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Addendum to "Attractivity, multistability, and bifurcation in delayed Hopfield's model with non-monotonic feedback" [J. Differential Equations 255 (11) (2013) 4244–4266]

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In our recent paper [2], we developed some criteria of global attraction for general systems of delay differential equations based on a link with a finite-dimensional discrete dynamical system. In this note, we give an example showing the necessity of the key assumption in our main result. For convenience of the reader, we recall its statement.

Theorem 1. *(See [2, Theorem 2.5].) Assume that $F = (F_1, \dots, F_s) : D \subset \mathbb{R}^s \to D$ is a continuous map defined on $D = (a_1, b_1) \times \cdots \times (a_s, b_s)$, and $z_* \in \mathbb{R}^s$ is a strong attractor in D for*

$$x(N+1) = F\big(x(N)\big), \quad N = 0, 1, 2, \dots \tag{1}$$

Then, for each $\phi \in \{x \in \mathcal{C}([-\tau, 0], \mathbb{R}^s) : x(t) \in D$ for all $t \in [-\tau, 0]\}$,

$$\lim_{t \to +\infty} x(t, \phi) = z_*,$$

where $x(t, \phi)$ is the solution of system

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$$x_i'(t) = -x_i(t) + F_i\big(x_1(t-\tau_{i1}), x_2(t-\tau_{i2}), \dots, x_s(t-\tau_{is})\big), \quad 1 \le i \le s, \tag{2}$$

with initial condition $x(t,\phi) = \phi(t)$, $t \in [-\tau, 0]$ ($\tau = \max\{\tau_{ij} : i, j = 1, \dots, s\}$).

Roughly speaking, $z_* \in \mathbb{R}^s$ is a strong attractor for system (1) if for every compact set $K \subset D$, there exists a nested family of closed sets $\{I_n\}_{n\in\mathbb{N}}$ converging to z_* such that $K \subset \mathrm{Int}(I_1)$, $z_* \in \mathrm{Int}(I_n)$, and I_n is the product of s compact intervals for each $n \in \mathbb{N}$ (cf. Definition 2.1 in [2]). A scalar version of Theorem 1 was originally obtained by Mallet-Paret and Nussbaum [3], and Ivanov and Sharkovsky [1]. As it can be seen in these references, in the one-dimensional case it is enough to assume that the equilibrium of the discrete equation is globally asymptotically stable. We recall that a strong attractor is always a globally asymptotically stable equilibrium, but although in one dimension both notions coincide, in dimensions higher than one this property is no longer true (see [2, Appendix A]).

In the light of this fact, the following question naturally arises: Does the conclusion of Theorem 1 hold true if the notion of strong attractor is replaced by the one of globally asymptotically stable equilibrium?

Next we give an example showing that the answer is negative, that is to say, in contrast with the scalar case, a globally asymptotically stable equilibrium of the discrete dynamical system (1) in general does not give a globally attracting equilibrium of the continuous system (2), even if the time lag is zero. Thus, our example illustrates the necessity of introducing the more restrictive notion of strong attractor for proving Theorem 1.

Example 1. Consider the map $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F(x,y) = \big(F_1(x,y), F_2(x,y)\big) = \begin{cases} (x(1+x), 0) & \text{if } y \ge 1, \\ (x(1+x), y(1-y)) & \text{if } \frac{1}{2} \le y \le 1, \\ (x(1+x), y - \frac{1}{4}) & \text{if } \frac{1}{4} \le y \le \frac{1}{2}, \\ (4yx(1+x), 0) & \text{if } y \le \frac{1}{4}. \end{cases}$$

It is easy to see that F is locally Lipschitz-continuous in $\mathbb{R}^2$. Moreover, $(0,0)$ is a globally asymptotically stable equilibrium (actually, a stable global attractor) for the discrete system

$$\begin{cases} x_{n+1} = F_1(x_n, y_n), \\ y_{n+1} = F_2(x_n, y_n). \end{cases} \tag{3}$$

This claim follows immediately from the fact that

$$F^3(x,y) = (0,0), \quad \text{for all } (x,y) \in \mathbb{R}^2. \tag{4}$$

Indeed, it is straightforward to check that the second iteration F^2 has the following expression:

$$F^2(x,y) = \begin{cases} (0,0) & \text{if } y \ge 1, \\ (4y(1-y)x(1+x)(1+x+x^2), 0) & \text{if } \frac{1}{2} \le y \le 1, \\ (4(y-\frac{1}{4})x(1+x)(1+x+x^2), 0) & \text{if } \frac{1}{4} \le y \le \frac{1}{2}, \\ (0,0) & \text{if } y \le \frac{1}{4}. \end{cases}$$

Since $F(x,0) = (0,0)$ for all $x \in \mathbb{R}$, Eq. (4) follows.

On the other hand, for $1/2 \le y \le 1$, the system of differential equations

$$\begin{cases} x'(t) = -x(t) + F_1\big(x(t), y(t)\big) \\ y'(t) = -y(t) + F_2\big(x(t), y(t)\big) \end{cases} \tag{5}$$

can be written as

$$\begin{cases} x'(t) = x^2(t) \\ y'(t) = -y^2(t). \end{cases} \tag{6}$$

This system has the solution

$$\big(x(t), y(t)\big) = \left(\frac{-1}{t-1}, \frac{1}{t+1}\right) \in \mathbb{R} \times \left(\frac{1}{2}, 1\right], \quad 0 \le t < 1.$$

Since this solution blows up at $t = 1$, it is clear that $(0, 0)$ is not a global attractor of (5).

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