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# A Note on the Global Stability of Generalized Difference Equations

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**Abstract**—In this note, we prove a discrete analogue of the continuous Halanay inequality and apply it to derive sufficient conditions for the global asymptotic stability of the equilibrium of certain generalized difference equations. The relation with some numerical schemes for functional delay differential equations is discussed. 

**Keywords**—Global asymptotic stability, Difference equations, Halanay inequality, Discretization of functional differential equations.

## 1. INTRODUCTION

In [1, Section 4.5], Halanay proved an asymptotic formula for the solutions of a differential inequality involving the "maximum" functional, and applied it in the stability theory of linear systems with delay. Such inequality was called *Halanay inequality* in several works [2–7], in which some generalizations and new applications can be found. In particular, in [6,8], the authors consider discrete Halanay-type inequalities in order to study some discretized versions of functional differential equations.

As it is pointed out in [6], although there are many numerical schemes to approximate the solutions of continuous type systems, the asymptotic behaviour of the two types of systems (discrete and continuous) do not often coincide. For other comments on the difference between the dynamics of a continuous time delay differential equation and its discrete version, see also [8].

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The purpose of this note is to give a simple discrete version of Halanay's lemma, and to apply it to obtain results on the global asymptotical stability of certain generalized difference equations. Moreover, we show that the classical criterion on the absolute stability (or delay-independent stability) in certain delay equations holds for the discretized equation using the Euler scheme if the discretization step is small enough.

## 2. DISCRETE HALANAY INEQUALITY

In this section, we give a discrete version of the original Halanay inequality. Although our result could be deduced from [6, Theorem 3.1], we include here our simpler proof in order to point out how the constant $\lambda_0$ involved in the asymptotic formula can be easily obtained.

First, we give some preliminaries on the difference equation

$$\Delta x_n = f(n, x_n, x_{n-1}, \dots x_{n-r}), \qquad n \in \mathbb{N}, \tag{1}$$

where $\Delta x_n = x_{n+1} - x_n$, and $f : \mathbb{N} \times \mathbb{R}^{r+1} \to \mathbb{R}$. Equation (1) is a class of generalized difference equation (see [9, Section 2]). The initial value problem for this equation requires the knowledge of initial data $\{x_{-r}, x_{-r+1}, \dots, x_0\}$. This vector is called *initial string* in [8]. For every initial string, there exists a unique solution $\{x_n\}_{n \geq -r}$ of (1) that can be calculated by the explicit recurrence formula

$$x_{n+1} = x_n + f(n, x_n, x_{n-1}, \dots x_{n-r}), \qquad n \geq 0. \tag{2}$$

THEOREM 1. *Let $r > 0$ be a natural number, and let $\{x_n\}_{n \geq -r}$ be a sequence of real numbers satisfying the inequality*

$$\Delta x_n \leq -a x_n + b \max\{x_n, x_{n-1}, \dots, x_{n-r}\}, \qquad n \geq 0. \tag{3}$$

*If $0 < b < a \leq 1$, then there exists a constant $\lambda_0 \in (0,1)$ such that*

$$x_n \leq \max\{0, x_0, x_{-1}, \dots x_{-r}\} \lambda_0^n, \qquad n \geq 0.$$

*Moreover, $\lambda_0$ can be chosen as the smallest root in the interval $(0,1)$ of equation*

$$\lambda^{r+1} + (a-1)\lambda^r - b = 0. \tag{4}$$

PROOF. Let $\{y_n\}$ be a solution of the difference equation

$$\Delta y_n = -a y_n + b \max\{y_n, y_{n-1}, \dots, y_{n-r}\}, \qquad n \geq 0. \tag{5}$$

Since $1 - a \geq 0$, $b > 0$, it is easy to prove that if $\{x_n\}$ satisfies (3) and $x_n \leq y_n$ for $n = -r, \dots, 0$, then $x_n \leq y_n$ for all $n \geq 0$.

Now, if $K > 0$, $\lambda \in (0,1)$, the sequence $\{y_n\}$ defined by $y_n = K\lambda^n$ is a solution of equation (5) if and only if $\lambda$ is a solution of (4). Define $F(\lambda) = \lambda^{r+1} + (a-1)\lambda^r - b$. $F$ is continuous on $(0,1]$, $\lim_{\lambda \to 0^+} F(\lambda) = -b < 0$, and $F(1) = a - b > 0$. Hence, there exists $\lambda_0 \in (0,1)$ such that $F(\lambda_0) = 0$ (we can choose the smallest value of $\lambda$ satisfying this equation since $F(\lambda)$ is a polynomial and it has at most $r+1$ real roots).

Thus, for this $\lambda_0$, $\{K\lambda_0^n\}$ is a solution of (5) for every $K > 0$. Finally, let $K = \max\{0, x_{-r}, \dots, x_0\}$. Clearly, $y_n \geq x_n$ for all $n = -r, \dots, 0$. Hence, using the first part of the proof, we can conclude that $x_n \leq y_n = K\lambda_0^n$ for all $n \geq 0$.

## 3. ASYMPTOTIC STABILITY OF GENERALIZED DIFFERENCE EQUATIONS

In this section, we consider the following generalized difference equation:

$$\Delta x_n = -ax_n + f(n, x_n, x_{n-1}, \dots x_{n-r}), \qquad a > 0. \tag{6}$$

Although for every initial string $\{x_{-r}, x_{-r+1}, \dots, x_0\}$, the solution $\{x_n\}$ of (6) can be explicitly calculated by a recurrence similar to (2), in general it is difficult to investigate the asymptotic behaviour of the solutions using that formula. The next result gives an asymptotic estimate by a simple use of the discrete Halanay inequality.

THEOREM 2. *Assume that $0 < a \leq 1$ and there exists a positive constant $b < a$ such that*

$$|f(n, x_n, \dots, x_{n-r})| \leq b\|(x_n, \dots, x_{n-r})\|_\infty, \qquad \forall\,(x_n, \dots, x_{n-r}) \in \mathbb{R}^{r+1}. \tag{7}$$

*Then there exists $\lambda_0 \in (0,1)$ such that for every solution $\{x_n\}$ of (6), we have*

$$|x_n| \leq \left(\max_{-r\leq i\leq 0} \{|x_i|\}\right) \lambda_0^n, \qquad n \geq 0,$$

*where $\lambda_0$ can be calculated in the form established in Theorem 1.*

*As a consequence, the trivial solution of equation (6) is globally asymptotically stable.*

PROOF. Let $\{x_n\}$ be a solution of equation (6). From [9, Section 1], we know that

$$x_n = x_0(1-a)^n + \sum_{i=0}^{n-1}(1-a)^{n-i-1} f(i, x_i, \dots, x_{i-r}), \qquad n \geq 0.$$

Thus, using (7), we obtain

$$|x_n| \leq |x_0|(1-a)^n + \sum_{i=0}^{n-1}(1-a)^{n-i-1} b \max\{|x_i|, \dots, |x_{i-r}|\}, \qquad n \geq 0.$$

Denote $v_n = |x_n|$ for $n = -r, \dots, 0$, and

$$v_n = |x_0|(1-a)^n + \sum_{i=0}^{n-1}(1-a)^{n-i-1} b \max\{|x_i|, \dots, |x_{i-r}|\},$$

for $n > 0$. We have that $|x_n| \leq v_n$, and hence,

$$\Delta v_n = -av_n + b\max\{|x_n|, \dots, |x_{n-r}|\} \leq -av_n + b\max\{v_n, \dots, v_{n-r}\}, \qquad n > 0.$$

Theorem 1 ensures that

$$|x_n| \leq v_n \leq \left(\max_{-r\leq i\leq 0}\{v_i\}\right)\lambda_0^n = \left(\max_{-r\leq i\leq 0}\{|x_i|\}\right)\lambda_0^n, \qquad n \geq 0,$$

with $\lambda_0$ as it was indicated in the statement of the theorem.

EXAMPLES. Equation (6) covers a variety of difference equations. For instance, we can mention equations

$$\Delta x_n = -ax_n + f(x_{n-k}), \qquad a > 0, \tag{8}$$

investigated recently in [10]. Theorem 2 ensures that if there exists $b > 0$ such that $|f(x)| \leq b|x|$ for all $x$, and $b < a \leq 1$, then all solutions of (8) converge to zero.

On the other hand, condition (7) is satisfied by some linear and nonlinear generalized difference equations. We can mention equation

$$x_{n+1} = \sum_{i=n-r}^{n} b_i x_i, \qquad b_i \in \mathbb{R}, \tag{9}$$

*for which Theorem 2 gives the global asymptotic stability of the equilibrium if*

$$\sup_{n\in\mathbb{N}} \sum_{i=n-r}^{n} |b_i| < 1, \tag{10}$$

since equation (9) can be rewritten in the form

$$\Delta x_n = -x_n + f(x_n, \dots, x_{n-r}),$$

with $f(x_n, \dots, x_{n-r}) = \sum_{i=n-r}^{n} b_i x_i$. For example, (10) is satisfied by equation $x_{n+1} = (1/p)(x_n + \dots + x_{n-r})$ if $p > r+1$.

For more general results on the asymptotic behaviour of generalized difference systems, we refer the reader to [9, Section 9].

We also point out that Theorem 2 applies to the linear difference equation $x_{n+1} = bx_n$. In this case, the condition provided by our theorem is precisely $|b| < 1$. Thus, with some reservations, we can affirm that our result is sharp in some sense.

REMARKS. A similar result to Theorem 2 can be obtained for systems of difference equations by using appropriate norms of vectors and matrices. Moreover, our result can be also extended to generalized difference equations with variable coefficients $a_n$ using the results in [6].

## 4. DISCRETIZATION OF DELAY EQUATIONS

Let us consider the functional differential equation

$$x'(t) = -ax(t) + b(t)f(t, x_t), \qquad x_0 = \phi, \tag{11}$$

where $\phi \in \mathcal{C}(I)$, $I = [-\tau, 0]$, $x_t \in \mathcal{C}(I)$ is defined by $x_t(s) = x(t+s)$ for $s \in I$, $f : \mathbb{R} \times \mathcal{C}(I) \to \mathbb{R}$ is a continuous functional, $a > 0$ and $b \in L^\infty(\mathbb{R}, \mathbb{R})$.

Using the continuous Halanay inequality, one can prove (see [4, Corollary 3.2]) that if $|f(t,\phi)| \le \|\phi\| = \max_{s\in I} |\phi(s)|$ for all $\phi \in \mathcal{C}(I)$, and $\operatorname{ess\,sup}|b(t)| = b < a$, then all solutions of (11) converge to zero.

We are interested in studying if the delay-independent condition $b < a$ for the global stability of (11) is preserved when we use a numerical scheme to approximate the solutions of (11). First, we propose the following idea to perform this approximation. We divide the interval $I$ into $r$ subintervals of the same length $h = \tau/r$ ($h$ will be the discretization step), and introduce the notations $t_0 = 0$, $t_{n+i} = t_n + ih$, $x(t_n) = x_n$, $x(t_n + ih) = x_{n+i}$, $i \in \mathbb{Z}$. Since the initial function $\phi$ is known, we can calculate $x_n = \phi(t_n)$ for $n = -r, \dots, 0$. On the other hand, given $r+1$ points $p_{n-r} = (t_{n-r}, x_{n-r}), \dots, p_n = (t_n, x_n)$, we define $\psi_n : [t_{n-r}, t_n] \to \mathbb{R}$ as the piecewise linear function connecting the points $p_{n-r}, \dots, p_n$, and $\varphi_n(t) = \psi_n(t - t_n)$. Since $\varphi_n \in \mathcal{C}(I)$, we can evaluate $f(t_n, \varphi_n)$.

Thus, we can use the explicit Euler discretization method to approximate the solutions of (5) in the form

$$\frac{x_{n+1} - x_n}{h} = -ax_n + \bar{f}(n, x_n, \dots, x_{n-r}), \qquad n > 0, \tag{12}$$

where $\bar{f}(n, x_n, \ldots, x_{n-r}) = b(t_n)f(t_n, \varphi_n)$. Now, we can rewrite (12) in the form $\Delta x_n = -ahx_n + h\bar{f}(n, x_n, \ldots, x_{n-r})$. Moreover,

$$\left|h\bar{f}(n, x_n, \ldots, x_{n-r})\right| = |hb(t_n)f(t_n, \varphi_n)| \leq hb\|\varphi_n\| = hb\|(x_n, \ldots, x_{n-r})\|_\infty.$$

Thus, Theorem 2 allows us to ensure that all solutions of (12) converge to zero if $bh < ah \leq 1$, that is, if $b < a$ and $h \leq 1/a$. Hence, for a sufficiently small size of the discretization step, we can affirm that the asymptotic stability properties of equation (11) are preserved for the generalized difference equation (12).

## REFERENCES


1. A. Halanay, *Differential Equations: Stability, Oscillations, Time Lags*, Academic Press, New York, (1966).
2. C.T.H. Baker and A. Tang, Generalized Halanay inequalities for Volterra functional differential equations and discretized versions, In *Volterra Equations and Applications. Stability Control: Theory, Methods and Applications, Volume 10*, pp. 39–55, Papers from the Volterra Centennial Symposium, University of Texas, Arlington, TX, May 23–24, 1996, (Edited by C. Corduneanu and I.W. Sandberg), Gordon and Breach, Amsterdam, (2000).
3. K. Gopalsamy, Stability and oscillations in delay differential equations of population dynamics, In *Mathematics and its Applications, Volume 74*, Kluwer, Dordrecht, (1992).
4. A. Ivanov, E. Liz and S. Trofimchuk, Halanay inequality, Yorke 3/2 stability criterion, and differential equations with maxima, *Tohoku Math. J.* (to appear).
5. E. Liz and S. Trofimchuk, Existence and stability of almost periodic solutions for quasilinear delay systems and Halanay inequality, *J. Math. Anal. Appl.* **248**, 625–644, (2000).
6. S. Mohamad and K. Gopalsamy, Continuous and discrete Halanay-type inequalities, *Bull. Aus. Math. Soc.* **61**, 371–385, (2000).
7. M. Pinto and S. Trofimchu, Stability and existence of multiple periodic solutions for a quasilinear differential equation with maxima, *Proc. Roy. Soc. Edinburgh Sect. A* **130**, 1103–1118, (2000).
8. K.L. Cooke and A.F. Ivanov, On the discretization of a delay differential equation, *J. Differ. Equations Appl.* **6**, 105–119, (2000).
9. R.P. Agarwal and P.J.Y. Wong, Advanced topics in difference equations, In *Mathematics and its Applications, Volume 404*, Kluwer, Dordrecht, (1997).
10. I. Györi and S. Trofimchuk, Global attractivity and persistence in a discrete population model, *J. Differ. Equations Appl.* **6**, 647–665, (2000).