2 or one of its preimages.

Theorem 10. *Let* $\alpha \in [0, \frac{1}{3})$. *Then the unique* 2-*cycle of* (11) *is globally attracting, that is, every orbit* $(h^n_\alpha(x))$ *either eventually equals* 2 (*the unique fixed point of* h_α) *or is attracted by the* 2-*cycle.*

Proof. Let $m = m_\alpha = 2 - \log(\alpha/(2 - 2\alpha))$ be the point at which h_α attains its minimum value and put $a = h_\alpha(m)$, $b = h_\alpha(a)$. If $b \leqslant m$, then the interval $[a, m]$ is invariant by h_α and every orbit $(h^n_\alpha(x))$ eventually falls into $[a, m]$. Since h_α is decreasing on $[a, m]$ every orbit must be attracted either by a 2-cycle or a fixed point. Since 2 is repelling for h_α and h_α has exactly one 2-cycle, the statement of the theorem follows. Thus in what follows we can assume, without loss of generality, that $b > m$.

To prove the theorem it is sufficient to show $|h'_\alpha(a)h'_\alpha(b)| < 1$. Indeed, assume that the inequality holds. Since h'_α is strictly increasing, we get $|h'_\alpha(x)h'_\alpha(y)| < 1$ whenever $x \in [a, m]$, $y \in [m, b]$. Then $c = h_\alpha(b) < m$, for otherwise there is $c' \in [m, b]$ such that $h_\alpha(c') = m$ and therefore

$$\begin{aligned}|b - a| &= |h_\alpha(a) - h_\alpha(m)| = |h'_\alpha(x)||a - m| = |h'_\alpha(x)||h_\alpha(m) - h_\alpha(c')| \\ &= |h'_\alpha(x)||h'_\alpha(y)||m - c'| < |h'_\alpha(x)||h'_\alpha(y)||b - a|\end{aligned}$$

for some appropriate $x \in [a, m]$, $y \in [m, b]$, a contradiction. Next, note that

$$\begin{aligned}|b - h^2_\alpha(b)| &= |h^2_\alpha(m) - h^2_\alpha(b)| = |h_\alpha(a) - h_\alpha(c)| = |h'_\alpha(x)||a - c| \\ &= |h'_\alpha(x)||h_\alpha(m) - h_\alpha(b)| = |h'_\alpha(x)||h'_\alpha(y)||m - b|\end{aligned}$$

for some appropriate $x \in [a, m]$, $y \in [m, b]$. Hence $|b - h^2_\alpha(b)| < |m - b|$, which together with $c < m$ implies that h^2_α maps monotonically the interval $[m, b]$ into itself. Since the intervals $[m, b]$ and $h_\alpha([m, b]) = [a, c]$ are pairwise disjoint, each of them contains one of the points of the 2-cycle. Clearly if an orbit does not end at the fixed point, then it eventually falls into $[a, c] \cup [m, b]$ and is attracted by the 2-cycle, as we desired to prove.

Thus it only rests to prove that $|h'_\alpha(a)h'_\alpha(b)| < 1$. A straightforward calculation shows that

$$h'_\alpha(a) = \alpha - \alpha^\alpha(2 - 2\alpha)^{1-\alpha}e^{2-3\alpha}.$$

Table 2
Global bifurcation analysis of Eq. (11)

Values of α	Behaviour
$1>\alpha>\alpha_*\approx 0.936621$	Globally stable equilibrium, monotone convergence
$\alpha_*\geqslant\alpha\geqslant 2/3$	Globally stable equilibrium, eventually monotone convergence
$2/3>\alpha\geqslant 1/3$	Globally stable equilibrium, oscillatory convergence
$1/3>\alpha\geqslant 0$	Unstable equilibrium, globally attracting 2-cycle

Using the obvious inequality $\alpha<\alpha^\alpha(1-\alpha)^{1-\alpha}$ whenever $\alpha<\frac{1}{3}$ we get in particular

$$|h'_\alpha(a)|=-h'_\alpha(a)<\alpha^\alpha(2-2\alpha)^{1-\alpha}e^{2-3\alpha}.$$

Further, recall that $m<b$, that is, $h'_\alpha(b)>0$. Since $h'_\alpha(x)<\alpha$ for every $x\in\mathbb{R}$ we have $|h'_\alpha(b)|<\alpha$. Thus, in order to finish the proof, it suffices to demonstrate

$$\alpha^{\alpha+1}(2-2\alpha)^{1-\alpha}e^{2-3\alpha}<1$$

or, equivalently,

$$\begin{aligned} g(\alpha) :=\ & \log(\alpha^{\alpha+1}(2-2\alpha)^{1-\alpha}e^{2-3\alpha}) \\ &=(\alpha+1)\log\alpha+(1-\alpha)\log(2-2\alpha)+2-3\alpha \\ &<0 \end{aligned}$$

for every $\alpha\in(0,\frac{1}{3})$.

We have

$$g''(\alpha)=\frac{2\alpha-1}{\alpha^2(1-\alpha)},$$

so g' is strictly decreasing in $(0,\frac{1}{3})$. Since

$$-0.2<g'(0.2)<0<g'(0.19)<0.2,$$

we see that the only zero z of g' in $(0,\frac{1}{3})$, which is the point at which g attains its maximum, satisfies

$$g(z)-g(0.2)<0.2\cdot 0.01=0.002.$$

This, together with $g(0.2)<-0.15$, gives $g(z)<-0.14$. We are done. □

A summary of our analysis for Eq. (11) is given in Table 2.

4.2. The three-dimensional model

We begin with the analysis of the characteristic equation (6), which in this case takes the form

$$\lambda^3-\alpha\lambda^2=-2(1-\alpha). \tag{12}$$

We analyze the position of the roots of (12) in the complex plane as α ranges from $\alpha=1$ to 0. For $\alpha=1$, the roots are $\lambda_1(1)=0$, $\lambda_2(1)=0$, $\lambda_3(1)=1$. As α varies from $\alpha=1$ to 0, $\lambda_1(\alpha)$ always remains in the real axis, and its value decreases from $\lambda_1(1)=0$ to $\lambda_1(0)=-2^{1/3}$.

The behavior of $\lambda_2(\alpha)$ and $\lambda_3(\alpha)$ is different. For α close to 1, $\lambda_2(\alpha)$ and $\lambda_3(\alpha)$ remain in the real axis, and they approximate to each other until (12) has a positive root with multiplicity 2. This value of α can be computed directly by imposing such a condition. It turns out that α is the real solution of equation $4\alpha^3-54(1-\alpha)=0$, with an approximate value of $\alpha_1=0.938725$. Thus, for $\alpha\in(\alpha_1,1)$, the equilibrium is asymptotically stable and the convergence is monotone at least locally. For $\alpha>\alpha_1$, Eq. (12) has a pair of complex conjugate roots, whose moduli are increasing as α decreases.

Table 3
Local stability analysis of Eq. (10)

Values of α	Roots of (12)	Behaviour
$\alpha_1 < \alpha < 1$	$-1 < \lambda_1 < 0, 0 < \lambda_2 < \lambda_3 < 1$	Equilibrium with local monotonic damping
$\alpha = \alpha_1 \approx 0.938725$	$-1 < \lambda_1 < 0, 0 < \lambda_2 = \lambda_3 < 1$	Border of monotonic and oscillatory damping
$\alpha_2 < \alpha < \alpha_1$	$-1 < \lambda_1 < 0, \|\lambda_2\| = \|\lambda_3\| < 1$	Equilibrium with local oscillatory damping
$\alpha = \alpha_2 \approx 0.633975$	$-1 < \lambda_1 < 0, \|\lambda_2\| = \|\lambda_3\| = 1$	Hopf bifurcation
$\alpha_3 < \alpha < \alpha_2$	$-1 < \lambda_1 < 0, \|\lambda_2\| = \|\lambda_3\| > 1$	Unstable equilibrium, no 2-cycles
$\alpha = \alpha_3 = 1/3$	$\lambda_1 = -1, \|\lambda_2\| = \|\lambda_3\| > 1$	Doubling-period bifurcation for h_α
$0 < \alpha < \alpha_3$	$\lambda_1 < -1, \|\lambda_2\| = \|\lambda_3\| > 1$	Unstable equilibrium, 2-cycles

Using Proposition 3, we find that $\lambda_2(\alpha)$ and $\lambda_3(\alpha)$ reach the unit circle in the complex plane for $\alpha=\alpha_2=\sin(3\theta)/\sin(2\theta)$, where $\theta \in (0, \pi/3)$ solves $\cos(\theta) = (1 + 3^{1/3})/4$. Thus $\alpha_2 \approx 0.633975$. Between α_1 and α_2, the equilibrium $p = 2$ is asymptotically stable with oscillatory damping, and it becomes unstable for $\alpha = \alpha_2$. Moreover, since $\lambda_2(\alpha)$ and $\lambda_3(\alpha)$ cross the unit circle in the increasing direction of their moduli, a Hopf bifurcation (also called Naimark–Sacker bifurcation, see, e.g., [31, Section 3.2C]) occurs, and an invariant curve is born. Next, for $\alpha_3 = \frac{1}{3}$, $\lambda_1(\frac{1}{3}) = -1$ and this gives place to the appearance of 2-cycles of the linearized equation of (10). From Theorem 5, we know that the family $h_\alpha(x) = \alpha x + (1 - \alpha)2e^{2-x}$ experiences a period-doubling bifurcation at this value of α. We show below that Eq. (10) actually possesses solutions of minimal period 2 if and only if $\alpha \in [0, \frac{1}{3})$.

For convenience of the reader, we summarize our analysis in Table 3. We emphasize that this local analysis does not depends on the particular form of h. The unique important point is that $c = h'(p) > 1$.

Next, the particular form of $h(x)$ in our example allows us to get more information on the dynamics of (10).

We have already proved in Theorem 9 that there is a unique 2-cycle $\{a, b\}$ of h which attracts all solutions of

$$x_{n+1} = h(x_n) = 2e^{2-x_n}, \tag{13}$$

with initial condition $x_0 \neq 2$. The approximate values of a, b are

$$a \approx 5.644 \times 10^{-5}; \quad b \approx 14.778. \tag{14}$$

This means that, for $\alpha = 0$, equation

$$x_{n+1} = h(x_{n-2}) = 2e^{2-x_{n-2}} \tag{15}$$

has only periodic solutions with minimal periods 1, 2 and 6. Moreover, it has exactly one periodic solution corresponding to each one of those periods. The dynamics of the solutions of (15) is easy to describe.

Theorem 11. *All solutions of* (15) *with initial conditions* $x_0, x_1, x_2 \neq 2$ *converge to the* 2-*cycle if* $(x_0 - 2)(x_1 - 2) < 0$ *and* $(x_1 - 2)(x_2 - 2) < 0$, *and converge to the* 6-*cycle otherwise*.

Proof. Let $\{a, b\}$ be the globally attracting 2-cycle of h. Then, Eq. (15) has a 2-cycle $\{a, b, a, b, a, b, \ldots\}$ and a 6-cycle $\{a, a, a, b, b, b, \ldots\}$. Since $g = h \circ h$ is increasing, it is easy to check that $g^n(x)$ converges monotonically to a if $x < 2$, and it converges monotonically to b if $x > 2$. Thus,

$$\lim_{k\to\infty} h^{2k}(x) = \begin{cases} a & \text{if } x < 2; \\ b & \text{if } x > 2; \end{cases} \qquad \lim_{k\to\infty} h^{2k+1}(x) = \begin{cases} b & \text{if } x < 2; \\ a & \text{if } x > 2. \end{cases} \tag{16}$$

The result follows from (16) and the observation that, given an initial vector (x_0, x_1, x_2), it is clear that $x_{3k+i} = h^k(x_i)$ for $i = 0, 1, 2$. □

For $\alpha > 0$, we can use Proposition 6 and Theorem 8. In particular, this last result shows that the situation described in Theorem 11 still holds for $\alpha > 0$ small enough.

Corollary 12. *If* $\alpha \geqslant 0$ *is small enough*, *then Eq.* (10) *has exactly one repelling equilibrium and two attractors*: *a* 2-*cycle and a* 6-*cycle. Moreover*, *they attract all solutions of* (10).

From Proposition 6 and Remark 1, we get the following consequence of Theorem 9:

Corollary 13. *Eq.* (10) *does not have any solution of minimal period* 2 *if* $\alpha \in [\frac{1}{3}, 1)$, *and it has a unique* 2-*cycle* $\{a_\alpha, b_\alpha\}$ *if* $\alpha \in [0, \frac{1}{3})$. *Furthermore*, $a_\alpha \to a$, $b_\alpha \to b$ *as* $\alpha \to 0$, *where* $\{a, b\}$ *is the* 2-*cycle of* (15) *given in* (14).

Remark 4. Numerical simulations indicate that the dynamics of (10) described by Corollary 12 holds for $\alpha < \alpha_4 \approx 0.067$. Also, between α_4 and $\alpha_3 = \frac{1}{3}$, it seems that there is an attracting invariant curve and an attracting 2-cycle. For α between α_3 and $\alpha_2 \approx 0.633975$, there is a globally attracting invariant curve. Finally, for $\alpha \in (\alpha_2, 1)$, the equilibrium seems to be globally attracting. We note that the observed invariant curve might hide a more complicated attractor for some values of the parameter. For example, for a range of values between $\alpha = 0.34$ and 0.35, it is observed a period 7 sink. However, there is probably a complicated invisible attractor associated to a resonance in the invariant curve. See [2] for further discussion.

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Appendix

In this appendix we provide a detailed proof of Theorem 8. We begin by stating two well-known results that will be useful in the sequel. See, e.g., [3, pp. 121–122 and Lemma 3, p. 26] for the proofs.

Proposition 14. *Let* I *be a compact interval and let* $f : I \to I$ *be a continuous map. Suppose that all periodic points of* f *are fixed points. Then* $f(x) > x$ (*respectively*, $f(x) < x$) *for a point* $x \in I$ *implies* $f^n(x) > x$ (*respectively*, $f^n(x) < x$) *for every positive integer* n.

Remark 5. In fact Proposition 14 implies that every orbit $(f^n(x))$ converges to a fixed point of f. The reason is the following. Say, for instance, that the orbit is neither monotone nor eventually a fixed point and $f(x) > x$. Let n_1 be the first number n such that $f^{n+1}(x) < f^n(x)$. Next, let n_2 be the first number n larger than n_1 such that $f^{n+1}(x) > f^n(x)$, let n_3 be the first number n larger than n_2 such that $f^{n+1}(x) < f^n(x)$, and so on. After applying Proposition 14 to $f^{n_1-1}(x)$, $f^{n_2-1}(x)$, $f^{n_3-1}(x), \ldots$ we get

$$\begin{aligned} x &< f(x) < \cdots < f^{n_1-1}(x) < f^{n_2}(x) < f^{n_2+1}(x) < \cdots \\ &< f^{n_3-1}(x) < f^{n_4}(x) < \cdots < \cdots < f^{n_4-1}(x) < \cdots \\ &< f^{n_3+1}(x) < f^{n_3}(x) < f^{n_2-1}(x) < \cdots < f^{n_1+1}(x) < f^{n_1}(x). \end{aligned} \tag{17}$$

Thus the orbit of x accumulates around two points $a \leqslant b$ satisfying $f(a) = b$ and $f(b) = a$. Since f has no 2-cycles, we conclude that $a = b$ is a fixed point and that the sequence $(f^n(x))$ converges to a.

Furthermore, notice that a consequence of the ordering described in (17) is that if a neighborhood U of the set of fixed points of f is given, then there is a number u with the property that if $x \in I$, then $f^n(x) \in U$ for some $n \leqslant u$.

Proposition 15. *Let* I *be an interval and let* $f : I \to \mathrm{Cl}(I)$ *be a continuous map. Let* J *and* K *be compact subintervals of* I *intersecting at most at one point. Suppose that there is some positive integer* r *such that* $h^i(J \cup K) \subset I$ *for each* $0 \leqslant i < r$ *and* $h^r(J) \cap h^r(K) \supset J \cup K$. *Then* h *has cycles of arbitrarily large minimal periods*.

In what follows we assume that h has a hyperbolic repelling fixed point p, a hyperbolic attracting 2-cycle $\{q_1, q_2\}$ with, say, $q_1 < p < q_2$, and no other cycles. We also assume that either h is unbounded at $(0, q_1]$ and $h^2([p, q_2])$ is well-defined, or h is bounded at $(0, q_1]$ and both $h^4(l)$ and $h^4(s)$ are well-defined (with $l = \liminf_{x\to 0} h(x)$, $s = \limsup_{x\to 0} h(x)$). Presently, no global differentiability assumptions are made on h.

Lemma 16. *We have $h^2(x) > x$ (respectively, $h^2(x) < x$) for every $x \in (0, q_1)$ (respectively, for every $x \in (q_2, \infty)$ such that $h^2(x)$ is well-defined).*

Proof. The statement $h^2(x) > x$ for every $x \in (0, q_1)$ follows from the fact that $h^2(x) \neq x$ for every such x and $h^2(q_1 - \varepsilon) > q_1 - \varepsilon$ whenever $\varepsilon > 0$ is small because the cycle $\{q_1, q_2\}$ is attracting. The other statement can be proved similarly.

Let $e = \max h([q_1, p])$. Then two possibility arise that must be separately considered:

(a) $0 \notin h([q_1, e])$. Then we define $a = \min h([q_1, e])$, $b = \max h([a, e])$.
(b) $0 \in h([q_1, e])$. Then we define $d = \sup h((0, p])$.

Lemma 17. *We have*

(i) *If possibility* (a) *holds, then $h([a, p) \cup (p, b]) = [a, p) \cup (p, b]$.*
(ii) *If possibility* (b) *holds, then $d < \infty$, $\{l\} \cup h((0, p) \cup (p, d]) \subset [0, p) \cup (p, d]$, and there is $c > 0$ such that $h^n((0, q_1]) \subset [c, d]$ for every $n \geqslant 1$.*

Proof. We prove (i). If $b = e$, then $h([a, b]) = [a, b]$ trivially. If $e < b$, then we have again $h([a, b]) = [a, b]$ unless there is $y \in [b, e]$ such that $h(y) < a$. But then we get that $h^2(x) < x$ for the point $x \in [a, q_1)$ such that $h(x) = y$, which contradicts Lemma 16.

Thus, to conclude the proof of (i) we must show that if $a \leqslant x < p$ (respectively, $p < x \leqslant b$), then $h(x) > p$ (respectively, $h(x) < p$). Assume that $h(x) = p$ for some $x < p$ and let y be the closest point to the left of p with this property. Note that there is such a point y because p is repelling. If $y < q_1$, then $h^2([y, p]) \supset [a, p]$ and we arrive at a contradiction with Proposition 15. If $y > q_1$ and we want to avoid contradicting Proposition 15 again, then we must have $h([y, p]) \subset [y, q_2)$ and $h(K) \subset K$ for the interval $K = [y, p] \cup h([y, p])$. This is impossible because while p is repelling, it should attract the orbits of all points from K due to Remark 5 (p is the only periodic point of the restriction of h to K). One can prove that $h(x) \neq p$ for every $x \in (p, d]$ in similar fashion. We just emphasize that if $h(y) = p$ for some $q_2 < y \leqslant b$, then $a \in h^3([p, y])$ so $h^4([p, y]) \supset [p, b]$ contradicting Proposition 15.

We prove (ii). First we show that $d < \infty$. In fact if $d = \infty$, then we have $0 \notin h([p, q_2])$ by hypothesis. Thus, since possibility (b) holds, there is $y \in (q_2, e]$ such that $h(y) = 0$. Now we use that h is unbounded at $(0, q_1)$ to find $x \in (0, q_1)$ such that $h(x) = y$ and hence $h^2(x) = 0 < x$, contradicting Lemma 16.

Trivially we have $[0, d) \subset h((0, d]) \subset [0, d]$. Moreover, arguing similarly as in (i) we get that $h((0, p) \cup (p, d)) \subset [0, p) \cup (p, d]$, hence $l \geqslant p$. Suppose $l = p$. If $s = p$, then h can be extended to a continuous map $\tilde{h} : [0, \infty) \to [0, \infty)$ by writing $\tilde{h}(0) = p$. Hence $\tilde{h}^2([0, p])$ covers $[0, p]$. Since $\tilde{h}$ has finitely many cycles, this contradicts Proposition 15.

Say $s - p = s - l = \delta > 0$. We use that p is repelling to find an iterate $f^r([p, p + \delta/2])$ of $[p, p + \delta/2]$ containing 0. In particular, if $\varepsilon > 0$ is small enough, then $h^r([p + \varepsilon, p + \delta/2])$ covers $[0, q_1]$. Find small disjoint intervals J and K close to 0 so that $h(J) = h(K) = [p + \varepsilon, p + \delta/2]$. Then $h^{r+1}(J) \cap h^{r+1}(K) \supset J \cup K$, contradicting Proposition 15.

Now we prove that $h(d) < p$. Suppose $h(d) = p$. By Proposition 15 we can assume $l < d = s$. Then $\liminf_{x \to 0} h^2(x) < \limsup_{x \to 0} h^2(x) = p$ and we argue to a contradiction as in the previous paragraph.

Only the proof of the last statement in (ii) is pending. We first show that

$$\text{if } x \in (0, q_1), \quad \text{then } h^n(x) \text{ is well-defined and } h^n(x) > x \text{ for every } n \geqslant 1. \tag{18}$$

We prove (18) inductively. Due to the inclusion $h((0, p) \cup (p, d]) \subset [0, p) \cup (p, d]$ it suffices to show that if $x \in (0, q_1)$, then $h^{2m}(x)$ is well-defined and satisfies $h^{2m}(x) > x$ for every $m \geqslant 1$. The statement follows from Lemma 16 for $m = 1$. Assume now that $h^{2r}(x) > x$ for some r. Then $h^{2(r+1)}(x)$ is well-defined. Moreover, if $h^{2(r+1)}(x) \leqslant x$, then we use that $h^{2(r+1)}(q_1 - \varepsilon) > q_1 - \varepsilon$ for a very small $\varepsilon > 0$ (recall that the cycle $\{q_1, q_2\}$ is attracting) to find a point $y \in [x, q_1 - \varepsilon)$ satisfying $h^{2(r+1)}(y) = y$. This is impossible, so (18) is proved.

Indeed, if $x \in (0, q_1]$, then the behavior of the orbit $(h^n(x))$ is further restricted by the fact that the map $g : [0, d] \to [0, d]$ defined by $g(y) = h(x)$ if $y \leqslant x$ and $g(y) = h(y)$ otherwise has p, q_1, and q_2 as its only periodic points. More precisely, notice that both orbits $(h^n(x))$ and $(g^n(x))$ coincide because of (18). Then, after applying Proposition 14 and Remark 5 to g^2, we see that $h^{2m}(x) \geqslant \min\{h^2(x), h^4(x)\}$ for every $m \geqslant 1$. Therefore $h^n(x) \geqslant \min\{h^2(x), h^4(x)\}$ for every $n \geqslant 1$.

Thus, in order to finish the proof it only rests to show that 0 belongs to the closure neither of $h^2((0, q_1])$ nor of $h^4((0, q_1])$. This easily follows from the facts that neither $h^2((0, q_1])$ nor $h^4((0, q_1])$ contain 0, and $h^n(l), h^n(s) \neq 0$ for every $n \leqslant 3$ by the hypothesis on h. $\square$

Lemma 18. *Fix small neighborhoods U_p, V_{q_1}, and W_{q_2} of the points p, q_1, and q_2. Then there is a number u such that if possibility* (a) *holds* (*respectively*, *possibility* (b) *holds*) *and $x \in [a, b]\backslash U_p$* (*respectively*, *$x \in (0, d]\backslash U_p$ and the orbit of x is well-defined*), *then $h^n(x) \in V_{q_1} \cup W_{q_2}$ for some $n \leqslant u$.*

Proof. We assume that possibility (b) holds; in the other case the argument is much simpler.

Let $g : [0, d] \to [0, d]$ be the continuous map defined by $g(x) = h(c)$ if $x \leqslant c$ and $g(x) = h(x)$ otherwise, with $c > 0$ the number defined in Lemma 17(ii). We emphasize that the property $h^n((0, q_1]) \subset [c, d]$ for every $n \geqslant 1$ implies that the only periodic points of g are the fixed point p and the 2-cycle $\{q_1, q_2\}$.

Since $\{l\} \cup h((0, p) \cup (p, d]) \subset [0, p) \cup (p, d]$ (Lemma 17(ii)) and p is repelling, there is a neighborhood O of p such that if $x \notin O$, then its orbit never visits O. Note that if $x \in O\backslash U_p$, then there is a number u_1 such that $h^n(x)$ escapes from O for some $n \leqslant u_1$. Also, recall that there is a number u_2 such that if $x \in [0, d]$, then $g^n(x) \in U_p \cup V_{q_1} \cup W_{q_2}$ for some $n \leqslant u_2$ (Remark 5).

Assume that $x \in (0, d]\backslash U_p$ has a well-defined orbit under h. Then $h^{n_1}(x) \in (0, d]\backslash O$ for some minimal number $n_1 \leqslant u_1$. If $h^{n+n_1}(x) \notin [0, c)$ for every $n \leqslant u_2$, then we have $h^{n_2+n_1}(x) = g^{n_2+n_1}(x) \in V_{q_1} \cup W_{q_2}$ for some $n_2 \leqslant u_2$. If $h^{i+n_1}(x) \in [0, c)$ for some $i \leqslant u_2$, then $h^{n+i+n_1+1}(x) = g^n(h^{i+n_1+1}(x))$ for every n and there is a number $n'_2 \leqslant u_2$ such that $h^{n'_2+i+n_1+1}(x) \in V_{q_1} \cup W_{q_2}$. We have shown that $u = u_1 + 2u_2 + 1$ does the job. $\square$

After recalling all the information we need about h we start dealing with Eq. (2). Observe that in the limit case $\alpha = 0$ the solutions of Eq. (4) can only accumulate at p, q_1, and q_2 due to Lemma 18. If $0 < \alpha < 1$, then it is still possible to get some useful information about the limiting behavior of the solutions of (2).

Lemma 19. *Let $0 < \alpha < 1$. Then Eq.* (2) *is permanent. Moreover*, *every solution of* (2) *eventually falls into* $[a, b]$ (*if possibility* (a) *holds*), *or into* $(0, d]$ (*if possibility* (b) *holds*).

Proof. First we show that (2) is permanent. If $h((0, p])$ is bounded, then (2) is permanent by Theorem 1. Hence we can assume that $h((0, p])$ is unbounded, when Lemma 17 implies that possibility (a) holds and $h([a, b]) = [a, b]$. Moreover, if $0 < x < a$, then the interval $h([x, a]) \cup [x, b]$ is invariant by h, that is, it is mapped by h into itself (Lemma 16). Since h is unbounded in $(0, p]$ we can construct an increasing sequence (J_m) of invariant compact intervals covering $(0, \infty)$, which immediately implies that (2) is permanent (because the initial vector of a given solution must be contained in some J_m and hence, by the nature of (2), the whole solution is contained in J_m).

Let (x_n) be a solution of (2) and write $L = \liminf_{n\to\infty} x_n$, $S = \limsup_{n\to\infty} x_n$. Find subsequences (n_j) and (n'_j) and numbers $L \leqslant L_0, L_k, S_0, S_k \leqslant S$ satisfying

$$\lim_{j\to\infty} x_{n_j+1} = L; \quad \lim_{j\to\infty} x_{n_j} = L_0; \quad \lim_{j\to\infty} x_{n_j-k} = L_k,$$

$$\lim_{j\to\infty} x_{n'_j+1} = S; \quad \lim_{j\to\infty} x_{n'_j} = S_0; \quad \lim_{j\to\infty} x_{n_j-k} = S_k.$$

Eq. (2) implies $L = \alpha L_0 + (1 - \alpha)h(L_k)$ which, in view of $L \leqslant L_0$, gives $h(L_k) \leqslant L$. Similarly, $h(S_k) \geqslant S$. Note that $L \leqslant S_k \leqslant p \leqslant L_k \leqslant S$.

Assume that possibility (a) holds. Suppose $L < a$ and recall that the interval $J = [L, b] \cup h([L, a])$ is invariant by h. Moreover, notice that L is the left endpoint of J. Since $S_k \leqslant p$ and $h(S_k) \geqslant S$, we get that $S \in J$. Then $L_k \leqslant S$ implies $L_k \in J$. In fact $h(L_k) \leqslant L$ forces $L_k > b$ (because $[a, b]$ is invariant and $L < a$) so there is $x \in [L, a)$ such that $h(x) = L_k$, which contradicts Lemma 16. We have shown $L \in [a, b]$. Since $L \leqslant S_k \leqslant p$ and $h(S_k) \geqslant S$, we also get $S \in [a, b]$.

Now we prove that the solution (x_n) eventually falls into $[a, b]$. Since $L, S \in [a, b]$ this is clearly the case if it accumulates at some interior point of $[a, b]$ (recall that $\alpha > 0$). Now, suppose that (x_n) only accumulates at $\{a, b\} = \{L, S\}$.

Then $S = S_0 = h(S_k) = b$. Furthermore, if the sequence $(x_{n'_j})$ is chosen so that $(x_{n'_j+r})$ converges for every $1 < r \leqslant k+1$, then the limit of each of these sequences is b, hence $b = \alpha b + (1-\alpha)h(b)$. We have arrived at the contradiction $h(b) = b$.

Assume finally that possibility (b) holds. Since $h((0, d]) \subset [0, d]$, we get $h((0, x]) \subset [0, x)$ for every $x > d$. Now, $S > d$ implies the existence of a number $S_k \leqslant p$ satisfying $h(S_k) \geqslant S > d$, a contradiction. We prove that the solution (x_n) eventually falls into $(0, d]$ similarly as before (d is not a fixed point of h). $\square$

In what follows we use the notation $B_\varepsilon(x) := (x - \varepsilon, x + \varepsilon)$.

Lemma 20. *Let $\varepsilon > 0$ be small enough. If $\alpha \geqslant 0$ is very small and (x_n) is a solution of* (2), *then every subsequence $(x_{(2k+2)m+r})_m$, $0 \leqslant r < 2k+2$, eventually falls into one of the intervals $B_\varepsilon(p)$, $B_\varepsilon(q_1)$, $B_\varepsilon(q_2)$.*

Proof. To begin with we fix $\varepsilon > 0$ taking advantage of the hyperbolicity of the 2-cycle $\{q_1, q_2\}$ to ensure that $h^2(\mathrm{Cl}(B_\varepsilon(q_i)) \subset B_\varepsilon(q_i)$. Next we define u after applying Lemma 18 to the neighborhoods $U_p = B_{\varepsilon/2}(x)$, $V_{q_1} = B_{\varepsilon/2}(q_1)$, $V_{q_2} = B_{\varepsilon/2}(q_2)$.

We prove that if α is very small and the sequence $(x_{(2k+2)m+r})_m$ accumulates outside $B_\varepsilon(p)$, then it eventually falls into $B_\varepsilon(q_1)$ or $B_\varepsilon(q_2)$. We again assume that the more difficult possibility (b) holds.

Suppose that $x_{(2k+2)t+r} \in (0, q_1]$ for some t. Then Lemma 17(ii), Lemma 18, the uniform continuity of $h|_{[c/2,d]}$, and the fact that α is very small, imply that $x_{(2k+2)(t+j)+r} \in B_\varepsilon(q_1)$ for some $j \leqslant s/(2k+2)$ and hence $x_{(2k+2)m+r} \in B_\varepsilon(q_1)$ for every $m \geqslant t + j$. If $x_{(2k+2)m+r} \in [q_1, d]$ for every m, then we find t such that $x_{(2k+2)t+r} \notin B_\varepsilon(p)$ and reason analogously to find $j \leqslant s/(2k+2)$ such that $x_{(2k+2)m+r} \in B_\varepsilon(q_1)$ or $x_{(2k+2)m+r} \in B_\varepsilon(q_2)$ for every $m \geqslant t + j$. $\square$

We are ready to demonstrate Theorem 8. Let $F_\alpha : K_\alpha \subset [0, \infty)^{k+1} \to [0, \infty)^{k+1}$ be defined by

$$F_\alpha(u_0, u_1, \ldots, u_k) = (u_1, \ldots, u_k, \alpha u_k + (1-\alpha)h(u_0)),$$

whenever the right-side expression makes sense. Note that there is a natural correspondence between the solutions of (2) and the orbits of the discrete dynamical system associated to F_α. Namely if $\mathbf{u} = (x_{-k}, \ldots, x_{-1}, x_0)$ and (x_n) is the solution of (2) with initial vector $\mathbf{u}$, then $F_\alpha^n(\mathbf{u}) = (x_{n-k}, \ldots, x_{n-1}, x_n)$ for every n. Hence (2) and F_α have the same dynamics. In particular, in order to finish the proof, it suffices to show that if $\alpha \geqslant 0$ is small enough, then $G_\alpha = F_\alpha^{2k+2}$ has finitely many fixed points (some of them becoming $(2k+2)$-periodic points for F_α) and every orbit under G_α converges to one of these fixed points.

We immediately see that the only periodic points for G_0 are those of the form $(a_0, a_1, \ldots, a_k)$ with $a_i \in \{p, q_1, q_2\}$ for every i. All these points are fixed points for G_0 (and, for instance, $(q_1, q_1, \ldots, q_1)$ is a $(2k+2)$-periodic point for F_0). Now we fix a very small $\varepsilon > 0$ and correspondingly assume that $\alpha \geqslant 0$ is small enough. Let $(G_\alpha^n(\mathbf{u}))$ be an arbitrary orbit under G_α. We already know (Lemma 20) that this orbit eventually falls into a set $O = B_\varepsilon(v_0) \times B_\varepsilon(v_1) \times \cdots \times B_\varepsilon(v_k)$ for some $\mathbf{v} = (v_0, v_1, \ldots, v_k) \in \{p, q_1, q_2\}^{k+1}$. This immediately implies that $(G_0^n(\mathbf{u}))$ converges to $\mathbf{v}$ and also that if n is large enough, then the ith coordinate of $G_0^n(\mathbf{u})$ is zero whenever $a_i = p$. In other words, for all large numbers n the points $G_0^n(\mathbf{u})$ belong to the local stable manifold $W_0^s(\mathbf{v})$ of the fixed point $\mathbf{v}$ of G_0, this manifold being defined by $W_0^s(\mathbf{v}) = W_0 \times W_1 \times \cdots \times W_k$ with $W_i = \{v_i\}$ or $W_i = B_\varepsilon(v_i)$ according to whether $v_i = p$ or $v_i \in \{q_1, q_2\}$.

If $\alpha > 0$ is small enough, then, starting from the hyperbolic fixed point $\mathbf{v}$, the implicit function theorem applies to guarantee that O contains exactly one fixed point $\mathbf{v}_\alpha$ of G_α (which has the same minimal period as $\mathbf{v}$ for respectively F_α and F_0) and no other periodic points. Moreover, the stable manifold theorem (see, e.g., [16, Chapter 5]) ensures that every orbit under G_α not escaping from O must eventually fall into the local stable manifold of $\mathbf{v}_\alpha$ and then converge to $\mathbf{v}_\alpha$. This finishes the proof.

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Corrigendum

Corrigendum to: "Periodic points and stability in Clark's delayed recruitment model" [Nonlinear Analysis: Real World Applications 9 (2008) 776–790]

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Although they do not affect the main results, it is necessary to fix two incorrect assertions concerning the example investigated in Section 4.2 of the above-referenced paper.

In this example we considered the following difference equation (labelled as (10) in the paper):

$$x_{n+1} = \alpha x_n + (1-\alpha)2e^{2-x_{n-2}}, \quad n = 2, 3, \ldots, \tag{1}$$

with $\alpha \in [0, 1)$, and initial conditions $(x_0, x_1, x_2) \in \mathbb{R}^3_+ = [0, \infty)^3$.

The statement of Corollary 12 in page 785 is incorrect. It should be replaced by the following one:

Corollary 12. *If $\alpha \geq 0$ is small enough, then Eq. (1) has exactly one repelling equilibrium, one attracting 2-cycle, and four 6-cycles (three saddles and one node). They attract all solutions of (1).*

The reason is that Eq. (1) with $\alpha = 0$, that is,

$$x_{n+1} = 2e^{2-x_{n-2}}, \quad n = 2, 3, \ldots, \tag{2}$$

has four solutions of minimal period six. One of them is an attractor defined by $\{a, a, a, b, b, b, \ldots\}$, where $\{a, b\}$ is the unique cycle of the map $h(x) = 2e^{2-x}$ with minimal period two. The other three are saddles, which combine the points a, b and the unstable fixed point $p = 2$ of h. Namely, they are given by $\{a, b, 2, b, a, 2, \ldots\}$, $\{a, a, 2, b, b, 2, \ldots\}$, and $\{a, 2, 2, b, 2, 2, \ldots\}$.

In fact, the region of attraction of these three cycles, considered as 6-periodic orbits of the map $F : \mathbb{R}^3_+ \to \mathbb{R}^3_+$ defined by $F(x, y, z) = (y, z, h(x))$, is the set

$$W = \{(x, y, z) \in \mathbb{R}^3_+ : (x-2)(y-2)(z-2) = 0\} \setminus \{(2, 2, 2)\}.$$

The set W is formed by the union of the stable manifolds of each point of these three orbits under the 6th iterate of F.

We emphasize that these periodic saddles are missing in the two lines preceding the statement of Theorem 11 in p. 785 of our paper. However, the statement of Theorem 11 is correct, because it excludes the solutions of (2) belonging to W.

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