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Clark's Equation: A Useful Difference Equation for Population Models, Predictive Control, and Numerical Approximations

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Abstract
A one-dimensional discrete-time dynamical system can be also seen as a recurrence, a difference equation, or an iteration scheme; and sometimes theoretical results come from different contexts. In this paper, I present a short survey about a particular family of one-dimensional maps that I have found in different situations. First, I introduce and explain the various motivations for the equation, and then I state some relevant results, with suitable references. Finally, I include some open problems and some words of caution about a series of recent poor-quality papers that, pretending to rediscover this equation, provide trivial results.

Keywords Clark model · Predictive control · One-dimensional maps · Quadratic map · Ricker map · Global stability

Mathematics Subject Classification Primary 39A30; Secondary 92D25

1 Introduction

The main object of this paper is Clark's equation, a family of one-dimensional maps that I have found in different contexts (see equation (1.2) below). This introductory section is devoted to explain the various motivations of the equation in the order I found them.

The rest of the paper is organized as follows: In Sect. 2, I establish some dynamical properties of Clark's equation, with special attention to the stability of the equilibrium. A particular case occurs when we consider the quadratic map $f(x) = rx(1 - x)$ in (1.2); in this respect, I discuss how some simple properties of this equation seemed

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to remain unnoticed by several authors in a series of papers of dubious quality (e.g., [1,2,18,38]). Section 3 is devoted to further discussions and open problems.

1.1 Population Dynamics: The Clark Model

One of the main motivations to study discrete-time one-dimensional maps comes from population dynamics. A simple first order difference equation relating population abundance in one generation with population abundance in the preceding generation was popularized by May in the seventies of the last century in a series of papers (e.g., [30]), but the idea goes back at least to the fifties, with pioneering work of Moran (1950) [32], Ricker (1954) [39], and Beverton and Holt (1957) [4], among others.

Ricker's paper [39] is oriented to fisheries, and for this reason he uses the word *stock* rather than population abundance. A key concept in his work is the *reproduction curve,* that represents graphically the relationship between an existing stock and the future stock. This relationship allows a simple formulation in terms of one-dimensional discrete-time models, namely

$$x_{n+1} = f(x_n),\ \ n \geq 0, \tag{1.1}$$

where $f : I \to I$ is a continuous function defined on a real interval I (often $I = [0, \infty)$ in population models). For oviparous fishes which spawn once a year, Ricker suggested x_n to represent the number of eggs after n generations, starting at x_0. The relationship given by equation (1.1) is known by various names: egg-recruit [4], stock-recruitment [7], spawner-recruit [37], or the more general between-year-dynamics [12], among others. Another interesting remark by Ricker (already stated by Moran [32]) is the main condition that a reproduction map f must satisfy:

(H) f is continuous, $f(0) = 0$, and f has a unique positive equilibrium p; moreover, $f(x) > x$ for all $x \in (0, p)$, and $f(x) < x$ for all $x > p$.

The equilibrium p is often referred to as the *carrying capacity.* The most known result from Ricker's paper [39] is the derivation of the exponential form for a reproduction curve $f(x) = xe^{r(1-x)}$, $r > 0$, known nowadays as the Ricker map. Ricker derived his now famous expression from simple assumptions involving cannibalism, which can be extended to more general forms of predation of juveniles by adults. Actually, a situation where the reproduction happens in a single reproductive burst at the beginning of the year and adults attack juveniles leads to a simple mechanistic derivation of the Ricker model [12, p. 336]. This derivation assumes that populations are *semelparous,* that is, each mature individual makes a single reproductive contribution, in contrast to *iteroparous* populations, which have members that make multiple reproductive contributions [37]. Iteroparous populations are age-structured, and assume at least two age classes: adults and juveniles (individuals with or without sexual maturity, respectively). Although age-structured populations usually increase the dimension of the model, an alternative derivation of the Ricker model assuming iteroparity (see, e.g., the monograph of Thieme [42]), leads to equation

$$x_{n+1} = \alpha x_n + (1 - \alpha) f(x_n),\ \ n \geq 0, \tag{1.2}$$

where $f(x)$ is the Ricker map, and $\alpha \in (0, 1)$ is an adult's probability of surviving one year including the reproductive season. The parameter α can also be interpreted as the fraction of energy invested into adult survivorship rather than reproduction.

As far as I know, the one-dimensional stock-recruitment model with adult survivorship (1.2) was first proposed by Clark [7], and hence it is usually referred to as Clark's equation (Ricker-Clark in the particular case when f is the Ricker map). Actually, Clark's equation is more general, since it often denotes a delayed version of (1.2), namely

$$x_{n+1} = \alpha x_n + (1-\alpha) f(x_{n-k}),\ \ n \geq 0, \tag{1.3}$$

which assumes a maturation delay $k \geq 1$, that is, recruitment to the breeding population takes place k years after birth [8] (see [11] for further comments and relevant references). Equation (1.3) was employed in fishery models, and in particular for whale populations, with different choices for f: Clark used the quadratic map $f(x) = rx(1-x), r > 1$, while the examples of Fisher [13] include the Beverton-Holt map $f(x) = \beta x/(1+x)$, $\beta > 1$, and a generalized logistic map $f(x) = rx(1-x)^z$, $z > 0$, already used by May [29].

Thus, Eq. (1.2) is the simplest form of Clark's model (individuals become adults in one reproductive season), and perhaps the simplest form of a population model with certain age structure. Sometimes the term $1 - \alpha$ does not appear explicitly in the equation, but of course it can be incorporated just changing f by $(1-\alpha)^{-1} f$; this is convenient because in this way (1.2) has the same equilibria as (1.1). See [21, Appendix A] to see how the term $1 - \alpha$ is incorporated to the population model in a natural way, using a balance equation.

We mention other contexts where Clark's equation (1.2) was used as a suitable mathematical model:

- Milton and Bélair [31] arrived at the same equation using a seasonal model to describe the growth of bobwhite quail populations; in this case, x_n is the quail population in the spring and f is the generalized Beverton-Holt map $f(x) = \beta x/(1 + x^m)$, $\beta > 1, m > 1$.
- Lasota proposed in 1977 [19] (see also [24]) a discrete model for the production of red blood cells (erythrocytes), which also leads to Clark's equation. In this case, x_n denotes the number of cells at time n, and $(1-\alpha)x_n$ is the number of cells destroyed in the time interval $[n, n+1]$. The function f gives the number of cells which are produced in the bone marrow during the same period, and has the form of a gamma-Ricker map [22] $f(x) = (cx)^\gamma e^{-x}$, where c, γ are positive constants.

It is also worth mentioning that the delayed Clark's equation (1.3) finds applications in other biological and physiological models, since it can be seen as a discretization of a relevant family of delay differential equations [27]; see, e.g., [11,15] for discussions on this topic.

1.2 Prediction-Based Control of Chaos

Mechanisms of control of chaos consist in introducing a perturbation in a chaotic system aiming at stabilizing a given unstable equilibrium or periodic orbit embedded in the chaotic attractor. There are many strategies of chaos control and many applications in various fields; for a review, see, e.g., [5]. In order to stabilize an equilibrium or a periodic orbit in the difference equation (1.1), there are some methods that consist in adding an external force (or control), leading to

$$x_{n+1} = f(x_n) + \alpha u_n, \ n \geq 0, \tag{1.4}$$

where the control input u_n can have several forms. Pyragas [36] suggested a delayed feedback control to stabilize a T-periodic orbit, choosing $u_n = x_{n-T} - x_n$. This work motivated that various authors developed a new form of control, named predictive control or prediction-based control (PBC for short). As far as I know, the first one was introduced by Sousa and Lichtenberg [9], who proposed the control input

$$u_n = x_n - f(x_n) \tag{1.5}$$

to stabilize an equilibrium of (1.1). Clearly, equations (1.4) and (1.5) result in Clark's equation (1.2). Further generalizations of this method are due to Ushio and Yamamoto [44], who used the term PBC for (1.4) with $u_n = x_n - f^T(x_n)$, aiming at stabilizing a T-periodic orbit of (1.1), and Polyak [35], who proposed the more general form $u_n = f^m(x_n) - f^{m+T}(x_n)$, $m \geq 0$, in order to choose a smaller value of α in (1.4) to stabilize the desired T-periodic orbit. An interesting idea in this paper is that the purpose of the author is not limited to chaos control; he suggests to use the control scheme as an iterative method to approximate a periodic orbit; in particular, for $T = 1$, it provides a numerical method to find a root p of the nonlinear equation $f(x) = x$ when the usual iterative method (1.1) does not work (because $|f'(p)| > 1$). This remark leads us to the third motivation of Clark's equation.

1.3 The Picard Method of Successive Approximations

In the framework of fixed point theory in functional spaces, if $F : C \subset X \to X$ is an operator defined in a convex subset of a Banach space X, it is usual to call the classical method of successive approximations given by

$$x_{n+1} = F(x_n), \ n \geq 0, \tag{1.6}$$

Picard iteration or the Picard iterative process. The Picard-Banach-Caccioppoli theorem ensures that the iterations of (1.6) converge to the unique fixed point of F if F maps C into itself and is strictly contractive, that is, $\|F(x) - F(y)\| \leq \rho\|x - y\|$, for all $x, y \in C$ and some positive constant $\rho < 1$. An idea to extend this result to a more general class of maps consists in considering the Picard iteration for the modified operator $F_\alpha = \alpha\,\mathrm{Id} + (1-\alpha)F$, where Id is the identity operator, and $\alpha \in (0,1)$, that is,

$$x_{n+1} = F_\alpha(x_n) = \alpha x_n + (1-\alpha)F(x_n),\ \ n \geq 0. \tag{1.7}$$

Once again, we arrive at (an abstract form of) Clark's equation. The iterative process (1.7) was first proposed by Krasnosel'skii [17] in 1955, with $\alpha = 1/2$, and later by Schaefer [41] in 1957 for general $\alpha \in (0,1)$. See [6] for further references and a survey of related results. Equation (1.7) is usually suggested as a simple method to achieve faster convergence in one-point iterative methods to solve a nonlinear equation (see, e.g., [34, pp. 205-206]).

In 1953, Mann [28] proposed another generalization of the Picard method, and in particular he proved that a sort of non-autonomous version of (1.7) provides a result of global stability for a continuous map $F : I \to I$ having a unique fixed point p, where $I = [a, b]$ is a compact real interval. Mann's iteration method is

$$x_{n+1} = \frac{n}{n+1}x_n + \frac{1}{n+1}F(x_n),\ \ n \geq 0. \tag{1.8}$$

We can see (1.8) as a nonautonomous version of (1.7), replacing α by a sequence $\alpha_n = n/(n+1)$. Mann's result was generalized to a class of functions defined in a convex subset of a Hilbert space by Johnson [16].

In summary, we can look at equation (1.2) at least from three different points of view: a discrete-time model for iteroparous populations, a particular example of prediction-based control, and a generalization of the simple iterative method to approximate a fixed point of a nonlinear function. In the following, I will refer to (1.2) as the Clark equation, though (for historical reasons) it could be termed the Schaefer equation. The reason why I chose the term Clark's equation is because most of my work on this equation was motivated by the biological context.

2 On Clark's Equation

In this section, I give a survey of some results concerning the dynamics of Clark's equation (1.2), assuming that $f : I \to I$ is a continuous function defined on a real interval I; more specific conditions will be set later.

Let us denote by $f_\alpha(x) = \alpha x + (1-\alpha)f(x)$ the map in the right-hand side of (1.2). It is clear that $f_\alpha : I \to I$ is well defined (because it is a convex combination of the identity map and f), and has the same fixed points as f. Local stability results are straightforward if f is $\mathcal{C}^1$. Assume that f has a fixed point $p \in I$, and p is unstable (under condition **(H)**, this means that $f'(p) \leq -1$). Since $f'_\alpha(p) = \alpha + (1-\alpha)f'(p)$, it follows that p becomes asymptotically stable for (1.2) if $\alpha > (f'(p)+1)/(f'(p)-1)$. This simple calculation shows the stabilizing effect of the PBC method (1.4)–(1.5). Next we see that, under some additional conditions, the stabilization is global.

2.1 Global Stability

In this subsection, I consider S-unimodal maps (see, e.g., [43]). More precisely, I assume the following conditions, which are motivated by some well-known discrete-time maps such as the quadratic (or logistic) and the Ricker maps (see [15,20,21,43] for further discussions and examples):

(A1) $f : I \to I$ (where $I = [0, b]$ or $I = [0, \infty)$) satisfies **(H)**.
(A2) f is a C^3 function and has a unique critical point c such that $f'(x) > 0$ for all $x \in (0, c)$, and $f'(x) < 0$ for all $x > c$.
(A3) $(Sf)(x) < 0$ for all $x \neq c$, where

$$(Sf)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

is the Schwarzian derivative of f.

We say that p is a global attractor for f (or a global attractor of (1.1)) if it is asymptotically stable and $\lim_{n\to\infty} f^n(x_0) = p$, for all $x_0 \in (0, b)$. Theorem 1 in [20] ensures that, under conditions (A1)–(A3), the positive equilibrium p is a global attractor of (1.2) if $f'_\alpha(p) \geq -1$, that is, if

$$\alpha \geq \frac{f'(p) + 1}{f'(p) - 1}, \tag{2.1}$$

and it is unstable otherwise. This means that local asymptotic stability implies global stability (the folklore "LAS implies GAS" statement). This result is not trivial because f_α, in general, does not inherit from f the unimodal shape and the negative sign of the Schwarzian derivative. We notice that the conditions assumed in the statement of Theorem 1 in [20] are slightly different, but it is easy to check that the proof remains valid under (A1)–(A3).

Applying the global stability result from [20] to the quadratic map $f(x) = rx(1-x)$ $(1 < r \leq 4)$, we obtain that the positive equilibrium $p = 1 - 1/r$ is a global attractor for f_α if and only if

$$1 < r \leq \frac{3 - \alpha}{1 - \alpha}. \tag{2.2}$$

In particular, when $\alpha = 0$, we get the usual global stability condition $1 < r \leq 3$ for the quadratic map.

A different global stability result for (1.2) has been recently proved in [23, Theorem 5]. Interestingly, such a result was motivated by equations of the form

$$x_{n+1} = x_n^\alpha\, g(x_n), \quad n = 0, 1, 2, \ldots, \tag{2.3}$$

where $g : [0, \infty) \to (0, \infty)$ is a smooth map and $\alpha > 0$. Equation (2.3) finds applications in various fields, like economics and ecology, among others. If $\alpha \in (0, 1)$, then the change of variables $y_n = -\ln(x_n/p)$ transforms (2.3) into the Clark equation (1.2). Thus, (2.3) with $\alpha \in (0, 1)$ can be seen as another motivation for studying Clark's equation. An example of (2.3) is the *gamma-Ricker* model [22,37]:

$$x_{n+1} = \beta x_n^{\gamma} e^{-\delta x_n}, \quad n = 0, 1, 2, \ldots, \tag{2.4}$$

where β, γ, δ are positive parameters.

The issue of global stability for the delayed Clark-type equation (1.3) has also received much attention, starting with the work of Fisher [13] (see [10,11] for more references). Although in the general case local stability does not imply global stability in (1.3) with $k \geq 3$ (as proved by Jiménez López and Parreño [15]), some "LAS implies GAS" results for $k = 1$ are available [3,14].

2.2 The Shape of f_α when f is Unimodal: The Case of the Quadratic Map

In this subsection, I answer the following question: is f_α unimodal if f is unimodal? This is a simple problem I dealt with in [21]. I focus on maps satisfying conditions (A1)–(A3); as explained in [21, Appendix B], f can have at most one inflexion point $\delta > c$, and, when it exists, $f'(\delta) = \min\{f'(x) : x \in I\} < 0$. Hence, we distinguish the following cases:

(a) If f is concave on $I = [0, b]$, then f_α is nondecreasing if $f'_\alpha(b) \geq 0$ (that is, $\alpha \geq -f'(b)/(1 - f'(b))$), and unimodal otherwise.
(b) If f has an inflexion point δ, then f_α is nondecreasing if $f'_\alpha(\delta) \geq 0$ (that is, $\alpha \geq -f'(\delta)/(1 - f'(\delta))$), and bimodal otherwise. In the latter case, there exist c_1, c_2 such that $0 < c_1 < \delta < c_2$, f_α attains a local maximum at c_1 and a local minimum at c_2.

An example of case (a) is the quadratic map $f(x) = rx(1 - x)$. In this case, f_α is unimodal if $0 \leq \alpha < r/(1 + r)$, and nondecreasing if $r/(1 + r) \leq \alpha < 1$.

An example of case (b) is the Ricker map $f(x) = xe^{r(1-x)}$, which has an inflexion point at $\delta = 2/r$, with $f'(\delta) = -e^{r-2}$. Hence, f_α is bimodal if $0 < \alpha < e^{r-2}/(1 + e^{r-2})$, and nondecreasing otherwise.

It is worth mentioning that the case of the quadratic map is special. As I pointed out in [21]:

> "The relation between the quadratic function $f(x) = rx(1 - x)$ and the map $f_\alpha(x) = \alpha x + (1 - \alpha) f(x)$ is only a rescaling of the parameter r after a change of variables. Indeed, the change $y = \gamma x$ with $\gamma = r(1 - \alpha)/(r(1 - \alpha) + \alpha)$ transforms equation
>
> $$x_{n+1} = \alpha x_n + (1 - \alpha) r x_n (1 - x_n) \tag{2.5}$$
>
> into $y_{n+1} = R y_n (1 - y_n)$, with $R = r(1 - \alpha) + \alpha$."

It follows from this simple observation that the dynamics of (2.5) can be translated to the well-known dynamics of the quadratic map (see, e.g., Thunberg's survey [43]). For example, since $p = 1 - 1/r$ is globally asymptotically stable for the quadratic map if $1 < r \leq 3$, it follows that p is globally asymptotically stable for (2.5) if $1 < R \leq 3$, which is equivalent to (2.2).

Surprisingly enough, this remark seemed to remain unnoticed by several authors, who devoted some papers to discuss the dynamics of (2.5). I give some examples: Kumari and Chugh [18] emphasize that their main result consists in showing experimentally that p is attracting for (2.5) with $\alpha = 0.95$ if $r \leq 32.1$; a direct application of (2.2) provides global attraction for $r \leq 41$. Moreover, notice that f_α is nondecreasing if $\alpha \geq r/(1+r)$; since $r \leq 4$, it follows that $r/(1+r) \geq 4/5 = 0.8$. Hence, if $\alpha \geq 0.8$, solutions of (2.5) starting at $x_0 \in (0, 1)$ converge monotonically to the equilibrium p. This remark gives a simple analytical improvement of the findings of Rani and Agarwal [38], who made an "experimental study" to conclude, among other trivial statements, that p is asymptotically stable for (2.5) if $\alpha > 0.83$. Papers [1,2], among other papers of Ashis, Cao and Chugh, are devoted to study different aspects of the dynamics of (2.5). Based on my previous considerations, all the results in those papers follow trivially from known results for the quadratic map.

2.3 Some Comments on Clark's Equation in the Bimodal Case

In contrast with the quadratic case, Clark's equation (1.2) with the Ricker map $f(x) = xe^{r(1-x)}$ or other unimodal maps is dynamically more challenging because f_α can have two critical points. Milton and Bélair [31] discussed some aspects about the role of the additional critical point; for example, increasing α not only helps to stabilization, but also creates a floor beneath which population abundance cannot fall, which have important consequences to prevent the risk of extinction in case of chaotic oscillations.

Together with some co-authors, I have studied the role of the parameter α in population models with harvesting in the case of constant harvesting [21] (a fixed quota is harvested every year), and proportional harvesting [26] (a proportion γx_n of the population is harvested, with $\gamma \in (0, 1)$). One of the more interesting phenomena in both cases is that, while increasing harvesting cannot destabilize a stable equilibrium in the Ricker model, destabilization is possible in the Clark-Ricker model. Increasing harvest intensity can destabilize the positive equilibrium via a period-doubling bifurcation, and larger harvesting efforts can lead to chaos, forming structures in the bifurcation diagram that we called chaotic bubbles, and are similar to paired period-doubling cascades studied by Sander and Yorke [40]. We have also studied in [25] the interplay between age structure and proportional harvesting using a simple 2-dimensional model of two age classes: juveniles and adults. The system can be reduced to a delayed Clark's equation (1.3), with $k = 1$. Interestingly, if adult-only harvest is considered, and adult survivorship s_a is equal to $1-s_j$, where s_j is juvenile survivorship (a common assumption due to intraspecific competition), then the effects of harvesting in (1.3) and (1.2) are qualitatively similar. This means that, in this case, the one-dimensional model (1.2) is a good tool to derive conclusions about an age-structured model, without increasing the dimension of the system.

In Lasota's model [19], we have also proved in [24] that increasing α can destabilize the equilibrium forming bubbles in the bifurcation diagram; however, in this case the map f is unimodal but it does not satisfy condition **(H)**.

3 Discussion

In this closing section, I would like to provide more comments about recent papers "rediscovering" Clark's equation, and also to state some open problems.

Rani and Agarwal refer to the sequence $\{x_n\}$ obtained from (1.1) as Picard iteration, and to the sequence $\{x_n\}$ obtained from (1.2) as *Mann iteration* or *superior sequence.* They do not provide any theoretical result, so their conclusions are heuristic and, as previously discussed, easy to improve by direct analytical results. But, unfortunately, this paper inspired some subsequent work, and it was even considered a *major breakthrough* by Ashis and Cao [1]. The paper [1] contains only trivial results and some numerical experiments; clearly all of them follow in a straightforward way from the above stated simple observation that, for the quadratic map, (1.2) is obtained from (1.1) by a linear change of variables. The authors write in their conclusions that they develop a "new fixed point iterative strategy", a statement which is far from being true, as I argued in this paper. I hope my arguments will help stopping the series of papers about Mann or superior iterations for the quadratic map.

But this paper aims also at making Clark's equation better known among readers not close to this ecological model or predictive control methods. In my opinion, there are still many open problems. For example, it is not clear whether or not, under conditions (A1)–(A3), local stability of the equilibrium implies global stability for the delayed Clark's equation (1.3) with $k = 1$ or $k = 2$. A very interesting discussion on this problem is given in [15, Section 2]. Another related problem suggested in the same paper is the following: if $k = 2$, then equation (1.3) with f satisfying (A1)–(A3) can have two coexisting attractors (see [11]); however, I do not know whether or not this is possible for $k = 1$, or even for $k = 0$ (in the cases for which f_α is bimodal). This is a relevant question, not only from a theoretical point of view; for population models, coexistence of two or more attractors results in multistability, that is, the long-term behavior of a population depends on its initial density.

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